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On the Stability of Sets Defined by a Finite Number of Equalities and Inequalities

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Abstract. Let a set be defined by a finite number of equalities and inequalities. For smooth data, the condition of Mangasarian and Fromovitz is known to be equivalent to the local stability—in a strong sense—of the set. We study here weaker forms of stability. Namely, we state a condition generalizing the one of Mangasarian and Fromovitz that, for some weak form of stability, is necessary. If the gradients of the equality constraints are linearly independent or if there is no equality constraint, this condition is also sufficient.

Key Words. Stability analysis, nonlinear constraints, constraint qualification, second-order analysis

1. Introduction

We consider a subset of $\mathbb{R}^n$ (n being a positive integer) defined in the following way:

$$D=\{x\in\mathbb{R}^n;\, g(x)=0 \text{ and } h(x)\leq 0\},$$

where g and h are smooth mappings from $\mathbb{R}^n$ into $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively; the vector order relation is taken componentwise. Let $\bar{x}$ be in D. Our problem is to study the stability of D (near $\bar{x}$) when g and h are subject to small perturbations. As we are interested in a local analysis, we will suppose throughout the paper that $h(\bar{x})=0$. Specifically, we consider perturbations of the form

$$D^\epsilon=\{x\in\mathbb{R}^n;\, g^\epsilon(x)=0 \text{ and } h^\epsilon(x)\leq 0\},$$

with $\epsilon\in\mathbb{R}^+$, $g^0=g$, $h^0=h$, and the mapping $(\epsilon, x) \to (g^\epsilon(x), h^\epsilon(x))$ smooth

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enough, and we call them smooth perturbations. The problem is to give an estimate of $\operatorname{dist}(\bar{x}, D^\epsilon)$ defined by

$$\operatorname{dist}(\bar{x}, D^\epsilon) = \min\{\|\bar{x} - x^\epsilon\|;\ x^\epsilon \in D^\epsilon\}$$

(here, $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^n$) with the help of the successive derivatives of g and h at $\bar{x}$.

Such problems have already been studied, mainly in connection with optimization problems. Linear systems of equalities and inequalities have been studied in Daniel (Refs. 1–2), Robinson (Ref. 3), and Mangasarian and Shiau (Ref. 4). The behavior of linear systems seems rather well understood. Nonlinear systems are studied in Robinson (Ref. 5). In particular, the main result of Ref. 5 is that the condition of Mangasarian and Fromovitz (Ref. 6), namely,

$$\nabla g(\bar{x})^t \text{ is surjective,} \tag{1a}$$

$$\text{there exists } d \text{ in } \mathbb{R}^n \text{ such that } \nabla g(\bar{x})^t d = 0 \text{ and } \nabla h(\bar{x})^t d < 0, \tag{1b}$$

is necessary and sufficient for some strong form of stability. Our aim is to study a weak form of stability, for which it is necessary to consider the second derivatives of the data. We state a new condition, which is a generalization of (1), necessary for weak stability. If $\nabla g(\bar{x})^t$ is surjective, this condition is also sufficient.

The paper is organized as follows. In Section 2, we study a new concept of stability, the k-stability, and we characterize it on some simple examples. In Section 3, we prove that the k-stability for $k<2$ implies the 1-stability. In Section 4, we establish a technical result concerning linear systems of equalities and inequalities. In Section 5, we give a necessary condition for 2-stability; we prove that, if the gradients of the equality constraints are linearly independent, this condition is also sufficient for 2-stability.

2. Concept of k-Stability

We define the k-stability, relate it to the result of Robinson (Ref. 5), and give a study of simple examples.

Definition 2.1. The family of perturbations D^ϵ is l-smooth (l positive integer) if the mapping $(\epsilon, x) \to (g^\epsilon(x), h^\epsilon(x))$ is l-times continuously differentiable.

Definition 2.2. Let $\bar{x}$ be in D. We say that D is k-stable at $\bar{x}$ ($k>0$, real) if g and h are C^l (l-times continuously differentiable), with $l \geq k$, and for any

l-smooth perturbation D^ϵ, there exists $K > 0$ such that, for all ϵ small enough,

$$\mathrm{dist}(\bar{x}, D^\epsilon) \leq K\epsilon^{1/k}.$$

The simplest form of stability is obviously when $k=1$; in this case, a complete characterization is known. See the following theorem.

Theorem 2.1. See Ref. 5, Theorems 1 and 2. The set D is 1-stable at $\bar{x}$ iff condition (1) holds.

It remains, however, to study the k-stability for $k > 1$. In order to get some intuition of the kind of result we might prove, let us consider two simple cases. As the results are easily obtained, we state them without proof.

(a) The dimensions are $n = p = 1$, $q = 0$, and g is C^∞. Then, if all derivatives of g are null at $\bar{x}$, there is no k-stability of D at $\bar{x}$. Otherwise, let k_0 be the index of the first nonnull derivative of g. If k_0 is even, there is no k-stability of D at $\bar{x}$. If k_0 is odd, then D is k_0 stable.

(b) The dimensions are $n = 1$, $p = 0$, $q = 1$, and h is C^∞. Then, if all derivatives of h are null, there is no k-stability of D at $\bar{x}$. Otherwise, let k_0 be the index of the first nonnull derivative of h at $\bar{x}$. If k_0 is odd, or if k_0 is even and $d^{(k_0)}h/dx^{(k_0)}(\bar{x}) < 0$, D is k_0-stable; otherwise, there is no k-stability of D at $\bar{x}$.

In view of these results, we may think that the integer values of k are of special importance and that, for integer values of k, the k-stability is strongly related to the derivatives of g and h at $\bar{x}$, up to the order k.

3. k-Stability for $1 < k < 2$

The study of this case reduces to the following theorem, showing again that integer values of k are of special importance.

Theorem 3.1. If g and h are C^2, the k-stability of D at $\bar{x}$ for $k < 2$ implies the 1-stability.

Before giving the proof, let us state some definitions and notations that will be useful in the sequel of this paper.

Definition 3.1. A perturbation D^ϵ of D is said to be elementary if it is of the form

$$g^\epsilon(x) = g(x) - \epsilon c,$$
$$h^\epsilon(x) = h(x) - \epsilon c',$$

with (c, c') given in $\mathbb{R}^p \times \mathbb{R}^q$

Let $\{x^\epsilon\}$ be a sequence in $\mathbb{R}^n$ converging to $\bar{x}$ when $\epsilon \to 0$. We notice that we may always write

$$x^\epsilon = \bar{x} + \alpha^\epsilon d^\epsilon,$$

with

$$\alpha^\epsilon = \|x^\epsilon - \bar{x}\|, \qquad \|d^\epsilon\| = 1.$$

Proof of Theorem 3.1. Let D be k-stable at $\bar{x}$ with $k<2$. We consider an elementary perturbation of the above form. Then, there exist $K>0$ and $\{x^\epsilon\}$, with $x^\epsilon \in D^\epsilon$, such that

$$\|x^\epsilon - \bar{x}\| \leq K\epsilon^{1/k}$$

This is equivalent, with the above notation, to

$$\alpha^\epsilon \leq K\epsilon^{1/k};$$

hence, $(\alpha^\epsilon)^2 = o(\epsilon)$. As $x^\epsilon = \bar{x} + \alpha^\epsilon d^\epsilon$ is in D^ϵ and (g, h) are C^2, we obtain

$$g(x^\epsilon) = \alpha^\epsilon \nabla g(\bar{x})' d^\epsilon + o(\epsilon) = \epsilon c,$$
$$h(x^\epsilon) = \alpha^\epsilon \nabla h(\bar{x})' d^\epsilon + o(\epsilon) \leq \epsilon c'.$$

The relation on g and the fact that c is arbitrary imply that the range of $\nabla g(\bar{x})'$ is dense in $\mathbb{R}^p$; hence, as any vector space of $\mathbb{R}^p$ is closed, $\nabla g(\bar{x})'$ is surjective. Now, suppose that $c=0$ and $c'<0$ (componentwise). Let r^ϵ be the minimum ($l2$) norm solution of

$$\nabla g(\bar{x})'(\alpha^\epsilon d^\epsilon + r^\epsilon) = 0$$

As $\nabla g(\bar{x})'$ is surjective,

$$r^\epsilon = o(\alpha^\epsilon \nabla g(\bar{x})' d^\epsilon) = o(\epsilon);$$

hence,

$$y^\epsilon = \epsilon^{-1}(\alpha^\epsilon d^\epsilon + r^\epsilon)$$

satisfies

$$\nabla g(\bar{x})' y^\epsilon = 0,$$
$$\nabla h(\bar{x})' y^\epsilon \leq c' + o(\epsilon)/\epsilon.$$

The right-hand side of the above inequality is strictly negative for ϵ small enough This proves that (1) holds; hence, by Theorem 2 1, D is 1-stable at $\bar{x}$. □

4. Some Results Concerning Linear Systems

Before going on with the study of the 2-stability, we have to state some technical results concerning linear systems of equalities and inequalities.

Consider the system

$$Az=e, \qquad Bz\leq f, \tag{2}$$

with A, B being matrices and z, e, f vectors of convenient (finite) dimensions Denote by B_i the ith row of B, and J being a set of indexes,

$$B_J=\{B_i;\ i\in J\}, \qquad f_J=\{f_i;\ i\in J\}.$$

We will have to consider the associated homogeneous system,

$$Az=0, \qquad Bz\leq 0. \tag{3}$$

Define

$$I=\{i;\ \text{there exists } z^i \text{ satisfying (3) and } B_iz^i<0\},$$

$$\bar{I}=\{i;\ \text{there exists no } z \text{ satisfying (3) and } B_iz<0\}$$

We denote

$$f^+=\max(f,0) \quad \text{and} \quad f^-=\min(f,0),$$

where the max and min are taken componentwise. The aim of this section is to prove the following result

Theorem 4.1. There exists $M>0$ such that, for any (e,f) such that (2) has a solution, (2) has a solution z satisfying

$$\|z\|\leq M(\|e\|+\|f_{\bar{I}}^+\|+\|f_I^-\|)$$

For clarity we state two preliminary lemmas. Our Lemma 4 2 is a corollary of Lemma 3.5 of Daniel (Ref. 1); we give a proof, different from the one of Ref. 1, for the convenience of the reader.

Lemma 4.1. If z satisfies

$$Az=0, \qquad B_{\bar{I}}z\leq 0,$$

then $B_{\bar{I}}z=0$

Proof. If $I=\varnothing$, then z satisfies (3); hence, from the definition of $\bar{I}$, $B_{\bar{I}}z=0$. Otherwise, let z^i be as in the definition of I. Define

$$z^0=\sum_{i\in I} z^i.$$

Then obviously,

$$Az^0=0, \qquad B_I z^0<0, \qquad B_{\bar{I}}z^0\le 0.$$

If Lemma 4.1 does not hold, there exist $j\in\bar{I}$ and w such that

$$Aw=0, \qquad B_{\bar{I}}w\le 0, \qquad B_j w<0.$$

Then, for $\alpha>0$ large enough,

$$z=w+\alpha z^0$$

satisfies (3) and $B_j z<0$; hence, $j\in I$, in contradiction with the definition of I. □

Lemma 4.2. There exists $M_1>0$ such that any solution (z, e, f) of

$$Az=e, \qquad B_{\bar{I}}z\le f_{\bar{I}} \tag{4}$$

satisfies

$$\|B_{\bar{I}}z\|\le M_1(\|e\|+\|f_{\bar{I}}^+\|).$$

Proof. If the conclusion is false, there exists $\{(z^k, e^k, f^k)\}$ satisfying (4) and

$$\|B_{\bar{I}}z^k\|>k(\|e^k\|+\|(f_{\bar{I}}^k)^+\|).$$

Normalizing the sequence if necessary, we may suppose that $\|B_{\bar{I}}z^k\|=1$; hence $e^k\to 0$ and $(f_{\bar{I}}^k)^+\to 0$. As $Az^k=e^k$ and $B_{\bar{I}}z^k$ are bounded, using a left pseudo-inverse of $(A, B_{\bar{I}})$ we deduce the existence of a bounded sequence $\bar{z}^k$, with

$$A\bar{z}^k=Az^k=e^k \quad \text{and} \quad B_{\bar{I}}\bar{z}^k=B_{\bar{I}}z^k\le(f_{\bar{I}}^k)^+.$$

Extracting a subsequence if necessary, we may suppose that $\bar{z}^k\to\bar{z}$. Passing to the limit, we obtain

$$\|B_{\bar{I}}\bar{z}\|=1, \qquad A\bar{z}=0, \qquad B_{\bar{I}}\bar{z}\le 0,$$

in contradiction with Lemma 4.1. □

Proof of Theorem 4.1. Let (z, e, f) satisfy (2). From Lemma 4.2, we deduce that

$$\|Az\| + \|B_{\bar{I}}z\| \leq (M_1+1)(\|e\| + \|f_{\bar{I}}^{+}\|).$$

Hence, applying a left pseudo-inverse of $(A, B_{\bar{I}})$ to the pair $(Az, B_{\bar{I}}z)$, we deduce the existence of $M_2 > 0$ independent of (z, e, f) such that there exists $\bar{z}$ satisfying

$$A\bar{z} = Az, \qquad B_{\bar{I}}\bar{z} = B_{\bar{I}}z,$$

and

$$\|\bar{z}\| \leq M_2(\|e\| + \|f_{\bar{I}}^{+}\|).$$

If $I = \varnothing$, the result is obtained. Otherwise, let z^0 be as in the proof of Lemma 4.1. Then,

$$z^\alpha = \bar{z} + \alpha z^0$$

satisfies, for any $\alpha \geq 0$,

$$Az^\alpha = e, \qquad B_{\bar{I}}z^\alpha \leq f_{\bar{I}},$$

and the inequality $B_I z^\alpha \leq f_I$ will be satisfied iff (as $B_i z^0 < 0$, for any i in I)

$$\alpha B_i z^0 \leq f_i - B_i\bar{z}, \qquad \forall i \in I,$$

and *a fortiori* if

$$\alpha B_i z^0 \leq f_i^- - B_i\bar{z}, \qquad \forall i \in I,$$

i.e.,

$$\alpha \geq (f_i^- - B_i\bar{z})/B_i z^0, \qquad \forall i \in I.$$

Denote

$$\beta = \min\{|B_i z^0|, i \in I\};$$

the above inequality will be satisfied if

$$\alpha \geq \beta^{-1}(|B_i\bar{z}| + f_i^-), \qquad \forall i \in I,$$

and this with the above estimate on $\bar{z}$ proves the result. □

Remark 4.1. To our knowledge, Theorem 4.1 is new; in particular, it is not a consequence of the results in Refs. 1–4.

5. Study of 2-Stability

The reduction of k-stability to the 1-stability for $k<2$ does not hold for $k=2$. Take, for instance, $n=2$, $p=1$, $q=0$ and $g(x)=(x_1)^2-(x_2)^2$.

For any element y of $\mathbb{R}^n$, we denote by $\nabla^2 g_i(\bar{x})$ the Hessian of g_i at $\bar{x}$ and by $y'\nabla^2 g(\bar{x})y$ the vector of $\mathbb{R}^p$ whose ith component is $y'\nabla^2 g_i(\bar{x})y$. Let us state the following condition: For any (c, c') in $\mathbb{R}^p \times \mathbb{R}^q$, there exists (y, z) in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\text{(i)}\quad \nabla g(\bar{x})'y=0, \qquad \nabla h(\bar{x})'y \leq 0, \tag{5a}$$

$$\text{(ii)}\quad \nabla g(\bar{x})'z+\tfrac{1}{2}y'\nabla^2 g(\bar{x})y=c, \tag{5b}$$

(iii) for any i in $(1, \ldots, q)$,

either $\nabla h_i(\bar{x})'y<0$

$$\text{or } \nabla h_i(\bar{x})'z+(1/2)y'\nabla^2 h_i(\bar{x})y \leq c_i' \tag{5c}$$

This new condition is a relaxation of (1) because, if in (5) we take $y=0$, then the above relations reduce to

$$\nabla g(\bar{x})'z=c, \qquad \nabla h(\bar{x})'z \leq c',$$

for any (c, c'), which is easily seen to be equivalent to (1). We will prove the following result.

Theorem 5.1. If D is 2-stable at $\bar{x}$, then condition (5) holds.

However, we will obtain it as a particular case of a more general result, involving uniform estimates with respect to a set of constants (c, c').

Definition 5.1. Let E be a subset of $\mathbb{R}^p \times \mathbb{R}^q$. We say that D is uniformly k-stable at $\bar{x}$ with respect to E if there exists $K>0$ such that, for any (c, c') in E, there exists for $\epsilon>0$ small enough a mapping $\epsilon \to x^\epsilon$ with $x^\epsilon \in D^\epsilon$ [elementary perturbation associated to (c, c')] and

$$\|\bar{x}-x^\epsilon\| \leq K\epsilon^{1/k}(\|c\|+\|c'\|)^{1/k}.$$

As an example, take

$$E=\{(\alpha c, \alpha c'), \alpha \in \mathbb{R}\}.$$

If D is k-stable at $\bar{x}$, the constant associated to (c, c') being K, then D is uniformly k-stable at $\bar{x}$ with respect to E, with the same constant K [this is a justification of the power $1/k$ associated to $\|c\|+\|c'\|$ in Definition 5.1]. We will prove the following result.

Theorem 5.2. Let D be uniformly 2-stable at $\bar{x}$ with respect to E. Then, there exists $K_1 > 0$ such that, to each (c, c') in E, there is associated (y, z) satisfying (5) and

$$\|z\| + \|y\|^2 \leq K_1(\|c\| + \|c'\|).$$

Proof. We consider the elementary perturbation associated to an element (c, c') of E. Let x^ϵ be an element of D^ϵ satisfying

$$\|x^\epsilon - \bar{x}\| \leq K\epsilon^{1/2}(\|c\| + \|c'\|)^{1/2}.$$

Writing, as in Section 3,

$$x^\epsilon = \bar{x} + \alpha^\epsilon d^\epsilon,$$

we obtain

$$g(x^\epsilon) = \alpha^\epsilon \nabla g(\bar{x})' d^\epsilon + [(\alpha^\epsilon)^2/2](d^\epsilon)' \nabla^2 g(\bar{x}) d^\epsilon + o(\epsilon) = \epsilon c,$$

$$h(x^\epsilon) = \alpha^\epsilon \nabla h(\bar{x})' d^\epsilon + [(\alpha^\epsilon)^2/2](d^\epsilon)' \nabla^2 h(\bar{x}) d^\epsilon + o(\epsilon) \leq \epsilon c'.$$

Case 1. An extracted sequence of α^ϵ/ϵ has a limit point γ. Dividing the above system by ϵ and passing to the limit for the extracted sequence, we obtain, d being a limit-point of $\{d^\epsilon\}$,

$$\gamma \nabla g(\bar{x})' d = c, \qquad \gamma \nabla h(\bar{x})' d \leq c'.$$

Then, statements (5) are satisfied with $z = \gamma d$, $y = 0$. Applying Theorem 4.1 to the linear system

$$\nabla g(\bar{x})' z = c, \qquad \nabla h(\bar{x})' z \leq c',$$

we see that (5) may be satisfied with

$$y = 0 \quad \text{and} \quad \|z\| \leq M(\|c\| + \|c'\|),$$

the constant M depending only on $\nabla g(\bar{x})$ and $\nabla h(\bar{x})$ [not on (c, c')].

Case 2. As $\epsilon \to 0$, $\alpha^\epsilon/\epsilon \to +\infty$. Dividing the above quadratic system by α^ϵ and passing to the limit, d being again a limit point of d^ϵ, we obtain

$$\nabla g(\bar{x})' d = 0, \qquad \nabla h(\bar{x})' d \leq 0.$$

Put

$$d^\epsilon = d + \beta^\epsilon v^\epsilon, \qquad \text{with } \beta^\epsilon = \|d^\epsilon - d\|.$$

We expand here d^ϵ in the same way we did for x^ϵ. For an extracted sequence, $d^\epsilon \to d$, hence $\beta^\epsilon \to 0$ and $(\alpha^\epsilon)^2 \beta^\epsilon = o(\epsilon)$ because of the uniform 2-stability.

Hence, for this extracted sequence, and using the relations on d, we have

$$\alpha^\epsilon \beta^\epsilon \nabla g(\bar{x})' v^\epsilon + [(\alpha^\epsilon)^2/2] d' \nabla^2 g(\bar{x}) d = \epsilon c + o(\epsilon),$$

and if $\nabla h_i(\bar{x})' d = 0$, then

$$\alpha^\epsilon \beta^\epsilon \nabla h_i(\bar{x})' v^\epsilon + [(\alpha^\epsilon)^2/2] d' \nabla^2 h_i(\bar{x}) d \leq \epsilon c_i' + o(\epsilon).$$

Dividing the above relations by ϵ and defining

$$y^\epsilon = \epsilon^{-1/2} \alpha^\epsilon d, \qquad z^\epsilon = \epsilon^{-1} \alpha^\epsilon \beta^\epsilon v^\epsilon,$$

we get

$$\nabla g(\bar{x})' z^\epsilon + (1/2)(y^\epsilon)' \nabla^2 g(\bar{x}) y^\epsilon = c + o(\epsilon)/\epsilon,$$

and if $\nabla h_i(\bar{x})' y^\epsilon = 0$, then

$$\nabla h_i(\bar{x})' z^\epsilon + (1/2)(y^\epsilon)' \nabla^2 h_i(\bar{x}) y^\epsilon \leq c_i' + o(\epsilon)/\epsilon$$

As

$$\epsilon^{-1/2} \alpha^\epsilon \leq K(\|c\| + \|c'\|)^{1/2},$$

y^ϵ is bounded and converges for some subsequence to some

$$y = \gamma d, \qquad \text{with } 0 \leq \gamma \leq K(\|c\| + \|c'\|)^{1/2};$$

hence,

$$\|y\|^2 \leq (K)^2(\|c\| + \|c'\|),$$

and y satisfies (5a).

As y^ϵ is bounded, we may, using Theorem 4.1, suppose that z^ϵ is also bounded and (extracting if necessary a new subsequence) converges toward some z; (y, z) satisfy (5b, c) and, applying Theorem 4.1 as in Case 1, we get [K_2 depending on $\nabla^2 g(\bar{x})$ and $\nabla^2 h(\bar{x})$]

$$\|z\| \leq M(\|c\| + \|c'\| + K_2 \|y\|^2) \leq M(1 + (K)^2 K_2)(\|c\| + \|c'\|),$$

which proves the theorem. □

We wonder whether condition (5), or perhaps the strengthened condition obtained by supposing that (y, z) satisfies the estimate of the conclusion of Theorem 5.2, with $E = \mathbb{R}^p \times \mathbb{R}^q$, is sufficient for 2-stability. This seems not easy to prove; the main difficulty is due to the lack of knowledge about the behavior of the solution of quadratic systems. Nevertheless, we can state the following partial converse result.

Theorem 5.3. Let $\bar{x}$ be a point of D at which condition (5) holds. If in addition g and h are C^3 and $\nabla g(\bar{x})'$ is surjective (or no equality constraint is present), then D is 2-stable at $\bar{x}$.

Proof. Let D^ϵ be a smooth perturbation of D associated to mappings g^ϵ and h^ϵ. We shall construct a point x^ϵ in D^ϵ sufficiently close to $\bar{x}$. Let us denote

$$c = -(d/d\epsilon)g^\epsilon(\bar{x})_{|\epsilon=0},$$

and define c' by

$$c'_i = \min(0, -(d/d\epsilon)h_i^\epsilon(\bar{x})_{|\epsilon=0}) - 1.$$

Let (y, z) be associated to (c, c') by condition (5). We shall choose x^ϵ of the form

$$x^\epsilon = \bar{x} + \epsilon^{1/2}y + \epsilon z + r(\epsilon).$$

As g is C^3 and $(\epsilon, x) \to g^\epsilon(x)$ is C^2, we find, using (5), that

$$\begin{aligned} &g^\epsilon(\bar{x} + \epsilon^{1/2}y + \epsilon z) \\ &= g(\bar{x} + \epsilon^{1/2}y + \epsilon z) - \epsilon c + O(\epsilon^2) \\ &= \nabla g(\bar{x})'(\epsilon^{1/2}y + \epsilon z) + (\epsilon/2)y'\nabla^2 g(\bar{x})y - \epsilon c + O(\epsilon^{3/2}) \\ &= O(\epsilon^{3/2}). \end{aligned}$$

As $\nabla g(\bar{x})'$ is surjective, we have the 1-stability for $\{x; g(x) = 0\}$ (this is a corollary of Theorem 2.1). Consequently, for ϵ small enough, we may find $r(\epsilon)$ such that $r(\epsilon) = O(\epsilon^{3/2})$ and

$$x^\epsilon = \bar{x} + \epsilon^{1/2}y + \epsilon z + r(\epsilon)$$

satisfies, for ϵ small enough,

$$g^\epsilon(x^\epsilon) = 0, \qquad \|x^\epsilon - \bar{x}\| \leq 2\|y\|\epsilon^{1/2}.$$

Now, consider the inequality constraints. We have

$$\begin{aligned} h^\epsilon(x^\epsilon) &= \nabla h(\bar{x})'(\epsilon^{1/2}y + \epsilon z) + (\epsilon/2)y'\nabla^2 h(\bar{x})y \\ &\quad + \epsilon(d/d\epsilon)h^\epsilon(\bar{x})_{|\epsilon=0} + O(\epsilon^{3/2}). \end{aligned}$$

If

$$\nabla h_i(\bar{x})'y < 0, \qquad \text{for some } i,$$

this implies that $h_i^\epsilon(x^\epsilon) < 0$, for some ϵ small enough. Otherwise, (5) implies that

$$\nabla h_i(\bar{x})'y = 0;$$

our choice of c' implies that

$$h_i^\epsilon(x^\epsilon) \leq -\epsilon + O(\epsilon^{3/2}),$$

which implies that the constraints are satisfied for ϵ small enough. □

Remark 5.1. Our Theorem 5.1 seems to be connected to some results of Frankowska (Ref. 7) involving inverse theorems for multifunctions. In particular, there seems to be a connection between our condition (5) and the surjectivity of the second-order contingent variations defined in Ref. 7.

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