JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 60, No. 1, JANUARY 1989

A Semistrong Sufficiency Condition for Optimality in Nonconvex Programming and Its Connection to the Perturbation Problem[1]

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Communicated by A. V. Fiacco

Abstract. What happens when a nonconvex program, having a local solution x^0 at which the gradients of the binding constraints are linearly independent, but without strict complementarity hypothesis, is perturbed? Under a relatively weak second-order assumption (some nonnegative second-order terms are supposed to be strictly positive), the perturbed problem has, in the neighborhood of x^0, a finite number of local minima, situated on curves that are connected to some pseudo-solutions of the tangent quadratic problem.

Key Words. Nonlinear programming, nonconvex optimization, perturbation of nonlinear programs, sufficiency conditions for optimality.

1. Introduction

Let us consider for every $\epsilon \in R$, the following nonlinear program:

$$\min \quad f^\epsilon(x), \qquad x \in R^n, \tag{1a$_\epsilon$}$$

$$\text{s.t.} \quad g_i^\epsilon(x) = 0, \qquad i = 1, \ldots, m, \tag{1b$_\epsilon$}$$

$$g_i^\epsilon(x) \leq 0, \qquad i = m+1, \ldots, p, \tag{1c$_\epsilon$}$$

where f^ϵ and g_i^ϵ, $\{i = 1, \ldots, p\}$ are C^2-functions from R^n to R, continuous with respect to $\epsilon \in R$, such that their gradients ∇f^ϵ and ∇g_i^ϵ are C^1 with respect to (x, ϵ). We denote

$$I^\epsilon(x) = \{i = i, \ldots, p;\ g_i^\epsilon(x) = 0\},$$

$$f(x) = f^0(x),$$

$$g_i(x) = g_i^0(x), \qquad i = 1, \ldots, p.$$

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Let x^0 be a local solution of $(1)_0$, and suppose that

$$\nabla g_i(x^0), \quad i=1,\dots,p, \quad \text{are linearly independent.} \tag{2}$$

Then there exists a unique Lagrange multiplier λ^0 in R^p such that

$$\text{(i)} \quad \nabla f(x^0)+\sum_{i=1}^{p} \lambda_i^0 \nabla g_i(x^0)=0, \tag{3a}$$

$$\text{(ii)} \quad g_i(x^0)=0, \qquad i=1,\dots,m, \tag{3b}$$

$$\text{(iii)} \quad g_i(x^0)\leqslant 0, \qquad \lambda_i^0 \geqslant 0, \qquad \lambda_i^0 g_i(x^0)=0, \qquad i=m+1,\dots,p. \tag{3c}$$

As we are interested in what happens in the neighborhood of x^0, we may suppose, to simplify the study, that

$$g_i(x^0)=0, \qquad i=1,\dots,p.$$

Let us denote

$$I_0^0=\{i=m+1,\dots,p;\ \lambda_i^0=0\},$$

$$H=\nabla^2 f(x^0)+\sum_{i=1}^{p} \lambda_i^0 \nabla^2 g_i(x^0).$$

A necessary condition for x^0 to be a local solution of $(1)_0$, called the second-order necessary condition, is that

$$\nabla g_i(x^0)'d=0, \qquad i\in(1,p)-I_0^0, \tag{4a}$$

$$\nabla g_i(x^0)'d\leqslant 0, \qquad i\in I_0^0, \tag{4b}$$

$$\text{imply} \quad d'Hd\geqslant 0. \tag{4c}$$

By changing an inequality into a strict inequality, we obtain the standard second-order sufficiency condition:

$$d\neq 0, \tag{5a}$$

$$\nabla g_i(x^0)'d=0, \qquad i\in(1,p)-I_0^0, \tag{5b}$$

$$\nabla g_i(x^0)'d\leqslant 0, \qquad i\in I_0^0, \tag{5c}$$

$$\text{imply} \quad d'Hd>0. \tag{5d}$$

Condition (5) ensures that x^0 is a strict local solution of $(1)_0$. Together with (2), it implies the following stability result (see Ref. 1, Theorem 3.1, and Corollary 4.3).

Theorem 1.1. Let x^0 be a local solution of $(1)_0$, and let λ^0 be an associated Lagrange multiplier such that (2), (3), and (5) hold. Then there exists $\varepsilon^0>0$ and $a_1>0$ such that, for $|\epsilon|<\epsilon^0$, problem $(1)_\epsilon$ possesses (at least) one local solution x^ϵ, with an associated Lagrange multiplier λ^ϵ, such that

$$\|x^\epsilon-x^0\|+\|\lambda^\epsilon-\lambda^0\|\leqslant a_1|\varepsilon|. \tag{6}$$

In addition, there exists $\alpha>0$ such that, for $|\epsilon|<\epsilon^0$, any local solution x^ϵ such that $\|x^\epsilon-x^0\|<\alpha$ has an associated Lagrange multiplier λ^ϵ that satisfies (6).

Stronger results have been obtained with the hypothesis that the following strong second-order sufficiency condition holds:

$$d\neq 0, \tag{7a}$$

$$\nabla g_i(x^0)'d=0, \qquad i\in(1,p)-I_0^0, \tag{7b}$$

$$\text{imply} \quad d'Hd>0. \tag{7c}$$

We notice that (7) implies (5). Using (7), Jittorntrum (Ref. 2) proved the following theorem.

Theorem 1.2. Assume that the hypotheses of Theorem 1.1 hold, with (7) instead of (5). Then for ϵ small enough, there is a unique solution x^ϵ of $(1)_\epsilon$ such that (6) holds, and the mapping $\epsilon\to(x^\epsilon,\lambda^\epsilon)$ has directional derivatives.

If I_0^0 is empty, i.e., the strict complementarity conditions are fulfilled, (5) and (7) are equivalent and the mapping $(x^\epsilon,\lambda^\epsilon)$ is C^1 (see Ref. 3). It is easy to see that, if card $(I_0^0)=1$, conditions (5) and (7) are also equivalent; but this is no longer true, in general, if card $(I_0^0)\geqslant 2$. The main drawback of the strong second-order sufficiency condition is that it assumes the strict positivity of an amount that can be strictly negative for an isolated minimum. Hence, it is very strong. In contrast, we will use an hypothesis in which we only assume the strict positivity of nonnegative amounts.

Definition 1.1. We will say that (I_1, I_2) is a pseudo-optimal subpartition of $(1, p)$, or shortly a pseudo-optimal subpartition, if

$$I_1\supset(1,p)-I_0^0,$$

$$I_2\subset(1,p)-I_1,$$

and

$$\nabla g_i(x^0)^t d = 0, \qquad i \in I_1,$$

$$\nabla g_i(x^0)^t d \leqslant 0, \qquad i \in I_2,$$

$$\text{imply} \quad d^t H d \geqslant 0.$$

Definition 1.2. We say that x^0 satisfies the semistrong second-order sufficiency condition if the following holds: For any pseudo-optimal subpartition (I_1, I_2), we have

$$d \neq 0, \tag{8a}$$

$$\nabla g_i(x^0)^t d = 0, \qquad i \in I_1, \tag{8b}$$

$$\nabla g_i(x^0)^t d \leqslant 0, \qquad i \in I_2, \tag{8c}$$

$$\text{imply} \quad d^t H d > 0. \tag{8d}$$

Let us remark that $((1, p) - I_0^0, I_0^0)$ is a pseudo-optimal subpartition. Hence, (8) is stronger than (5). On other hand, the semistrong second-order sufficiency condition is obviously weaker than the strong second-order sufficiency condition. If the problem $(1)_0$ is convex (i.e., if f and g_i, $i = m+1, \ldots, p$, are convex and g_i, $i = 1, \ldots, m$, are linear), then $((1, p) - I_0^0, \varnothing)$ is a pseudo-optimal subpartition. Hence, the strong and semistrong conditions are equivalent for convex problems. We stress the fact that the hypothesis is made on f and g, but is independent of the perturbations.

The paper is organized as follows. In Section 2, we assume only that the standard second-order sufficiency condition holds at x^0 and, considering a primal–dual local solution $(x^\epsilon, \lambda^\epsilon)$ of $(1)_\epsilon$, we study the limit points of $\epsilon^{-1}(x^\epsilon - x^0, \lambda^\epsilon - \lambda^0)$. We show that the limit points do not necessarily correspond to the local minima of the tangent quadratic problem (an example where this curious phenomenon occurs is given in Section 4). We call these limit points Pseudo-solutions of the tangent quadratic problem. In Section 3, we study how to compute the local minima associated with a given nondegenerate pseudo-solution of the tangent quadratic problem, and we deduce the characterization of the finite set of the local minima of $(1)_\epsilon$, close to x^0, when the semistrong second-order sufficiency condition holds.

2. Some Results Using Only the Standard Second-Order Sufficiency Condition

As we are interested in the sequel in the directional derivatives of the solutions of $(1)_\epsilon$, we limit our analysis to the case $\varepsilon \geqslant 0$. We define a branch to be a mapping $\epsilon \to (x^\epsilon, \lambda^\epsilon)$, defined for ϵ small enough, such that $(x^\epsilon, \lambda^\epsilon) \to (x^0, \lambda^0)$ when $\varepsilon \to 0$. We will speak of a branch of solutions if x^ϵ is a local solution of $(1)_\epsilon$ and λ^ϵ is its (uniquely defined for ϵ small and x^ϵ close to x^0) associated Lagrange multiplier, i.e.,

$$\text{(i)} \quad \nabla f^\epsilon(x^\epsilon) + \sum_{i=1}^{p} \lambda_i^\epsilon \nabla g_i^\epsilon(x^\epsilon) = 0, \tag{9a}$$

$$\text{(ii)} \quad g_i^\epsilon(x^\epsilon) = 0, \qquad i = 1, \ldots, m, \tag{9b}$$

$$\text{(iii)} \quad g_i^\epsilon(x^\epsilon) \leqslant 0, \qquad \lambda_i^\epsilon \geqslant 0, \qquad \lambda_i^\epsilon g_i^\epsilon(x^\epsilon) = 0, \qquad i = m+1, \ldots, p. \tag{9c}$$

Set

$$d^\epsilon = \epsilon^{-1}(x^\epsilon - x^0), \qquad \mu^\epsilon = \epsilon^{-1}(\lambda^\epsilon - \lambda^0).$$

Under the hypotheses of Theorem 1.1, $(d^\epsilon, \mu^\epsilon)$ is bounded. Denote

$$c_0 = \partial\Big(\nabla f^\epsilon(x^0) + \sum_{i=1}^{p} \lambda_i^0 \nabla g_i^\epsilon(x^0)\Big)/\partial\epsilon(\epsilon = 0),$$

$$c_i = \partial g_i^\epsilon(x^0)/\partial\epsilon(\epsilon = 0), \qquad i = 1, \ldots, p.$$

The usual linearization techniques applied to the system (9) lead to the following system (see Ref. 2):

$$Hd + c_0 + \sum_{i=1}^{p} \mu_i \nabla g_i(x^0) = 0, \tag{10a}$$

$$\nabla g_i(x^0)^t d + c_i = 0, \qquad i \in (1, p) - I_0^0, \tag{10b}$$

$$\nabla g_i(x^0)^t d + c_i \leqslant 0, \qquad \mu_i \geqslant 0, \qquad \mu_i(\nabla g_i(x^0)^t d + c_i) = 0, \qquad i \in I_0^0. \tag{10c}$$

For any solution (d, μ) of (10), we will denote

$$J(d) = \{i \in (1, p);\ \nabla g_i(x^0)^t d + c_i = 0\},$$

$$J_0(d) = \{i \in J(d) \cap I_0^0;\ \mu_i = 0\}.$$

Some simple computations involving the difference between (3) and (9) imply the following proposition.

Proposition 2.1. We assume that (2), (3), and (5) hold. Let (d, μ) be a limit point of $(d^\epsilon, \mu^\epsilon)$ when $\epsilon \to 0$. Then, (d, μ) satisfies (10).

Now, let us analyze what supplementary information on (d, μ) is given by the second-order necessary condition satisfied by $(x^\epsilon, \lambda^\epsilon)$. We denote

$$I_0^\epsilon(x^\epsilon) = \{i = m+1, \ldots, p;\ g_i^\epsilon(x^\epsilon) = 0,\ \lambda_i^\epsilon = 0\}.$$

The second-order necessary condition for x^ϵ is

$$\nabla g_i^\epsilon(x^\epsilon)^t d = 0, \qquad i \in I^\epsilon(x^\epsilon) - I_0^\epsilon(x^\epsilon), \tag{11a}$$

$$\nabla g_i^\epsilon(x^\epsilon)^t d \leq 0, \qquad i \in I_0^\epsilon(x^\epsilon), \tag{11b}$$

$$\text{imply} \quad d^t H^\epsilon d \geq 0, \tag{11c}$$

where

$$H^\epsilon = \nabla^2 f^\epsilon(x^\epsilon) + \sum_{i=1}^{p} \lambda_i^\epsilon \nabla^2 g_i^\epsilon(x^\epsilon).$$

Proposition 2.2. Assume that (2), (3), and (5) hold. Let $\{\epsilon^k\}_{k\in N}$ be a sequence such that $\epsilon^k \to 0$, and let $\{x^{\epsilon^k}\}$ be a sequence of local solutions of $(1)_{\epsilon^k}$ with associated unique Lagrange multiplier λ^ϵ. Let (I_1, I_2) be such that

$$(I_1, I_2) = (I^{\epsilon^k}(x^{\epsilon^k}) - I_0^{\epsilon^k}(x^{\epsilon^k}),\ I_0^{\epsilon^k}(x^{\epsilon^k})).$$

Then, (I_1, I_2) is a pseudo-optimal subpartition; and, if (d, μ) is a limit point of $(\epsilon^k)^{-1}(x^{\epsilon^k} - x^0, \lambda^{\epsilon^k} - \lambda^0)$, then

$$J(d) - J_0(d) \subset I_1 \subset J(d), \tag{12a}$$

$$I_2 \subset J_0(d) - I_1. \tag{12b}$$

Proof. Since (2) holds, the condition (11) holds at x^{ϵ^k}, and thus

$$\nabla g_i(x^0)^t \delta = 0, \qquad i \in I,$$

$$\nabla g_i(x^0)^t \delta \leq 0, \qquad i \in I_2,$$

$$\text{imply} \quad \delta^t H \delta \geq 0.$$

Let (d, μ) be a limit point of

$$(d^{\epsilon^k}, \mu^{\epsilon^k}) = (\varepsilon^k)^{-1}(x^{\epsilon^k} - x^0, \lambda^{\epsilon^k} - \lambda^0).$$

We have that $I_1 \cup I_2 \subset J(d)$, and $\lambda_i^{\epsilon^k} \neq 0$, for ϵ^k small enough, if $i > m$ and $i \in J(d) - J_0(d)$. Hence, $J(d) - J_0(d) \subset I_1$. As $I_1 \cap I_2 = \emptyset$, we deduce that

$$I_2 \subset J(d) - I_1 = J_0(d) - I_1.$$

□

It is usual to connect the system (10) to the following tangent quadratic problem (TQP):

$$\min\ c_0^t d + \tfrac{1}{2} d^t H d,$$

$$\nabla g_i(x^0)^t d + c_i = 0, \qquad i \in (1, p) - I_0^0,$$

$$\nabla g_i(x^0)^t d + c_i \leq 0, \qquad i \in I_0^0.$$

It is easy to verify that the standard second-order sufficiency condition (5) together with (2) imply that (TQP) has at least one local solution.

The first-order optimality conditions of (TQP) are precisely relations (10). As (TQP) is not convex, the first-order necessary conditions (10) are not sufficient for optimality. Notice however that, as (TQP) is quadratic, the second-order necessary conditions of (TQP) are also sufficient for local optimality.

Proposition 2.2 does not imply that a branch of local solutions of $(1)_\epsilon$ is connected to a solution of the (TQP). In fact, we give an example in Section 4, where a branch of local solution is not connected to a solution of the (TQP), but only to some $d \in R^n$ such that (10) is satisfied for some μ and (12) is satisfied for some pseudo-optimal subpartition.

Definition 2.1. We say that d is a pseudo-solution of the tangent quadratic problem associated with (I_1, I_2) if (12) is satisfied and there exists some μ such that (d, μ) satisfy (10).

Remark 2.1. (i) A pseudo-solution of the tangent quadratic problem associated with (I_1, I_2) is a local solution of what we may call a pseudo-tangent quadratic problem:

$$\min\ c_0^t d + \tfrac{1}{2} d^t H d, \tag{13a}$$

$$\nabla g_i(x^0)^t d + c_i = 0, \qquad i \in I_1, \tag{13b}$$

$$\nabla g_i(x^0)^t d + c_i \leq 0, \qquad i \in I_2. \tag{13c}$$

Problem (13) is obtained from (TQP) by discarding some inequality constraints and by converting some others into equality constraints.

(ii) We remark that a pseudo-solution of the (TQP) may be associated with several pseudo-optimal subpartitions.

(iii) A pseudo-solution of the (TQP) is a stationary point of (TQP), since it satisfies (10); but the converse is not always true, as shown in Example 4.2.

3. Nondegenerate Pseudo-Solutions of the (TQP) and the Semistrong Second-Order Sufficiency Condition

This section contains the main results. We first present some results using a nondegeneracy hypothesis on a given pseudo-solution of the (TQP). Then, we characterize the set of solutions of the perturbed problem under the semistrong second-order sufficiency condition.

Definition 3.1. A pseudo-optimal subpartition (I_1, I_2) is nondegenerate if the following condition holds:

$$\delta \neq 0, \tag{14a}$$

$$\nabla g_i(x^0)^t \delta = 0, \qquad i \in I_1, \tag{14b}$$

$$\nabla g_i(x^0)^t \delta \leq 0, \qquad i \in I_2, \tag{14c}$$

$$\text{imply} \quad \delta^t H \delta > 0. \tag{14d}$$

If, in addition, (I_1, I_2) is associated with a pseudo-solution d of the (TQP), we say that d is a nondegenerate pseudo-solution of the (TQP) and that (d, I_1, I_2) is nondegenerate.

Remark 3.1. There is at most one pseudo-solution of the (TQP) associated with a nondegenerate pseudo-optimal subpartition (I_1, I_2) because it is a part of the solution (d, μ) of the linear system

$$Hd + \sum_{i \in I_1 I_2} \mu_i \nabla g_i(x^0) = -c_0,$$

$$\nabla g_i(x^0)^t d = -c_i, \qquad i \in I_1 \cup I_2,$$

whose solution is unique, since (2) and (14) hold.

We consider the related nonlinear system

$$\nabla f^\epsilon(x^\epsilon) + \sum_{i \in I_1 \cup I_2} \lambda_i^\epsilon \nabla g_i^\epsilon(x^\epsilon) = 0, \tag{15a}$$

$$g_i^\epsilon(x^\epsilon) = 0, \qquad i \in I_1 \cup I_2. \tag{15b}$$

Lemma 3.1. We assume that (2), (3), and (14) hold. Then, for ε small enough, there is a unique C^1-branch $(x^\epsilon, \lambda^\epsilon)$ such that (15) is satisfied and $\lambda_i^\epsilon = 0$, $i \in (1, p) - (I_1 \cup I_2)$.

Proof. The system (15) is satisfied at $\varepsilon = 0$ by (x^0, λ^0). By Remark 3.1, the Jacobian of the system (15) with respect to $(x, \lambda_i, i \in I_1 \cup I_2)$, is nondegenerate. Hence, by the implicit function theorem, the system (15) has, for ϵ small, a unique solution $(x^\epsilon, \lambda_i^\epsilon, i \in I_1 \cup I_2)$. Taking $\lambda_i^\epsilon = 0$, $i \in (1, p) - (I_1 \cup I_2)$, we get the result. □

We will say that this branch is associated with (d, I_1, I_2). The following proposition gives a means to recognize whether this branch corresponds to the solutions of $(1)_\epsilon$.

Proposition 3.1. We assume that (2) and (3) hold. Let $(x^\epsilon, \lambda^\epsilon)$ be a branch associated with a nondegenerate triple (d, I_1, I_2). Then, for ϵ small enough, necessary conditions for x^ϵ to be a local minimum of $(1)_\epsilon$ are

$$g_i^\epsilon(x^\epsilon) \leq 0, \qquad i \in J_0(d) - (I_1 \cup I_2), \tag{16a}$$

$$\lambda_i^\epsilon \geq 0, \qquad i \in I_0^0 \cap (I_1 \cup I_2), \tag{16b}$$

$$(I^\epsilon(x^\epsilon) - I_0^\epsilon(x^\epsilon), I_0^\epsilon(x^\epsilon)) \text{ is a pseudo-optimal subpartition;} \tag{17}$$

and sufficient conditions are that (16) holds and

$$(I^\epsilon(x^\epsilon) - I_0^\epsilon(x^\epsilon), I_0^\epsilon(x^\epsilon)) \text{ is a nondegenerate pseudo-optimal subpartition.} \tag{18}$$

Proof. The definition of $J(d)$ implies that, for ϵ small, $g_i^\epsilon(x^\epsilon) < 0$, if $i \notin J(d)$; and, by the definition of that branch and (12), $g_i^\epsilon(x^\epsilon) = 0$, for i in $J(d) - J_0(d)$ and $I_1 \cup I_2$. Hence, x^ϵ will be feasible for $(1)_0$ iff (16a) holds. In the same way, we see that as $\lambda_i^\epsilon = 0$, if $i \notin I_1 \cup I_2$, and $\lambda_i^\epsilon > 0$, if $i \geq m+1$ and $i \notin I_0^0$, then λ^ϵ will be feasible iff (16b) holds. As λ^ϵ is the only possible multiplier associated with x^ϵ, condition (16) is necessary. If x^ϵ is a local solution of $(1)_\epsilon$, the necessity of (17) for ϵ small enough is a consequence of Proposition 2.20. Let us now prove that (16) and (18) are sufficient conditions. If x^ϵ satisfies (16) but is not a local solution, then for some $\delta^\epsilon \neq 0$:

$$\nabla g_i(x^\epsilon)^t \delta^\epsilon = 0, \qquad i \in I^\epsilon(x^\epsilon) - I_0^\epsilon(x^\epsilon),$$

$$\nabla g_i(x^\epsilon)^t \delta^\epsilon \leq 0, \qquad i \geq I_0^\epsilon(x^\epsilon),$$

$$(\delta^\epsilon)^t H^\epsilon \delta^\epsilon \leq 0.$$

We may suppose that $\|\delta\|^\epsilon = 1$. We take a converging subsequence of such $\{\delta^\epsilon\}$, when $\epsilon \to 0$. Passing to the limit in the above relations, we deduce that no limit set of $(I^\epsilon(x^\epsilon) - I_0^\epsilon(x^\epsilon), I_0^\epsilon(x^\epsilon))$ can be a nondegenerate pseudo-optimal subpartition. This proves that (16) and (18) are sufficient conditions for optimality. □

Corollary 3.1. We assume that (2), (3), and the semistrong second-order sufficiency condition (8) hold at x^0. Let $(x^\epsilon, \lambda^\epsilon)$ be as in Proposition 3.1. Then, for ϵ small enough and x^ϵ close to x^0, (16) and (18) are a necessary and sufficient condition for x^ϵ to be a local solution of $(1)_\epsilon$.

Proof. It is sufficient to notice that (8) is equivalent to the requirement that any pseudo-optimal subpartition be nondegenerate. □

We define

$$X_{\epsilon,\alpha}=\{x^\epsilon\in R^n;\ x^\epsilon \text{ is a local solution of (1) and } \|x^\epsilon-x^0\|<\alpha\}.$$

Now, if the semistrong second-order sufficiency condition holds, we have seen in Section 2 that any branch of solutions of $(1)_\epsilon$ is associated with at least one pseudo-solution of the (TQP). Corollary 3.1 gives a means to compute the unique solution associated with a given couple of pseudo-solutions and associated pseudo-optimal subpartitions. As there is a finite number of nondegenerate pseudo-optimal subpartitions, there is also a finite number of such couples. Hence, we have a constructive means to compute all solutions of $(1)_\epsilon$ close to x^0.

Theorem 3.1. We assume that hypotheses (2), (3), and the semistrong second-order sufficiency condition (8) hold. Let a_1 be as in Theorem 1.1. Then, for α small enough and $\varepsilon<a_1\alpha$, the set $X_{\epsilon,\alpha}$ is finite and nonempty. To compute it, it is sufficient to compute the elements of the finite number of branches associated with the pseudo-solutions of the (TQP) and to check whether they satisfy conditions (16) and (18).

The following procedure can be used to check the semistrong sufficiency condition and to compute the local solutions close to x^0 of the perturbed problem.

Step 1. Find x^0 together with λ^0 satisfying (2), (3), and (5).

Step 2. Find all pseudo-optimal subpartitions (there is a finite number of candidates) and check if they are nondegenerate [condition (14)]. If all pseudo-optimal subpartitions are nondegenerate, the semistrong sufficiency condition holds.

Step 3. For any nondegenerate pseudo-optimal subpartition (I_1, I_2): (a) find if the (TQP) has a pseudo-solution (d, μ) associated to (I_1, I_2); (b) if yes, compute branch $(x^\epsilon, \lambda^\epsilon)$ associated to (d, μ), using the system (15); (c) check if (16) and (18) hold for a given (small enough) ε; if yes, then x^ϵ is a local solution of $(1)_\epsilon$ and λ^ϵ is an associated Lagrange multiplier.

4. Examples

Example 4.1. We give an example in which a branch of solutions of the perturbed problem is associated with a pseudo-solution of the (TQP)

that is not a local solution of the (TQP). We consider the following problem:

$$\min x_1^2-x_2^2+(\epsilon^2-2/3\epsilon)(2x_1+x_2), \tag{19a$_\epsilon$}$$

$$2x_1-x_2-\epsilon\leq 0, \tag{19b$_\epsilon$}$$

$$-2x_1-x_2-\varepsilon\leq 0. \tag{19c$_\epsilon$}$$

For $\epsilon=0$, the problem reduces to

$$\min x_1^2-x_2^2;\ x_2\geq 2|x_1|. \tag{20}$$

It has a unique minimum $x^0=(0,0)^t$ with an associated unique multiplier $\lambda^0=(0,0)^t$.

Let us consider the tangent quadratic problem at $\epsilon=0$:

$$\min -2/3(2d_1+d_2)-d_1^2+d_2^2, \tag{21a}$$

$$2d_1-d_2-1\leq 0, \tag{21b}$$

$$-2d_1-d_2-1\leq 0. \tag{21c}$$

The first-order optimality conditions for problem (21) are

$$\text{(i)}\quad \begin{bmatrix}-2d_1\\ 2d_2\end{bmatrix}-\begin{bmatrix}\frac{4}{3}\\ \frac{2}{3}\end{bmatrix}+\mu_1\begin{bmatrix}2\\1\end{bmatrix}+\mu_2\begin{bmatrix}-2\\-1\end{bmatrix}=0; \tag{22a}$$

$$\text{(ii)}\quad 2d_1-d_2-1\leq 0,\qquad \mu_1\geq 0,\qquad \mu_1(2d_1-d_2-1)=0; \tag{22b}$$

$$\text{(iii)}\quad -2d_1-d_2-1\leq 0,\qquad \mu_2\geq 0,\qquad \mu_2(-2d_1-d_2-1)=0. \tag{22c}$$

Equations (22) have two solutions: (I) a solution for which only the first constraint is binding; we get

$$\bar d=(\tfrac{10}{9},\tfrac{11}{9})^t,\qquad \bar\mu=(\tfrac{16}{9},0)^t;$$

and (II) a solution for which only the second constraint is binding; we get

$$\bar d=(-\tfrac{2}{3},\tfrac{1}{3})^t,\qquad \bar\mu=(0,0)^t.$$

The first solution $\bar d$ is in fact the only local minimum of (21) and there is a branch of solutions of $(19)_\epsilon$ associated with $\bar d$:

$$\bar x^\epsilon=(10\epsilon/9-2\epsilon^2/3,\ 11\epsilon/9-4\epsilon^2/3)^t,$$

$$\bar\lambda^\epsilon=(16\epsilon/9-5\epsilon^2/3,\ 0)^t.$$

In contrast, the second solution $\tilde d$ is only a saddle point of problem (21). It is also a pseudo-solution associated to the subpartition $(\{2\},\varnothing)$. There is actually a branch of solutions associated with $\tilde d$:

$$\tilde x^\epsilon=(-2\epsilon/3,\ \epsilon/3)^t,$$

$$\tilde\lambda^\epsilon=(0,\ \epsilon^2)^t.$$

Example 4.2. In this example, the (TQP) has a stationary point th
is not a pseudo-solution. The problem is

$$\min x_1^2 - x_2^2,$$

$$2x_1 - x_2 - \epsilon \leq 0,$$

$$-2x_1 - x_2 - \epsilon \leq 0.$$

At $\epsilon = 0$, the problem still reduces to (20). The tangent quadratic problem

$$\min d_1^2 - d_2^2,$$

$$2d_1 - d_2 - 1 \leq 0,$$

$$-2d_1 - d_2 - 1 \leq 0.$$

The point $d = (0, 0)^t$ is obviously a stationary point of this probler We prove that it is not a pseudo-solution: $J(d)$ is empty; hence, by (12 the only subpartition that could be associated with d is $(I_1, I_2) = (\varnothing, \varnothing$ But the Hessian of the Lagrangian is not positive; hence, $(\varnothing, \varnothing)$ cann be a pseudo-optimal subpartition.

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JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 60, No. 1, JANUARY 1989

A Minimax Theorem for Vector-Valued Functions

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Communicated by P. L. Yu

Abstract. In this work, as usual in vector-valued optimization, we consider the partial ordering induced in a topological vector space by a closed and convex cone. In this way, we define maximal and minimal sets of a vector-valued function and consider minimax problems in this setting. Under suitable hypotheses (continuity, compactness, and special types of convexity), we prove that, for every

$$\alpha \in \operatorname{Max} \bigcup_{s \in X_0} \operatorname{Min}_w f(s, Y_0),$$

there exists

$$\beta \in \operatorname{Min} \bigcup_{t \in Y_0} \operatorname{Max} f(X_0, t),$$

such that $\beta \leq \alpha$ (the exact meanings of the symbols are given in Section 2).

Key Words. Minimax theorems, vector-valued optimization.

1. Introduction

Minimax theorems for real-valued (or extended real-valued) functions $f: X_0 \times Y_0 \to R$ [or $R \cup \{-\infty, +\infty\}$] state that, under suitable hypotheses of compactness, convexity, and continuity, the equality

$$\inf_{y \in Y_0} \sup_{x \in X_0} f(x, y) = \sup_{x \in X_0} \inf_{y \in Y_0} f(x, y)$$

holds. References 1–4 discuss on this subject. See also Ref. 5 for a survey and extensive bibliographical references. The numerous studies of vector-valued optimization in recent years (e.g., see Refs. 6–9) seem to lead, in a natural way, to the investigation of minimax problems in this more general

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