LOCAL ANALYSIS OF NEWTON TYPE METHODS FOR VARIATIONAL INEQUALITIES AND NONLINEAR PROGRAMMING (FINAL VERSION, TO APPEAR IN APPLIED MATHEMATICS AND OPTIMIZATION) COMMUNICATED BY J. STOER

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Abstract. This paper presents some new results in the theory of Newton type methods for variational inequalities, and their application to nonlinear programming. A condition of semi-stability is shown to ensure the quadratic convergence of Newton's method and the superlinear convergence of some quasi-Newton algorithms, provided the sequence defined by the algorithm exists and converges. A partial extension of these results to nonsmooth functions is given. The second part of the paper considers some particular variational inequalities with unknowns (x, λ), generalizing optimality systems. Here only the question of superlinear convergence of $\{x^k\}$ is considered. Some necessary or sufficient conditions are given. Applied to some quasi-Newton algorithms they allow to obtain the superlinear convergence of $\{x^k\}$. The application of the previous results to nonlinear programming allows to strenghten the known results, the main point being a characterization of the superlinear convergence of $\{x^k\}$ assuming a weak second-order condition without strict complementarity.

Running head : Local analysis of Newton type methods

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1. Introduction. This paper is devoted to the local study of Newton type algorithms for variational inequalities. Variational inequalities have been studied for a long time (see Lions and Stampacchia [16]) mainly because of their applications to mechanical systems. The operators in that field are often monotone, and a large theory of monotone operators has been developed (see Brézis [6]); several algorithms for convex programming, including duality methods, have been extended to this framework (see Gabay [11]). Some problems in economy as well as optimality systems of nonlinear programming problems can also be represented by variational inequalities (see Robinson [21] and Harker and Pang [13]). Consequently the strenght and large use of Newton type algorithms for nonlinear programming, the so-called successive quadratic programming (see Bertsekas [2] and Fletcher [10]) suggests to develop a theory of Newton type methods for variational inequalities (we will not speak here of some different approaches of Newton type algorithms for variational inequalities – reviewed in the survey by Harker and Pang [13]). Some early (but unpublished) works in this direction due to Josephy [14, 15] give a local analysis using the concept of strong regularity (Robinson [19]). Josephy obtains a quadratic rate of convergence for Newton's method and superlinear convergence for some quasi-Newton algorithms. In the case of nonlinear programming problems, assuming the gradients of active constraints to be linearly independent, the strong regularity reduces to some strong second-order sufficient condition.

The quadratic rate of convergence under the weak second order sufficiency condition for nonlinear programming problems, and assuming the linear independence of the gradients of active constraints, has been recently obtained by the author [4]. This

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suggests that the theory of Newton type methods for variational inequalities can be extended. For this purpose we use the new concept of semi-stability. We say that a solution $\bar{x}$ of a variational inequality is semi-stable if, given a small perturbation in the right-hand side, a solution x of the perturbed variational inequality that is sufficiently close to $\bar{x}$, is such that the distance of x to $\bar{x}$ is of the order of the magnitude of the perturbation. This does not imply the existence of a solution for the perturbed problem. Indeed, we give a counterexample showing that existence for a small perturbation does not always hold under the semi-stability hypothesis. We use a "hemi-stability" hypothesis in order to prove the existence of the sequence satisfying the Newton type steps, then we show that semi-stability allows to obtain in a simple way quadratic convergence for Newton's method and superlinear convergence for a large class of Newton type algorithms (here we extend the Dennis and Moré [9] sufficient condition for superlinear convergence). This allows us to adapt Grzegòrski's[12] theory in order to derive the superlinear convergence of a large class of quasi-Newton updates including Broyden's one. For polyhedral convex sets we may characterize semi-stability : it reduces to the condition that the solution $\bar{x}$ is an isolated solution of the variational inequality linearized at $\bar{x}$. An equivalent condition is the "strong positivity condition" of Reinoza [18]. We also check that for non-differentiable data the theory can be extended using point-based approximations (reminiscent of those of Robinson [23]) that play the role of linearized function.

The second part of this paper is devoted to a special class of variational inequalities generalizing optimality systems. The unknowns here are couples (x, λ) and we try to obtain conditions related to superlinear convergence of $\{x^k\}$ alone. Indeed we give a characterization of the superlinear convergence of $\{x^k\}$, valid under a second-order hypothesis satisfied by optimality systems for which the weak second-order sufficiency condition holds. This allows us to extend to inequality constrained problems the characterization of Boggs, Tolle and Wang [3] for equality constrained problems (this improves some previous results of the author [4] in which some necessary or sufficient conditions are given) ; our result assumes only that the gradients of active constraints are linearly independent and the weak second-order sufficient condition holds, but includes no strict complementarity hypothesis. We apply this result in order to obtain superlinear convergence for a large class of quasi-Newton updates . We note that these results can be used in order to formulate some globally convergent algorithms having fast convergence rates (see [5]).

2. Newton type methods for variational inequalities. Let φ be a continuously differentiable mapping from $\mathbb{R}^q$ into $\mathbb{R}^q$. Given a closed convex subset K of $\mathbb{R}^q$ we consider the variational inequality

$$\langle \varphi(z), y - z \rangle \geq 0, \quad \forall y \in K; \quad z \in K. \tag{2.1}$$

We may define the (closed convex) cone of outward normals to K at a point $z \in K$

$$N(z) := \{x \in \mathbb{R}^q; \quad \langle x, y - z \rangle \leq 0, \quad \forall y \in K\},$$

and if $z \notin K$, $N(z) := \emptyset$. A relation equivalent to (2.1) is then

$$\varphi(z) + N(z) \ni 0. \tag{2.2}$$

When $K = \mathbb{R}^q$, $N(z) = \{0\}$ and we recover the equation $\varphi(z) = 0$. A natural extension of the Lagrange-Newton method for nonlinear programming (see Fletcher [10]) is what we will call the Newton-type algorithm :

Algorithm 1.

0) Choose $z^0 \in \mathbb{R}^n$, $\quad k \leftarrow 0$.

1) While z^k is not solution of (2.2) : choose M^k, $q \times q$ matrix, and compute z^{k+1} solution of

$$\varphi(z^k) + M^k(z^{k+1} - z^k) + N(z^{k+1}) \ni 0. \tag{2.3}$$

We define Newton's method as Algorithm 1 when $M^k = \varphi'(z^k)$. In order to obtain estimates of the rate of convergence of $\{z^k\}$ we essentially use the following concept.

DEFINITION 2.1. *A solution $\bar{z}$ of (2.2) is said to be semi-stable if there exist $c_1 > 0$ and $c_2 > 0$ such that, for all $(z, \delta) \in \mathbb{R}^q \times \mathbb{R}^q$ solution of*

$$\varphi(z) + N(z) \ni \delta,$$

and $\|z - \bar{z}\| \leq c_1$, then $\|z - \bar{z}\| \leq c_2\|\delta\|$.

REMARK 2.1. *(i) Note that this definition involves only those δ for which $\|\delta\| \leq c_1/c_2$, because otherwise $\|z - \bar{z}\| \leq c_2\|\delta\|$ is always satisfied whenever $\|z - \bar{z}\| \leq c_1$; hence taking c_1 small enough, we can restrict δ to an arbitrary neighbourhood of 0.*

(ii) If $K = \mathbb{R}^q$ this condition reduces to the invertibility of $\varphi'(\bar{z})$; this will be obtained as a consequence of theorem 3.1.

THEOREM 2.1. *Let $\bar{z}$ be a semi-stable solution of (2.1), and let $\{z^k\}$ computed by Algorithm 1 converge towards $\bar{z}$. Then*

(i) If $(\varphi'(\bar{z}) - M^k)(z^{k+1} - z^k) = o(z^{k+1} - z^k)$ then $\{z^k\}$ converges superlinearly.

(ii) If $(\varphi'(\bar{z}) - M^k)(z^{k+1} - z^k) = O(\|z^{k+1} - z^k\|^2)$ and φ' is locally Lipschitz then $\{z^k\}$ converges quadratically.

Proof. Define $\theta^k := (\varphi'(\bar{z}) - M^k)(z^{k+1} - z^k)$. We can write the Newton type step (2.3) as

$$\varphi(z^{k+1}) + N(z^{k+1}) \ni r^k \tag{2.4}$$

with,

$$\begin{aligned} r^k &:= \theta^k + \varphi(z^{k+1}) - \varphi(z^k) - \varphi'(\bar{z})(z^{k+1} - z^k) \\ &= \theta^k + o(z^{k+1} - z^k). \end{aligned}$$

If $\theta^k = o(z^{k+1} - z^k)$ then from the semi-stability of $\bar{z}$ and (2.4) we get

$$z^{k+1} - \bar{z} = O(r^k) = o(z^{k+1} - z^k) = o(\|z^{k+1} - \bar{z}\| + \|z^k - \bar{z}\|),$$

hence $z^{k+1} - \bar{z} = o(z^k - \bar{z})$, i.e. $\{z^k\}$ converges superlinearly. This proves (i). If φ' is locally Lipschitz and $\theta^k = O(\|z^{k+1} - z^k\|^2)$ we already know that $\{z^k\}$ converges superlinearly, hence $\|z^{k+1} - z^k\|/\|z^k - \bar{z}\| \to 1$. Let L be a Lipschitz constant of φ' at $\bar{z}$. We have, for k large enough :

$$\begin{aligned} \|\varphi(z^{k+1}) - \varphi(z^k) - \varphi'(\bar{z})(z^{k+1} - z^k)\| &= \|\int_0^1 \left[\varphi'(z^k + \sigma(z^{k+1} - z^k)) - \varphi'(\bar{z})\right](z^{k+1} - z^k)d\sigma\| \\ &\leq L \max(\|z^{k+1} - \bar{z}\|, \|z^k - \bar{z}\|)\|z^{k+1} - z^k\| \\ &\leq 2L\|z^{k+1} - z^k\|^2, \end{aligned}$$

hence

$$z^{k+1} - \bar{z} = O(r^k) = O(\|z^{k+1} - z^k\|^2) = O(\|z^k - \bar{z}\|^2),$$

from which the quadratic convergence follows. □

REMARK 2.2. *Taking $K = \{0\}$ we see that the conditions of Theorem 2.1 are not necessary in general. However, when $K = \mathbb{R}^q$ (case of a nonlinear equation) it is known that condition (i) is a characterization of superlinear convergence (Dennis-Moré [8]).*

COROLLARY 2.1. *If $\{z^k\}$ computed by Algorithm 1 converges toward a semi-stable solution $\bar{z}$ of (2.1) then*

(i) If $M^k \to \varphi'(\bar{z})$, $\{z^k\}$ converges superlinearly,

(ii) If φ' is locally Lipschitz and $M^k = \varphi'(\bar{z}) + O(z^k - \bar{z})$ (which is the case for Newton's method under the hypothesis of Lipschitz continuity of φ') then $\{z^k\}$ converges quadratically.

Until now we assumed the existence of a converging sequence instead of giving the hypotheses that imply its existence. Our point of view is that it is clearer to do so ; indeed, if we want now to prove that the sequence is well defined, for say, Newton's method with a good starting point, we just have to posit the following definition :

DEFINITION 2.2. *We will say that $\bar{z}$ is a hemi-stable solution of (2.1) if for all $\alpha > 0$ there exists $\varepsilon > 0$ such that, given $\hat{z} \in \mathbb{R}^q$, the variational inequality (in z)*

$$\varphi(\hat{z}) + M(z - \hat{z}) + N(z) \ni 0$$

has a solution z satisfying $\|z - \bar{z}\| \leq \alpha$, whenever $\|\hat{z} - \bar{z}\| + \|M - \varphi'(\bar{z})\| < \varepsilon$. Then using Corollary 2.1, we obtain

THEOREM 2.2. *(Local analysis of Newton's method). If $\bar{z}$ is a semi-stable and hemi-stable solution of (2.1), there exists $\varepsilon > 0$ such that if $\|z^o - \bar{z}\| \leq \varepsilon$, then :*

(i) *At each step k there exists z^{k+1} solution of the Newton step satisfying $\|z^{k+1} - z^k\| \leq 2\varepsilon$,*

(ii) *The sequence $\{z^k\}$ defined in this way converges superlinearly (quadratically if φ' is locally Lipschitz) towards $\bar{z}$.*

Proof. We just have to prove (i) and the convergence of $\{z^k\}$ towards $\bar{z}$; then (ii) will follow from Corollary 2.1. Assume φ' merely continuous at $\bar{z}$. Take $\varepsilon_0 \leq \min(c_1, 1/3c_2)$ where c_1, c_2 are given by the semi stability condition. From the hemi-stability condition we have that for some $\varepsilon \in (0, c_1)$, $\|z^k - \bar{z}\| \leq \varepsilon$ implies the existence of z^{k+1} such that $\|z^{k+1} - \bar{z}\| \leq \varepsilon_0$ and

$$\varphi(z^k) + \varphi'(z^k)(z^{k+1} - z^k) + N(z^{k+1}) \ni 0. \tag{2.5}$$

Now $\varphi(z^{k+1}) + N(z^{k+1}) \ni \delta^k$ where

$$\delta^k := \varphi(z^{k+1}) - \varphi(z^k) - \varphi'(z^k)(z^{k+1} - z^k). \tag{2.6}$$

From differential calculus we obtain, reducing ε_0 and ε if necessary, that

$$\|\delta^k\| \leq \frac{1}{3c_2}\|z^{k+1} - z^k\|. \tag{2.7}$$

As $\varepsilon_0 \leq c_1$ the semi stability condition gives

$$\|z^{k+1} - \bar{z}\| \leq \frac{1}{3}\|z^{k+1} - z^k\| \leq \frac{1}{3}\|z^{k+1} - \bar{z}\| + \frac{1}{3}\|z^k - \bar{z}\|,$$

hence

$$\|z^{k+1} - \bar{z}\| \leq \frac{1}{2}\|z^k - \bar{z}\| \tag{2.8}$$

and

$$\|z^{k+1} - z^k\| \leq \|z^{k+1} - \bar{z}\| + \|z^k - \bar{z}\| \leq 2\varepsilon.$$

This proves (i) and the linear convergence of $\{z^k\}$. □

REMARK 2.3. *The condition $\|z^{k+1} - z^k\| \leq 2\varepsilon$ in Theorem 2.2 is constructive in the sense that, if we choose the solution of (2.5) closest to z^k, then, if the starting point z^0 is close enough to $\bar{z}$, the condition is satisfied and conclusion of Theorem 2.2 follows.*

REMARK 2.4.

(i) *Semi-stability does not imply hemi-stability, as shown by the following example. Consider the variational inequality with $K = \mathbb{R}^+$:*

$$-z + N(z) \ni 0,$$

corresponding to the optimality system of the badly posed optimization problem

$$\min\{-z^2/2;\ z \geq 0\}.$$

Here

$$N(z) = \begin{cases} \emptyset & \text{if } z < 0, \\ \mathbb{R}^+ & \text{if } z = 0, \\ 0 & \text{if } z > 0. \end{cases}$$

We have that $\bar{z} = 0$ is the unique solution. Now the perturbed variational inequality

$$-z + N(z) \ni \delta$$

has a solution iff $\delta \leq 0$ and this solution is $z = -\delta$, hence semi-stability holds although the variational inequality may have no solution for $\|\delta\|$ arbitrarity small.

Let us prove now that hemi-stability does not hold. Here $\varphi(z) = -z$ and $\varphi'(\bar{z}) = -1$; take $\hat{z} = \varepsilon$ and $M = \varepsilon - 1$ with $\varepsilon \in (0,1)$; we discuss the solvability near 0 of

$$-\varepsilon + (\varepsilon - 1)(z - \varepsilon) + \partial\mathbb{R}^+(z) \ni 0.$$

If z is solution, either $z = 0$, but then $-\varepsilon + (\varepsilon - 1)(z - \varepsilon) = -\varepsilon^2 < 0$, impossible ; or $-\varepsilon + (\varepsilon - 1)(z - \varepsilon) = 0$, i.e. $z = \varepsilon^2/(\varepsilon - 1) < 0$, which is also impossible. Hence the perturbed variational inequality has no solution, although $(\hat{z}, M)$ is arbitrarily close to $(\bar{z}, \varphi(\bar{z}))$.

(ii) *A sufficient condition for semi and hemi-stability is the strong regularity of Robinson [19]. Indeed strong regularity amounts to say that the equation*

$$\varphi(\bar{z}) + \varphi'(\bar{z})(z - \bar{z}) + N(z) \ni \delta$$

is such that there exist $\varepsilon > 0$, $\alpha > 0$, $\beta > 0$ such that if $\|\delta\| \le \varepsilon$, there exist a unique solution z such that $\|z - \bar{z}\| \le \alpha$, and this z satisfies $\|z - \bar{z}\| \le \beta\|\delta\|$. Now let z solve the perturbed variational inequality $\varphi(z) + N(z) \ni \delta$. Then

$$\varphi(\bar{z}) + \varphi'(\bar{z})(z - \bar{z}) + N(z) \ni \delta + o(z - \bar{z}).$$

Strong regularity implies that $z - \bar{z} = O(\delta) + o(z - \bar{z})$, hence $z - \bar{z} = O(\delta)$, i.e. the semi-stability holds.
Also if $\bar{z}$ is a strongly regular solution of (2.2) it is obviously a strongly regular solution of the linearized variational inequality

$$\varphi(\bar{z}) + \varphi'(\bar{z})(z - \bar{z}) + N(z) \ni 0.$$

We apply Thm 2.1 of Robinson [19]. If $\|\hat{z} - \bar{z}\| + \|M - \varphi'(\bar{z})\|$ is small enough, the variational inequality

$$\varphi(\hat{z}) + M(z - \hat{z}) + N(z) \ni 0$$

has a solution and

$$\|z - \bar{z}\| = O(\varphi(\hat{z}) - \varphi(\bar{z}));$$

this implies hemi-stability.

(iii) *We will see later that in the case of optimality systems for local solutions of nonlinear programming problems, semi-stability and hemi-stability are equivalent.*

Theorem 2.1 may also be used in order to derive superlinear convergence of some quasi-Newton algorithm. By quasi-Newton algorithm we mean a Newton type algorithm with M^{k+1} satisfying the so-called quasi-Newton equation

$$M(z^{k+1} - z^k) = \varphi(z^{k+1}) - \varphi(z^k). \tag{2.9}$$

A typical situation is when a closed convex subset $\mathcal{K}$ of the space of $q \times q$ matrices is known to satisfy

$$\varphi'(z) \in \mathcal{K}, \ \forall z \in \mathbb{R}^q. \tag{2.10}$$

Then M^{k+1} is taken as a solution of

$$\min \|M - M^k\|_\sharp \ ; \ M \in \mathcal{K} \quad \text{and} \quad M \text{ satisfies (2.9)}. \tag{2.11}$$

Here $\|\cdot\|_\sharp$ is a matrix norm that we will assume to be associated to a scalar product. If $\|.\|_\sharp$ is the Frobenius norm we recover Broyden's update when $\mathcal{K}$ is the space of $q \times q$ matrices, the PSB update when $\mathcal{K}$ is the space of symmetric matrices, etc.; see Grzegòrski [12]. We first quote

LEMMA 2.1. *Under the hypotheses of Theorem 2.1, if $\{M^k\}$ satisfies the quasi-Newton equation and*

$$(M^{k+1} - M^k)(z^{k+1} - z^k) = o(z^{k+1} - z^k),$$

then $\{z^k\}$ converges superlinearly.

Proof. Using (2.9) we get

$$\begin{aligned}(M^{k+1} - M^k)(z^{k+1} - z^k) &= \varphi(z^{k+1}) - \varphi(z^k) - M^k(z^{k+1} - z^k),\\ &= (\varphi'(\bar{z}) - M^k)(z^{k+1} - z^k) + o(z^{k+1} - z^k).\end{aligned}$$

The conclusion is then obtained with Theorem 2.1. □

THEOREM 2.3. *Let φ' be locally Lipschitz, $\bar{z}$ be a semi-stable and hemi-stable solution of (2.2). We assume that (2.9)-(2.11) hold. Then there exists $\varepsilon > 0$ such that, if*

$$\|z^0 - \bar{z}\| + \|M^0 - \varphi'(\bar{z})\|_\sharp < \varepsilon,$$

then

(i) At each step k there exists z^{k+1} solution of the Newton type step satisfying $\|z^{k+1} - z^k\| \le 2\varepsilon$.

(ii) The sequence $\{z^k\}$ defined in this way converges superlinearly towards $\bar{z}$.

Proof. : Define

$$S^k := \{M \in \mathcal{K};\ M(z^{k+1} - z^k) = \varphi(z^{k+1}) - \varphi(z^k)\}.$$

Then M^{k+1} is the projection of M^k onto S^k (with the $\|\cdot\|_\sharp$ norm), hence for all $M \in S^k$ we have (see Grzegòrski [12], thm 1)

$$\|M^{k+1} - M^k\|_\sharp^2 + \|M^{k+1} - M\|_\sharp^2 \le \|M^k - M\|_\sharp^2, \tag{2.12}$$

and a fortiori

$$\|M^{k+1} - M\|_\sharp \le \|M^k - M\|_\sharp. \tag{2.13}$$

Define

$$\psi^k := \int_0^1 \varphi'(z^k + \sigma(z^{k+1} - z^k))d\sigma, \tag{2.14}$$

$$\nu^k := \max(\|z^{k+1} - \bar{z}\|, \|z^k - \bar{z}\|). \tag{2.15}$$

Then ψ^k is an element of S^k and, for k large enough we have, L being a Lipschitz constant of φ' in a neighbourhood of $\bar{z}$ in the $\|\cdot\|_\sharp$ norm :

$$\|\psi^k - \varphi'(\bar{z})\|_\sharp = \|\int_0^1 \left[\varphi'(z^k + \sigma(z^{k+1} - z^k)) - \varphi'(\bar{z})\right] d\sigma\|_\sharp \le L\nu^k,$$

hence, taking $M = \psi^k$ in (2.13), and using the previous inequality, we get :

$$\|M^{k+1} - \varphi'(\bar{z})\|_\sharp \le \|M^k - \varphi'(\bar{z})\|_\sharp + 2L\nu^k. \tag{2.16}$$

We prove in Lemma 2.2 below that this bounded deterioration result implies that (for ε small enough) $z^k \to \bar{z}$ linearly and $\|M^k - \varphi'(\bar{z})\|_\sharp$ converges. As $\psi^k \to \varphi'(\bar{z})$, $\|M^k - \psi^k\|_\sharp$ and $\|M^{k+1} - \psi^k\|_\sharp$ also converge towards the same limit. Taking $M = \psi^k$ in (2.12) we deduce that $\|M^{k+1} - M^k\| \to 0$; this and Lemma 2.1 imply the conclusion. □

LEMMA 2.2. *(Linear convergence under bounded deterioration). Let $\bar{z}$ be as in Theorem 2.3. Let $\{z^k\}$ be computed by a Newton type algorithm such that $\{M^k\}$ satisfies (2.16). Then for any θ in $(0,1)$ there exists $\varepsilon > 0$ such that if $\|z^0 - \bar{z}\| + \|M^0 - \varphi'(\bar{z})\|_\sharp < \varepsilon$, then*

(i) At each step k there exists z^{k+1} solution of the Newton type step, satisfying $\|z^{k+1} - z^k\| \le 2\varepsilon$.

(ii) $z^k \to \bar{z}$ linearly with speed θ, i.e. $\|z^{k+1} - \bar{z}\| \le \theta\|z^k - \bar{z}\|$.

(iii) $\|M^k - \varphi'(\bar{z})\|_\sharp$ converges.

Proof. Writing (2.3) as

$$\varphi(z^k) + \varphi'(\bar z)(z^{k+1} - z^k) + N(z^{k+1}) \ni (\varphi'(\bar z) - M^k)(z^{k+1} - z^k),$$

and using

$$\varphi(z^{k+1}) = \varphi(z^k) + \varphi'(\bar z)(z^{k+1} - z^k) + \int_0^1 [\varphi'(z^k + \sigma(z^{k+1} - z^k)) - \varphi'(\bar z)](z^{k+1} - z^k) d\sigma,$$

we deduce that

$$\varphi(z^{k+1}) + N(z^{k+1}) \ni \delta^k$$

with (ν^k being defined in (2.15) and using the canonical norm of $L(\mathbb{R}^n)$) :

$$\|\delta^k\| \le (\|\varphi'(\bar z) - M^k\| + L\nu^k)\|z^{k+1} - z^k\|,$$

and from the semi-stability hypothesis we deduce

$$\|z^{k+1} - \bar z\| \le c_2(\|\varphi'(\bar z) - M^k\| + L\nu^k)\|z^{k+1} - z^k\|.$$

Using the triangle inequality

$$\|z^{k+1} - z^k\| \le \|z^{k+1} - \bar z\| + \|z^k - \bar z\|$$

we deduce that whenever

$$\|\varphi'(\bar z) - M^k\| + L\nu^k < 1/c_2,$$

then

$$\|z^{k+1} - \bar z\| \le \theta_1 \|z^k - \bar z\|$$

with

$$\theta_1 = \frac{c_2(\|\varphi'(\bar z) - M^k\| + L\nu^k)}{1 - c_2(\|\varphi'(\bar z) - M^k\| + L\nu^k)}.$$

Using the hemi-stability hypothesis in order to estimate ν^k we see that there exists $\varepsilon_0 > 0$ such that $\theta_1 \le \theta$ whenever

$$\|\varphi'(\bar z) - M^k\|_\sharp + \|z^{k+1} - \bar z\| \le \varepsilon_0. \tag{2.17}$$

If $\varepsilon \le \varepsilon_0$ this is the case for $k = 0$. Now assume that (2.17) is satisfied for $k = 0, \dots, \bar k$. Then with (2.16) and using the linear convergence of $\{z^k\}$, we get

$$\|z^k - \bar z\| \le \theta^k \varepsilon, \qquad k = 0 \text{ to } \bar k + 1,$$

$$\nu^k \le 2 \sum_{k=0}^{\bar k+1} \|z^k - \bar z\| \le 2 \sum_{k=0}^{\infty} \theta^k \|z^0 - \bar z\| \le \frac{2\varepsilon}{1-\theta},$$

$$\|\varphi'(\bar z) - M^{\bar k+1}\|_\sharp \le \|\varphi'(\bar z) - M^0\|_\sharp + 2L\sum_{k=0}^{\bar k+1} \nu^k$$
$$\le \varepsilon + \frac{4L\varepsilon}{1-\theta},$$

hence

$$\|\varphi'(\bar z) - M^{\bar k+1}\|_\sharp + \|z^{\bar k+1} - \bar z\| \le 2\frac{\varepsilon + 4L\varepsilon}{1-\theta} \le \frac{4L+2}{1-\theta}\varepsilon. \tag{2.18}$$

We now choose

$$\varepsilon = \frac{1-\theta}{4L+2}\varepsilon_0.$$

For this value it appears that (2.17) is satisfied also for $k = \bar k + 1$ hence (by recurrence) for all $k \in \mathbb{N}$. This proves the linear convergence with speed θ. Also for all $k \in \mathbb{N}$ and $\ell < k$:

$$\|\varphi'(\bar z) - M^k\|_\sharp \le \|\varphi'(\bar z) - M^\ell\|_\sharp + 2L\sum_{i=\ell}^{k-1} \nu^i$$
$$\le \|\varphi'(\bar z) - M^\ell\|_\sharp + \frac{2L\theta^\ell}{1-\theta},$$

hence

$$\overline{\lim}\|\varphi'(\bar z) - M^k\|_\sharp \le \|\varphi'(\bar z) - M^\ell\|_\sharp + \frac{2L\theta^\ell}{1-\theta}.$$

When $\ell \to \infty$ we deduce

$$\overline{\lim}\|\varphi'(\bar z) - M^k\|_\sharp \le \underline{\lim}\|\varphi'(\bar z) - M^k\|_\sharp,$$

i.e. $\|\varphi'(\bar z) - M^k\|_\sharp$ converges.

REMARK 2.5. *The hemi-stability hypothesis is needed only to insure existence of z^{k+1} close to $\bar z$. The rest of the analysis relies upon the semi-stability hypothesis.*

3. Characterization of semi-stability when K is polyhedral. We assume here that K is polyhedral, i.e. defined by a finite number of linear equalities and inequalities. This allows us to give several characterizations of semi-stability.

THEOREM 3.1. *If K is polyhedral and $\bar z$ is a solution of (2.1), $\bar z$ is semi-stable iff one of the following hypotheses holds :*

(a) $\bar z$ is an isolated solution of the linearization at $\bar z$ of (2.2) :

$$\varphi(\bar z) + \varphi'(\bar z)(z - \bar z) + N(z) \ni 0. \tag{3.19}$$

(b) One has $\langle z - \bar z, \varphi'(\bar z)(z - \bar z)\rangle > 0$ for all $z \in K$ different of $\bar z$ solution of

$$\text{(3.20)} \qquad \begin{aligned} &\langle \varphi(\bar z), z - \bar z\rangle = 0, &(i)\\ &\varphi(\bar z) + \varphi'(\bar z)(z - \bar z) + N(\bar z) \ni 0. &(ii)\end{aligned}$$

(c) The conditions below have no solution but $\bar{z}$:

$$N(z) \subset N(\bar{z}), \quad (i)$$

$$\langle \varphi(\bar{z}), z - \bar{z} \rangle = 0, \quad (ii) \tag{3.21}$$

$$\alpha\varphi(\bar{z}) + \varphi'(\bar{z})(z - \bar{z}) + N(z) \ni 0, \text{ for some } \alpha \geq 0. \quad (iii)$$

REMARK 3.1.

(i) In the case of a nonlinear equation it follows from condition (a) that semi-stability is equivalent to the inversibility of the Jacobian, which in turn is also equivalent to hemi-stability.

(ii) Reinoza ([18], thm 2.1) already proved the equivalence of (a) and (b). He called condition (b) a strong positivity condition, although in the context of nonlinear programming we will see that it corresponds to weak second-order sufficient conditions ; hence it might be better to call it a weak positivity condition.

Proof. of Theorem 3.1 : We will prove that

$$\{\bar{z} \text{ is semi-stable}\} \Rightarrow \text{(a)} \Rightarrow \text{(b)} \Rightarrow \text{(c)} \Rightarrow \{\bar{z} \text{ is semi-stable}\}.$$

a) Proof of $\{\bar{z}$ semi-stable$\} \Rightarrow$ (a). If z is solution of (3.19) then from the first order expansion of φ at $\bar{z}$:

$$\varphi(z) + N(z) \ni o(z - \bar{z}),$$

hence if $\bar{z}$ is semi-stable and $\|z - \bar{z}\| \leq c_1$, we get $\|z - \bar{z}\| = o(z - \bar{z})$ and this implies $z = \bar{z}$ for z close enough to $\bar{z}$; hence (a) holds.

b) Proof of (a) $\Rightarrow$ (b). Let z in K contradict (b), i.e. $z \neq \bar{z}$, z satisfies (3.20) but $\langle z - \bar{z}, \varphi'(\bar{z})(z - \bar{z})\rangle \leq 0$. From (3.20) we get

$$0 \leq \langle \varphi(\bar{z}) + \varphi'(\bar{z})(z - \bar{z}), z - \bar{z} \rangle = \langle z - \bar{z}, \varphi'(\bar{z})(z - \bar{z})\rangle$$

hence

$$\langle z - \bar{z}, \varphi'(\bar{z})(z - \bar{z})\rangle = 0. \tag{3.22}$$

For α in $]0, 1[$ define $z^\alpha := \bar{z} + \alpha(z - \bar{z})$. From (3.2 ii), (2.2) and the convexity of $N(\bar{z})$ we deduce that

$$\varphi(\bar{z}) + \varphi'(\bar{z})(z^\alpha - \bar{z}) + N(\bar{z}) \ni 0,$$

hence with (3.2 i) and (3.22), for all $y \in K$:

$$\begin{aligned} 0 &\leq \langle \varphi(\bar{z}) + \varphi'(\bar{z})(z^\alpha - \bar{z}), y - \bar{z} \rangle, \\ &= \langle \varphi(\bar{z}) + \varphi'(\bar{z})(z^\alpha - \bar{z}), y - z^\alpha \rangle, \end{aligned}$$

that is

$$\varphi(\bar{z}) + \varphi'(\bar{z})(z^\alpha - \bar{z}) + N(z^\alpha) \ni 0,$$

hence z^α is a solution of (3.19). Also $z^\alpha \to \bar{z}$ when $\alpha \searrow 0$; this contradicts (a).

c) Proof of (b) $\Rightarrow$ (c). Assume that (c) does not hold and let $z \in K$, $z \neq \bar{z}$ be a solution of (3.21). From (3.3ii) and (3.3 iii) we deduce that

$$\langle z - \bar{z}, \varphi'(\bar{z})(z - \bar{z})\rangle \leq 0.$$

As (3.2 i) coincide with (3.3 ii) it remains to derive (3.2 ii) in order to get a contradiction with (b). If $\alpha \leq 1$, multiplying relation (2.2) by $(1-\alpha)$, adding it to (3.3 iii) and using (3.3 i) we get (3.2 ii). If $\alpha > 1$ we may check similarly, dividing (3.3 iii) by α, that $y^\alpha := \bar{z} + \frac{1}{\alpha}(z - \bar{z})$ contradicts (b).

d) Proof of (c) $\Rightarrow$ $\{\bar{z}$ is semi-stable$\}$. If $\bar{z}$ is not semi-stable let $z^k \to \bar{z}$ and $\delta^k \to 0$ in $\mathbb{R}^n$ be such that

$$\varphi(z^k) + N(z^k) \ni \delta^k, \tag{3.23}$$

and $\|\delta^k\|/\|z^k - \bar{z}\| \to 0$. Define $\beta^k := \|z^k - \bar{z}\|^{-1}$ and $w^k := \beta^k(z^k - \bar{z})$. Then substituting $\varphi(\bar{z}) + \varphi'(\bar{z})(z^k - \bar{z}) + o(z^k - \bar{z})$ to $\varphi(z^k)$ in (3.23) we get after multiplication by β^k

$$\beta^k\varphi(\bar{z}) + \varphi'(\bar{z})w^k + N(z^k) \ni \beta^k\delta^k + \beta^k o(z^k - \bar{z}). \tag{3.24}$$

This right-hand side of (3.24) has limit 0. As K is a polyhedron we may extract without loss of generality a subsequence such that $N(z^o) = N(z^k)$ for all k; also $\|w^k\| = 1$ hence $\{w^k\}$ has at least a limit-point w (for the same subsequence) with $\|w\| = 1$. Again as K is a polyhedron, the set $N^0 := N(z^o) + \mathbb{R}^+\varphi(\bar{z})$ is the cone of exterior normals at z^o to the set

$$K^0 := K \cap \{z \in \mathbb{R}^q; \langle z - z^o, \varphi(\bar{z})\rangle \leq 0\}.$$

Hence N^0 is closed. By (3.24) and the closedness of N^0 we have

$$\mathbb{R}^+\varphi(\bar{z}) + \varphi'(\bar{z})w + N(z^o) \ni 0. \tag{3.25}$$

Also as $\beta^k \geq 0$ and the vectors $\bar{z} + (\beta^k)^{-1}w^k = z^k$, $z^k - (\beta^k)^{-1}w^k = \bar{z}$ are elements of K, we get from (2.1) and (3.23) :

$$\left\{\begin{array}{rcl} \langle w^k, \varphi(\bar{z})\rangle & = & \beta^k\langle z^k - \bar{z}, \varphi(\bar{z})\rangle \geq 0, \\ -\langle w^k, \varphi(z^k)\rangle & = & \beta^k\langle \bar{z} - z^k, \varphi(z^k)\rangle \geq \beta^k\langle \bar{z} - z^k, \delta^k\rangle \to 0. \end{array}\right. \tag{3.26}$$

As $z^k \to \bar{z}$, $\varphi(z^k) \to \varphi(\bar{z})$. This, (3.26) and $w^k \to w$ imply

$$\langle w, \varphi(\bar{z})\rangle = 0. \tag{3.27}$$

Now, as K is a polyhedron, $\bar{z} + \varepsilon w$ is in K for $\varepsilon > 0$ small enough. Let us check that $N(\bar{z} + \varepsilon w) \supset N(z^o)$. It is sufficient to check that any linear inequality constraint defining K that is active at z^o is also active at $\bar{z} + \varepsilon w$. Here we say that a constraint $\langle a, z\rangle \leq b$ is active at z if $\langle a, z\rangle = b$. Extracting again if necessary a subsequence we may assume that the set of active constraints is the same for all $\{z^k\}$. Then for the subsequence considered here we have $\langle a, z^k\rangle = b$, hence $\langle a, \bar{z}\rangle = b$ and $\langle a, w^k\rangle = 0$, from which $\langle a, w\rangle = 0$, and finally $\langle a, \bar{z} + \varepsilon w\rangle = b$. This proves that $N(\bar{z} + \varepsilon w) \supset N(z^o)$. This and (3.25) (multiplied by $\varepsilon > 0$) imply

$$\mathbb{R}^+\varphi(\bar{z}) + \varepsilon\varphi'(\bar{z})w + N(\bar{z} + \varepsilon w) \ni 0. \tag{3.28}$$

Also for $\varepsilon > 0$ small enough and as K is a polyhedron, $N(\bar{z}+\varepsilon w) \subset N(\bar{z})$.This, (3.27), (3.28) and the fact that $z = \bar{z} + \varepsilon w$ is in K give a contradiction to (c). □

REMARK 3.2. *The proof of*

$$\{\bar{z} \textit{ is semi-stable } \} \Rightarrow (a) \Rightarrow (b) \Rightarrow (c)$$

does not use the fact that K is polyhedral.

4. Extension of the theory to nonsmooth data. Although we are mainly interested in this paper by finite dimensional variational inequalities with smooth data we will give here a partial extension of the previous results to problems in a Hilbert space with nonsmooth data. Let K be a closed convex subset of a Hilbert space Z, $N(z)$ the cone of outward normals to K at z and φ a mapping from Z into itself. In order to define an extension of Algorithm 1 for the problem

$$\varphi(z) + N(z) \ni 0, \tag{4.29}$$

we use a concept of point-based approximation (PBA) close to the one of Robinson [23].

DEFINITION 4.1. *We say that $\psi : Z \times Z \to Z$ is a PBA to φ if for any two sequences $\{y^k\}, \{z^k\}$ converging to the same point the following holds :*

$$\|\varphi(y^k) - \psi(z^k, y^k)\| \leq r(y^k, z^k), \tag{4.30}$$

with $r(y^k, z^k)/\|y^k - z^k\| \to 0$.

Here $\psi(z^k, .)$ can be seen as a generalization of the linearization of φ at z^k (see Remark 4.1 below). We now define a somewhat abstract Newton type method as the following algorithm :

Algorithm 2
0) Choose $z^0 \in Z; k \leftarrow 0$.

1) While z^k does not satisfy (4.1) : choose a mapping $\Xi^k : \quad Z \to Z$, approximation of $\psi(z^k, .)$. Compute z^{k+1} solution of

$$\Xi^k(z^{k+1}) + N(z^{k+1}) \ni 0. \tag{4.31}$$

We define semi-stability as in section 2.

THEOREM 4.1. *If $\{z^k\}$ computed by Algorithm 2 converges towards a semi-stable solution $\bar{z}$ of (4.1), then*

(i) If $\psi(z^k, z^{k+1}) - \Xi^k(z^{k+1}) = o(z^{k+1} - z^k)$, then $\{z^k\}$ converges superlinearly.

(ii) If $\psi(z^k, z^{k+1}) - \Xi^k(z^{k+1}) = 0(\|z^{k+1} - z^k\|^2)$ and for some $c_1 > 0$ and all (y, z) close enough to $\bar{z}$ the function r in (4.2) satisfies $r(y, z) \leq c_1\|y - z\|^2$, then $\{x^k\}$ converges quadratically.

Proof. Writing the step (4.3) as

$$\psi(z^k, z^{k+1}) + N(z^{k+1}) \ni \psi(z^k, z^{k+1}) - \Xi^k(z^{k+1})$$

and using (4.2), we deduce that

$$\varphi(z^{k+1}) + N(z^{k+1}) \ni \psi(z^k, z^{k+1}) - \Xi^k(z^{k+1}) + o(z^{k+1} - z^k).$$

In case (i) it follows from semi-stability that $z^{k+1} - \bar{z} = o(z^{k+1} - z^k)$, hence z^k converges superlinearly. In case (ii) we similarly obtain $z^{k+1} - \bar{z} = 0(\|z^{k+1} - z^k\|^2)$,which implies the quadratic convergence. □

REMARK 4.1. *Theorem 4.1 can be seen as an extension of Theorem 2.1. Indeed if φ is continuously differentiable and*

$$\psi(z^k, z^{k+1}) = \varphi(z^k) + \varphi'(z^k)(z^{k+1} - z^k),$$

$$\Xi^k(z^{k+1}) = \varphi(z^k) + M^k(z^{k+1} - z^k),$$

for some $q \times q$ matrix M^k, then

$$\begin{aligned}\psi(z^k, z^{k+1}) - \Xi^k(z^{k+1}) &= (\varphi'(z^k) - M^k)(z^{k+1} - z^k)\\ &= (\varphi'(\bar{z}) - M^k)(z^{k+1} - z^k) + o(z^{k+1} - z^k),\end{aligned}$$

hence point (i) of Theorem 4.1 reduces to point (i) of Theorem 2.1. Similarly if φ' is locally Lipschitz we have

$$(\varphi'(z^k) - M^k)(z^{k+1} - z^k) = (\varphi'(\bar{z}) - M^k)(z^{k+1} - z^k) + 0(\|z^k - \bar{z}\|.\|z^{k+1} - z^k\|),$$

the last term being $0(\|z^{k+1} - z^k\|)^2$ as $\|z^{k+1} - z^k\|/\|z^k - \bar{z}\| \to 1$ because of the superlinear convergence, hence point (ii) of Theorem 4.1 reduces to point (ii) of Theorem 2.1.

We define the directional derivatives $\varphi'(.,.)$ of φ as the limit

$$\varphi'(z, d) := \lim_{\alpha \searrow 0} \frac{1}{\alpha}[\varphi(z + \alpha d) - \varphi(z)].$$

We will state in Theorem 4.2 below an extension of Theorem 3.1. This Theorem 4.2 applies to B-differentiable mappings (here B stands for Bouligand), as defined in Robinson [22], i.e. mappings having the following property : φ is locally Lipschitz has directional derivatives and $d \to \varphi'(x, d)$ is Lipschitz. Then it is known that (for given x) $\varphi(x + d) = \varphi(x) + \varphi'(x, d) + o(d)$ (see also Shapiro [24]).

THEOREM 4.2. *Assume that $Z = \mathbb{R}^q$, φ is a B-differentiable mapping , K is polyhedral and $\bar{z}$ is a solution of (4.1). Then $\bar{z}$ is semi-stable iff one of the following hypotheses holds :*

(a) $\bar{z}$ is an isolated solution of the linearization at $\bar{z}$ of (4.1) defined as follows :

$$\varphi(\bar{z}) + \varphi'(\bar{z}, z - \bar{z}) + N(z) \ni 0.$$

(b) One has $\langle z - \bar{z}, \varphi'(\bar{z}, z - \bar{z})\rangle > 0$ for all z different of $\bar{z}$ solution of

$$\langle \varphi(\bar{z}), z - \bar{z}\rangle = 0,$$

$$\varphi(\bar{z}) + \varphi'(\bar{z}, z - \bar{z}) + N(\bar{z}) \ni 0.$$

(c) The relation below has no solution but $\bar{z}$:

$$N(z) \subset N(\bar{z}),$$

$$\langle \varphi(\bar{z}), z - \bar{z}\rangle = 0,$$

$$\alpha\varphi(\bar{z}) + \varphi'(\bar{z}, z - \bar{z}) + N(z) \ni 0 \quad \textit{for some } \alpha \geq 0.$$

The proof is the same as the one of Theorem 3.1, replacing first order variations by directional derivatives.

5. Convergence analysis for some structured variational inequalities. We now specialize our study to a particular case of variational inequalities. In the next section we will apply the results of this section to nonlinear programming problems. Let F, g be smooth (resp. C^1 and C^2) mappings : $\mathbb{R}^n \to \mathbb{R}^n$ and $\mathbb{R}^n \to \mathbb{R}^p$, respectively. Let I, J be a partition of $\{1, \ldots, p\}$. By $g(x) \ll 0$ we mean

$$g_i(x) \leq 0, \forall i \in I,$$
$$g_j(x) = 0, \forall j \in J.$$

We now consider the system (in which $\lambda \in \mathbb{R}^p$)

$$(5.32) \qquad \left\{ \begin{array}{l} F(x) + g'(x)^* \lambda = 0, \\ \\ g(x) \ll 0, \ \lambda_I \geq 0, \ \lambda_i g_i(x) = 0, \ \forall i \in I. \end{array} \right.$$

As observed in Robinson [23] we may embed (5.1) into (2.1) in the following way. Put $q := n + p$, $z := (x, \lambda)$ and

$$\varphi(x, \lambda) := \begin{pmatrix} F(x) + g'(x)^* \lambda \\ -g(x) \end{pmatrix};$$

$$K_1 := \{\lambda \in \mathbb{R}^p, \lambda_I \geq 0\} \ ; \ K := \mathbb{R}^n \times K_1;$$

so that K is polyhedral and

$$N(x, \lambda) = \{0\} \times N_1(\lambda),$$

with $N_1(\lambda)$ normal cone (or cone of outwards normals) to K_1 at λ, i.e.

$$N_1(\lambda) = \left\{ \begin{array}{l} \emptyset \text{ if } \lambda \text{ is not in } K_1, \text{ otherwise} \\ \{\mu \in \mathbb{R}^p; \ \mu_J = 0; \ \mu_I \leq 0; \ \mu_i = 0 \text{ if } \lambda_i > 0, \forall i \in I\}. \end{array} \right.$$

The corresponding variational inequality can be written in the following way :

$$(5.33) \qquad \left\{ \begin{array}{lll} F(x) & +g'(x)^* \lambda & = 0, \\ \\ -g(x) & +N_1(\lambda) & \ni 0. \end{array} \right.$$

Let us denote

$$H(x, \lambda) := F'(x) + \sum_{i=1}^{p} \lambda_i \nabla^2 g_i(x).$$

Then we have

$$(5.34) \qquad \varphi'(x, \lambda) = \begin{pmatrix} H(x, \lambda) & g'(x)^* \\ \\ -g'(x) & 0 \end{pmatrix},$$

and

$$(5.35) \qquad \langle (y, \mu), \varphi'(x, \lambda)(y, \mu) \rangle = \langle y, H(x, \lambda) y \rangle.$$

Taking (5.2)-(5.3) and Theorem 3.1 in account, we see that semi-stability for (5.2) (expressed at some point $(\bar{x}, \bar{\lambda})$ solution of (5.2)) can be stated as

$$
(5.36) \qquad \begin{cases} (y,\mu) = 0 \text{ is an isolated solution of} \\ (i) \quad H(\bar{x},\bar{\lambda})y + g'(\bar{x})^*\mu = 0, \\ (ii) \quad g(\bar{x}) + g'(\bar{x})y \in N_1(\bar{\lambda}+\mu). \end{cases}
$$

For any $\hat{I} \subset I$ by $z \overset{\hat{I}}{\ll} 0$ we mean $z_J = 0$ and $z_i \leq 0$ for all i in $\hat{I}$. Let us define

$$
\begin{aligned}
\bar{I} &:= \{i \in I; g_i(\bar{x}) = 0\}, \\
I^+ &:= \{i \in \bar{I}; \bar{\lambda}_i > 0\}, \\
I^0 &:= \bar{I} - I^+ = \{i \in \bar{I};\ \bar{\lambda}_i = 0\}, \\
I^* &:= J \cup I^+.
\end{aligned}
$$

It may be convenient to define the so-called "critical cone" (or cone of critical directions) :

$$
C = \{y \in \mathbb{R}^n;\ g'(\bar{x})y \overset{\bar{I}}{\ll} 0;\ g'_{I^+}(\bar{x})y = 0\}.
$$

PROPOSITION 5.1. *Semi-stability of (5.2) is equivalent to*

$$
(5.37) \quad \begin{cases} (y,\mu) = 0 \textit{ is the unique solution of} \\ (i) \quad H(\bar{x},\bar{\lambda})y + g'(\bar{x})^*\mu = 0, \\ (ii) \quad y \in C,\ \mu_{I^0} \geq 0;\ \mu_i = 0 \textit{ if } g_i(\bar{x}) < 0, \forall i \in I;\ \mu_i g'_i(\bar{x})y = 0,\ \forall i \in I^0. \end{cases}
$$

Proof. We have to prove the equivalence of (5.5) and (5.6). The set of solutions of (5.6.i-ii) is a cone. Hence it is equivalent to state that $(y,\mu) = 0$ is the unique solution of (5.6i-ii) or to state that $(y,\mu) = 0$ is an isolated solution of (5.6i-ii)). Now it is sufficient to prove the equivalence of (5.5 ii) and (5.6 ii) when (y,μ) is small enough. If μ is sufficiently close to zero and $i \in I^+$ then $\bar{\lambda}_i + \mu_i > 0$ hence by (5.5 ii) $g'_i(\bar{x})y = 0$. On the other hand if (5.5ii) holds μ_{I^0} must be nonnegative and $\mu_i > 0$ for some $i \in I_0$ implies $g'_i(\bar{x})y = 0$. Also if $g_i(\bar{x}) < 0$, then $g_i(\bar{x}) + g'_i(\bar{x})y < 0$ for y sufficiently close to 0. For that reason (5.5 ii) is equivalent (when (y,μ) is small enough) to

$$
\begin{cases}
g'_i(\bar{x})y = 0, \quad \forall i \in I^+, \\
g'_i(\bar{x})y \leq 0, \quad \mu_i \geq 0, \quad \mu_i g'_i(\bar{x})y = 0,\ \forall i \in I^0, \\
\mu_i = 0 \quad \text{if } g_i(\bar{x}) < 0,
\end{cases}
$$

and this is easily shown to be equivalent to (5.6ii). □

Let us now consider Newton's method applied to (5.2). The subproblem to be solved at step k is, denoting by d^k the increment in x, i.e. $d^k = x^{k+1} - x^k$:

$$\begin{cases} F(x^k) & +H(x^k, \lambda^k)d^k + g'(x^k)^*\lambda^{k+1} = 0, \\ g(x^k) & +g'(x^k)d^k \in N_1(\lambda^{k+1}). \end{cases}$$

As the evaluation of $g'(x^k)$ is already necessary in order to evaluate $\varphi(x^k, \lambda^k)$ the only part of the Jacobian that perhaps needs to be approximated is $H(x^k, \lambda^k)$. We obtain then the Newton type algorithm

Algorithm 3

0) Choose $(x^0, \lambda^0) \in \mathbb{R}^n \times \mathbb{R}^p$. $k \leftarrow 0$.

1) While (x^k, λ^k) is not solution of (5.2) : Choose $M^k, n \times n$ matrix and compute (d^k, λ^{k+1}) solution of

$$\begin{cases} F(x^k) & +M^k d^k + g'(x^k)^*\lambda^{k+1} = 0, \\ g(x^k) & +g'(x^k)d^k \in N_1(\lambda^{k+1}). \end{cases}$$

and put $x^{k+1} \leftarrow x^k + d^k$.

When $M^k = H(x^k, \lambda^k)$, applying Corollary 2.1 and Proposition 5.1, we easily obtain

THEOREM 5.1. *(Convergence of Newton's method). Let $\{x^k, \lambda^k\}$ be computed by Algorithm 3 with $M^k = H(x^k, \lambda^k)$ converge toward $(\bar{x}, \bar{\lambda})$ satisfying (5.2) and (5.6). If $x \to (F(x), g'(x))$ is C^1 (resp. C^1 with a locally Lipschitz derivative) then $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ superlinearly (resp. at a quadratic rate).*

We now consider conditions related to the superlinear convergence of $\{x^k\}$ alone. We are looking for necessary and/or sufficient conditions of the following type : at each iteration k we define

$$\begin{aligned} &E^k \quad \text{closed convex subset of } \mathbb{R}^n, \\ &P^k \text{orthogonal projection onto } E^k, \\ &h^k \quad := P^k[(H(\bar{x}, \bar{\lambda}) - M^k)d^k]. \end{aligned}$$

The condition will be

$$h^k = o(d^k). \tag{5.38}$$

As a particular case of our results we will recover the characterization of Boggs, Tolle and Wang [3] concerning equality constrained nonlinear programming problems, and we will be able to extend the characterization to variational inequalities satisfying the assumption

$$d^t H(\bar{x}, \bar{\lambda})d > 0 \quad \text{for all } d \text{ in } C - \{0\}. \tag{5.39}$$

All our results, however will need the following qualification hypothesis (linear independence of the gradients of active constraints)

$$\{\nabla g_i(\bar{x})\}_{i \in \bar{I} \cup J} \quad \text{surjective.} \tag{5.40}$$

On the other hand we do not need any strict complementary hypothesis.

THEOREM 5.2. *We assume $x \to (F(x), g'(x))$ to be C^1. Let $\{(x^k, \lambda^k)\}$ be computed by Algorithm 3 converge towards $(\bar{x}, \bar{\lambda})$, semi-stable solution of (5.2) satisfying (5.9). Then :*

(i) *Condition (5.7) is sufficient for superlinear convergence when E^k is defined as*

$$E_1^k := \{d \in \ker g'_{I^*}(x^k); g'_i(x^k)d \geq 0, \forall i \in I^0 \quad \textit{such that } g_i(x^k) + g'_i(x^k)d^k = 0\}$$

.

(ii) *Condition (5.7) is necessary, and also sufficient for superlinear convergence if in addition (5.8) holds, when E^k is defined as*

$$E_2^k := \{d \in \ker g'_{I^*}(x^k); g'_{I^0}(x^k)d \leq 0\}.$$

REMARK 5.1. *If the strict complementarity hypothesis holds, i.e. $I^0 = \phi$, then $E_1^k = E_2^k = \ker g'_{I^*}(x^k)$ and, with this choice of E^k, condition (5.7) is necessary and sufficient for superlinear convergence of $\{x^k\}$, under the hypotheses of semi-stability of $(\bar{x}, \bar{\lambda})$.* **Proof of Theorem 5.2**

a) Preliminaries. Writing the Kuhn-Tucker conditions for the projection problem defining h^k we get the existence of $\eta^k \in \mathbb{R}^p$ satisfying

$$h^k - [H(\bar{x}, \bar{\lambda}) - M^k]d^k + g'(x^k)^*\eta^k = 0 \qquad (5.41)$$

and

$\eta_i^k = 0 \quad \text{if } i \in I - \bar{I},$

$\eta_{I^0}^k \leq 0, \eta_i^k = 0 \quad \text{for all } i \text{ in } I^0 \text{ such that } g_i(x^k) + g'_i(x^k)d^k < 0, \text{ if } E^k = E_1^k,$

$\eta_{I^0}^k \geq 0 \quad \text{if } E^k = E_2^k.$

Substracting the first relation defining the Newton type step from (5.10) we get

$$h^k - F(x^k) - H(\bar{x}, \bar{\lambda})d^k + g'(x^k)^*(\eta^k - \lambda^{k+1}) = 0. \qquad (5.42)$$

Expanding $F(x^k)$ up to the first order and taking (5.2) in account we have

$$\begin{aligned} -F(x^k) &= -F(\bar{x}) - F'(\bar{x})(x^k - \bar{x}) + o(x^k - \bar{x}), \\ &= g'(\bar{x})^*\bar{\lambda} - F'(\bar{x})(x^k - \bar{x}) + o(x^k - \bar{x}), \\ &= g'(x^k)^*\bar{\lambda} - H(\bar{x}, \bar{\lambda})(x^k - \bar{x}) + o(x^k - \bar{x}), \end{aligned}$$

hence with (5.11) :

$$(5.43) h^k - H(\bar{x}, \bar{\lambda})(x^k + d^k - \bar{x}) + g'(x^k)^*(\bar{\lambda} + \eta^k - \lambda^{k+1}) = o(x^k - \bar{x}).$$

Let us define

$$\delta^k := \|h^k\| + \|x^k - \bar{x}\| + \|d^k\|.$$

Then $(\delta^k)^{-1}(h^k, x^k + d^k - \bar{x}, \lambda^{k+1} - \eta^k - \bar{\lambda})$ is bounded, the boundedness of the third term being a consequence of (5.9) and (5.12). Let (h, z, ζ) be a limit-point of this sequence, i.e. a limit for a subsequence $k \in S \subset \mathbb{N}$. Then $\zeta_i = 0$ if $i \in I - \bar{I}$ and from (5.12) we deduce that

$$h - H(\bar{x}, \bar{\lambda})z - g'(\bar{x})^* \zeta = 0. \tag{5.44}$$

Also, expanding g as follows :

$$g(x^k) + g'(x^k)d^k = g(\bar{x}) + g'(\bar{x})(x^k + d^k - \bar{x}) + o(x^k - \bar{x}),$$

we deduce from the fact that d^k is a Newton type step associated to a multiplier λ^{k+1} that

$$\begin{cases} g'(\bar{x})z \overset{\bar{I}}{\ll} 0, \\ g'_i(\bar{x})z = 0, \quad \text{if } i \in I \text{ is such that } \lambda_i^{k+1} > 0 \text{ for all } k \in S \\ \qquad\qquad\qquad \text{(which is the case if } \bar{\lambda}_i > 0, \text{ i.e. } i \in I^+). \end{cases} \tag{5.45}$$

The above relations imply

$$z \in C. \tag{5.46}$$

We also have from the definition of h^k :

$$g'_{I^*}(\bar{x})h = 0, \tag{5.47}$$

$$h \in C \quad \text{when } E^k = E_2^k. \tag{5.48}$$

b) Proof of case i). If $h^k = o(d^k)$, then a fortiori $h = 0$. With (5.13) we deduce that

$$H(\bar{x}, \bar{\lambda})z + g'(\bar{x})^* \zeta = 0. \tag{5.49}$$

If $i \in I^0$ is such that $g'_i(\bar{x})z < 0$, then $g_i(x^k) + g'_i(x^k)d^k < 0$, $\eta_i^k = 0$ and $\lambda_i^{k+1} = 0$ for k in S large enough, hence $\zeta_i = 0$. This implies

$$\zeta_i g'_i(\bar{x})z = 0 \quad \text{for all } i \text{ in } I^0.$$

But this with the fact that $\zeta_{I^0} \geq 0$ (due to $\eta_{I^0}^k \leq 0$), (5.15), (5.18) and (5.6) imply $z = 0$, i.e.

$$x^{k+1} - \bar{x} = o(\|h^k\| + \|x^k - \bar{x}\| + \|d^k\|).$$

This and (5.7) imply

$$x^{k+1} - \bar{x} = o(\|x^k - \bar{x}\| + \|x^{k+1} - x^k\|) = o(\|x^k - \bar{x}\| + \|x^{k+1} - \bar{x}\|)$$

which implies $x^{k+1} - \bar{x} = o(x^k - \bar{x})$, i.e. x^k converges superlinearly.

c) Proof of case ii). If x^k converges superlinearly then $z = 0$. Computing the scalar product of (5.13) by h we get

$$\|h\|^2 = \langle \zeta, g'(\bar{x})h \rangle.$$

Using (5.17), the nonnegativity of $\lambda_{I^0}^{k+1}$ and the complementarity condition $\eta_i^k g'_i(x^k)h^k = 0$, for i in I^0, we deduce that the right hand side of the above relation is non positive ; hence $h = 0$, i.e.

$$h^k = o(\|h^k\| + \|x^k - \bar{x}\| + \|d^k\|),$$

which implies $h^k = o(\|x^k - \bar{x}\| + \|d^k\|)$. However the superlinear convergence of $\{x^k\}$ implies $\|d^k\|/\|x^k - \bar{x}\| \to 1$, hence (5.7) holds.

We now prove that if (5.7) and (5.8) hold, $\{x^k\}$ converges superlinearly. As (5.7) implies $h = 0$, computing the scalar product of (5.13) with z we get

$$\langle z, H(\bar{x}, \bar{\lambda})z\rangle + \langle \zeta, g'(\bar{x})z\rangle = 0. \tag{5.50}$$

As $z \in C, g_i'(\bar{x})z = 0$ if $\lambda_i^{k+1} \neq 0$, and $\eta_{I^0}^k \geq 0$, the second term of (5.19) is non negative. This and (5.8) imply that $z = 0$ i.e.

$$x^{k+1} - \bar{x} = o(\|h^k\| + \|x^k - \bar{x}\| + \|d^k\|).$$

Using (5.7) and the relation $\|d^k\| \leq \|x^{k+1} - \bar{x}\| + \|x^k - \bar{x}\|$ we deduce that

$$x^{k+1} - \bar{x} = o(\|x^k - \bar{x}\| + \|x^{k+1} - \bar{x}\|).$$

which implies the superlinear convergence of $\{x^k\}$.

With the help of Theorem 5.2 we may obtain the superlinear convergence of $\{x^k\}$ when M^k is updated using ideas of quasi-Newton algorithms.

We define the quasi-Newton equation (for M^{k+1}) as follows

$$M(x^{k+1} - x^k) = F(x^{k+1}) - F(x^k) + \left[g'(x^{k+1}) - g'(x^k)\right]^* \lambda^{k+1}. \tag{5.51}$$

We assume that there exists a closed convex subset $\mathcal{K}$ of the space of $n \times n$ matrices such that

$$H(x, \lambda) \in \mathcal{K},\ \forall (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^p, \tag{5.52}$$

and we choose M^{k+1} solution of

$$\min \|M - M^k\|_\sharp;\ M \in \mathcal{K};\ M \quad \text{satisfies (5.20)}, \tag{5.53}$$

where as before $\|\cdot\|_\sharp$ is a norm associated to a scalar product.

LEMMA 5.1. *Under the hypotheses of Theorem 5.2, if M^{k+1} satisfies (5.20) and*

$$(M^{k+1} - M^k)(x^{k+1} - x^k) = o(x^{k+1} - x^k), \tag{5.54}$$

then $\{x^k\}$ converges superlinearly.

Proof. As M^{k+1} satisfies (5.20) we have

$$\begin{aligned} M^{k+1}(x^{k+1} - x^k) &= F(x^{k+1}) - F(x^k) + (g'(x^{k+1}) - g'(x^k))^*\bar{\lambda} + (g'(x^{k+1}) - g'(x^k))^*(\lambda^{k+1} - \bar{\lambda}) \\ &= H(\bar{x}, \bar{\lambda})(x^{k+1} - x^k) + o(x^{k+1} - x^k). \end{aligned}$$

Hence if M^{k+1} satisfies (5.20), and (5.23) holds, then

$$(H(\bar{x}, \bar{\lambda}) - M^k)(x^{k+1} - x^k) = o(x^{k+1} - x^k).$$

Now let h^k be the projection of $(H(\bar{x}, \bar{\lambda}) - M^k)(x^{k+1} - x^k)$ onto E_1^k. As E_1^k is a cone, and the projector operator being non expansive we obtain $\|h^k\| \leq \|(H(\bar{x}, \bar{\lambda}) - M^k)(x^{k+1} - x^k)\| = o(x^{k+1} - x^k)$. This and Theorem 5.2 (case i) imply the conclusion. □

THEOREM 5.3. *Let $(\bar{x}, \bar{\lambda})$ be a semi-stable and hemi-stable solution of (5.1). Then there exists $\varepsilon > 0$ such that, if $\|x^0 - \bar{x}\| + \|M^0 - H(\bar{x}, \bar{\lambda})\|_\sharp < \varepsilon$, then :*

At each step k there exists (x^{k+1}, λ^{k+1}) solution of the Newton type step satisfying $\|x^{k+1} - x^k\| < 2\varepsilon$.
The sequence $\{x^k\}$ defined in this way converges superlinearly towards $\bar{x}$.

Proof. Define

$$S^k := \{M \in \mathcal{K}; M \quad \text{satisfies (5.20)}\},$$

$$A^k := \int_0^1 H(x^k + \sigma(x^{k+1} - x^k), \lambda^{k+1}) d\sigma.$$

Then A^k is an element of S^k and for some $c_1 > 0$

$$\|A^k - H(\bar{x}, \bar{\lambda})\|_\sharp \le c_1 \nu^k, \tag{5.55}$$

with here

$$\nu^k := \|x^{k+1} - \bar{x}\| + \|x^k - \bar{x}\| + \|\lambda^{k+1} - \bar{\lambda}\|.$$

As M^{k+1} is the projection of M^k onto S^k, we have from Grzegòrski [12]

$$\|M^{k+1} - M^k\|_\sharp^2 + \|M^{k+1} - A^k\|_\sharp^2 \le \|M^k - A^k\|_\sharp^2, \tag{5.56}$$

hence with (5.24)

$$\|M^{k+1} - H(\bar{x}, \bar{\lambda})\|_\sharp \le \|M^k - H(\bar{x}, \bar{\lambda})\|_\sharp + 2c_1 \nu^k. \tag{5.57}$$

As a consequence the approximation of the Jacobian at step k is

$$\hat{M}^k := \begin{pmatrix} M^k & g'(x^k)^* \\ g'(x^k) & 0 \end{pmatrix},$$

and approximates

$$\bar{M} := \begin{pmatrix} H(\bar{x}, \bar{\lambda}) & g'(\bar{x})^* \\ g'(\bar{x}) & 0 \end{pmatrix}.$$

We define a new norm as follows. To

$$\hat{M} := \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & 0 \end{pmatrix}$$

is associated

$$\|\hat{M}\|_\$:= \|M_{11}\|_\sharp + \|M_{12}\|$$

with $\|.\|$ an arbitrary norm. From (5.26) we deduce that for some $c_2 > 0$

$$\|\hat{M}^{k+1} - \bar{M}\|_\$ \le \|\hat{M}^k - \bar{M}\|_\$ + c_2 \nu^k.$$

Applying Lemma 2.2 (for which we may assume that $\lambda^0 = \bar{\lambda}$) we deduce that if (x^0, M^0) is close enough to $(\bar{x}, H(\bar{x}, \bar{\lambda}))$, then (x^k, λ^k) is well defined and converges linearly to $(\bar{x}, \bar{\lambda})$ and that $\|\hat{M}^k - \bar{M}\|_\$$ converges. This implies that $\|M^k - H(\bar{x}, \bar{\lambda})\|_\sharp$ converges. As $A^k \to H(\bar{x}, \bar{\lambda})$, $\|M^{k+1} - A^k\|_\sharp$ and $\|M^k - A^k\|_\sharp$ do converge to the same limit, and with (5.25) this implies $\|M^{k+1} - M^k\| \to 0$. The conclusion is then a consequence of Lemma 5.1. □

6. Application to nonlinear programming. In this section we particularize some of our results to nonlinear programming problems, and we will see that it allows to get some improvements with respect to known results. By nonlinear programming problem we mean

$$\min f(x); g(x) \ll 0 \tag{6.58}$$

where f is a smooth mapping $\mathbb{R}^n \to \mathbb{R}$, and g as well as the relation "$\ll$" are as in section 5. Let us recall some well-known facts of optimization theory (see e.g. Fletcher [10]). To problem (6.1) is associated the first-order optimality system

$$\begin{cases} \nabla f(x) + g'(x)^* \lambda = 0, \\ g(x) \ll 0, \ \lambda_I \geq 0, \ \lambda^t g(x) = 0, \end{cases} \tag{6.59}$$

which is formally equivalent to (5.1) if we define $F(x) := \nabla f(x)$. In this case the mapping $H(x, \lambda)$ can be interpretated as the Hessian with respect to x of the Lagrangian $L(x, \lambda) := f(x) + \lambda^t g(x)$. We will say that λ is a Lagrange multiplier associated to x if (x, λ) satisfies (6.2). We recall the results involving second-order conditions with a unique multiplier.

PROPOSITION 6.1. *(see e.g. Ben-Tal [1])*

(i) *(Second-order necessary condition). Let $\bar{x}$ be a local solution of (6.1) to which is associated a unique multiplier $\bar{\lambda}$. Then $d^t H(\bar{x}, \bar{\lambda}) d \geq 0$ for all critical directions d .*

(ii) *(Second-order sufficiency condition). Let $(\bar{x}, \bar{\lambda})$ satisfying (6.2) be such that $d^t H(\bar{x}, \bar{\lambda}) d > 0$ for all non zero critical directions d . Then $\bar{x}$ is a local solution of (6.1).*

We now make the link between semi-stability and the second-order sufficiency condition.

PROPOSITION 6.2. *Let $(\bar{x}, \bar{\lambda})$ be an isolated solution of (6.2) such that $\bar{x}$ is a local solution of (6.1). Then $(\bar{x}, \bar{\lambda})$ is semi-stable iff it satisfies the second-order sufficiency condition.*

Proof. Characterization (3.2) of semi-stability applied to the variational inequality in form (5.1), and using (5.4), gives

$$\langle d, H(\bar{x}, \bar{\lambda}) d \rangle > 0 \text{ for all } (d, \mu) \neq 0 \text{ solution of}$$

$$\begin{aligned} \lambda_I + \mu_I &\geq 0 \\ H(\bar{x}, \bar{\lambda}) d + g'(\bar{x})^* \mu &= 0, \\ g(\bar{x})^t (\mu - \lambda) &= 0, \\ g(\bar{x}) + g'(\bar{x}) d &\in N_1(\lambda), \end{aligned}$$

the last relation implying that d is critical. Hence the second-order sufficiency optimality condition implies semi-stability. Conversely, let us assume that the second-order sufficiency condition does not hold. By Proposition 6.1 there exists a critical direction $\bar{d} \neq 0$ with $\bar{d}^t H(\bar{x}, \bar{\lambda}) \bar{d} = 0$, and $\bar{d}$ is a solution of the quadratic homogeneous problem

$$\min \frac{1}{2} d^t H(\bar{x}, \bar{\lambda}) d; \ d \in C,$$

where the critical cone is

$$C := \{d; g'(\bar{x})d \overset{\bar{I}}{\ll} 0; g_i'(\bar{x})d = 0 \quad \text{if } \lambda_i > 0, \forall i \in I\}.$$

Writing the optimality system of this problem, we find that to $\bar{d}$ is associated a multiplier μ such that $(\bar{d}, \mu)$ satisfies (5.6 i-ii). By Proposition 5.1 this contradicts semi-stability. □

PROPOSITION 6.3. *Let $(\bar{x}, \bar{\lambda})$ be a semi-stable solution of (6.2) such that $\bar{x}$ is a local solution of (6.1). Then $(\bar{x}, \bar{\lambda})$ is hemi-stable.*

Proof. Semi-stability implies the uniqueness of the multiplier, hence also the hypothesis of Mangasarian and Fromovitz. By Proposition 6.2 the second-order sufficiency condition also holds for problem (6.1) at $(\bar{x}, \bar{\lambda})$. Let us consider the following problem

$$\min_d \nabla f(\bar{x})^t d + \frac{1}{2} d^t H(\bar{x}, \bar{\lambda}) d; \; g(\bar{x}) + g'(\bar{x})d \ll 0. \tag{6.60}$$

Obviously $\bar{d} = 0$ satisfies the first order optimality condition associated to the unique multiplier λ. Also $\bar{d} = 0$ satisfies the second order sufficiency condition (their formulation for (6.3) at $(\bar{d} = 0, \bar{\lambda})$ coincides with the one for (6.1) at $(\bar{x}, \bar{\lambda})$). Hence if we make a small perturbations in the data of this problem there exists a local solution whose distance to $\bar{d} = 0$ is of the order of the perturbation (see e.g. Robinson [20], Thm 4.1). Hence hemi-stability holds. □

From Theorem 2.2 and Proposition 6.3 we deduce

THEOREM 6.1. *Assume that f and g are C^2 with Lipschitz second derivatives, $\bar{x}$ is a local solution of (6.1), $\bar{\lambda}$ is the unique Lagrange multiplier associated to $\bar{x}$, and the second-order sufficiency condition holds. Then there exists $\varepsilon > 0$ such that if $\|x^0 - \bar{x}\| + \|\lambda^0 - \bar{\lambda}\| < \varepsilon$, and (x^{k+1}, λ^{k+1}) is chosen so that $\|x^{k+1} - x^k\| + \|\lambda^{k+1} - \lambda^k\| < 2\varepsilon$, then Algorithm 3 with $M^k = H(x^k, \lambda^k)$, i.e. Newton's method, is well defined and converges at a quadratic rate to $(\bar{x}, \bar{\lambda})$.*

REMARK 6.1. *That Newton's method converges at a quadratic rate when the starting point is close to a solution $(\bar{x}, \bar{\lambda})$ of (6.2), assuming x is a local solution of (6.1), the gradients of active constraints linearly independent, and strict complementarity, is well known. Recently the author [4] relaxed the strict complementarity hypothesis. Here we improve the result of [4] by assuming that the multiplier is unique instead of the linear independence of the gradients of active constraints.*

We now apply the results of section 5 on the superlinear convergence of $\{x^k\}$ only. From Theorem 5.2 and the fact that condition (5.8) coincides with the second-order sufficient condition, we get:

THEOREM 6.2. *Let $\bar{x}$ be a local solution of (6.1) such that the gradients of active constraints are linearly independent, $\bar{\lambda}$ be a multiplier associated to $\bar{x}$ and the second-order sufficient condition holds. Then if (x^k, λ^k) computed by Algorithm 3 converge to $(\bar{x}, \bar{\lambda})$, then $\{x^k\}$ converges superlinearly iff*

$$P^k[(H(\bar{x}, \bar{\lambda}) - M^k)d^k] = o(d^k),$$

with P^k orthogonal projection on the set E_2^k defined in Theorem 5.2.

REMARK 6.2. *If no inequality constraint is present, Theorem 6.2 reduces to a theorem of Boggs, Tolle and Wang [3]. Some necessary or sufficient conditions (but not the characterization given here) for problems with equalities and inequalities, without strict complementarity have been given by the author in [4].*

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