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EXACT PENALIZATION WITH A SMALL NONSMOOTH TERM

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Abstract

With the standard nonlinear programming problem, we associate a potential which is the sum of the Lagrangian and of a nonsmooth penalization term. We show that, for some values of the multiplier used in the potential, the coefficient of the nonsmooth term can be taken very small. The result is valid without second order assumptions but assumes uniqueness of the Lagrange multiplier. It has relation to some results in perturbation theory and to the study of Newton type algorithms for constrained optimization.

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1.- Introduction

We consider the standard nonlinear programming problem of minimizing a cost function over $\mathbb{R}^n$, under a finite number of equality and inequality constraints. This may be written as

$$\min f(x) \ ; \ g(x) \ll 0 \qquad (P)$$

where $x \in \mathbb{R}^n$ and f and g are functions from $\mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^p$, respectively, and the symbol "$\ll 0$" is defined as follows. Given a certain partition (I, J) of $\{1, \ldots, p\}$ we say that $z \ll 0$ iff

$$z_i = 0, \ i \in I, \quad z_j \leq 0, \ j \in J.$$

In the sequel we assume f, g to be of class C^1, i.e. continuously Fréchet differentiable. Let $\bar{x}$ be a local solution of (P). We assume that to $\bar{x}$ is associated a *unique* Lagrange multiplier, i.e. a vector $\bar{\lambda} \in \mathbb{R}^p$ such that

$$\nabla f(\bar{x}) + \sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0; \quad g(\bar{x}) \ll 0 \,;\, \bar{\lambda}_j \geq 0,\ \bar{\lambda}_j g_j(\bar{x}) = 0,\ j \in J. \quad (1.1)$$

As a consequence of results of Han and Mangasarian [8] and Ioffe [9], it is known that the function

$$\theta_r(x) := f(x) + r\|g(x)^\sharp\|$$

where, for $z \in \mathbb{R}^p$, $z^\sharp$ is defined by

$$z_i^\sharp := z_i,\ i \in I, \quad z_i^\sharp := z_i^+,\ j \in J,$$

has a minimum at $\bar{x}$, for r large enough. The motivation for considering such exact penalty functions, for which an early reference is Pietrzykowski [12], comes both from the theory of successive quadratic programming (see Pschenichny et al. [13] and Han [7]) and from the strong relation between exact penalization and a certain stability of the cost (Clarke [4], Burke [3]). The concept of exact penalization is also discussed in Bonnans [2].

An important issue is to estimate the value of the coefficient r. For this we have to use the concept of dual norm: $\|z\|_* := \max\{z^t y \,;\, \|y\| \leq 1\}$. From the above relation, it follows that $|z^t y| \leq \|y\|\|z\|_*$. It is proved in Charalambous [5] and Han and Mangasarian [8] that, under the additional assumption of a second order sufficient condition, involving a multiplier not depending on the critical direction, $\theta_r(x)$ has a local minimum at $\bar{x}$ whenever $r > \|\bar{\lambda}\|_*$. The result is still true if the multiplier depends on the critical direction [2]. We prove in this paper that the second order assumption is unnecessary, whenever the Lagrange multiplier is unique. This is a particular case of our main result that states some exact penalization properties for a nondifferentiable augmented Lagrangian of the type

$$\mathcal{L}_r(x, \lambda) := f(x) + \lambda^t g(x) + r\|g(x)^\sharp\|. \quad (1.2)$$

Here λ will be an approximation of $\bar{\lambda}$ satisfying conditions (2.3) below and r will be a *small* number. The interest of having a small coefficient for the nondifferentiable term is twofold. First it may allow, as in Bonnans [1], to remove the so-called Maratos effect that prevents the standard sequential quadratic algorithm to accept the unit stepsize. The second interest is in connection with perturbation theory, as there is a close connection between exact potentials and under estimates of the cost function for a problem with a perturbation in the right hand side of the constraints.

The paper is organized as follows: In section 2 we state the main result. Section 3 discusses some restoration operators, i.e. mappings in $\mathbb{R}^n$ whose image is included in the feasible set for (P). We prove the main result in section 4. Finally the link with perturbation theory is established in section 5.

2.- Main Result

Throughout the paper, $\bar{x}$ is a local solution of (P) associated with a *unique* multiplier $\bar{\lambda}$. Given $\alpha > 0$, we denote by $B(\bar{x}, \alpha)$ the *closed* ball of center $\bar{x}$ and radius α, and define the localized problem

$$\min f(x) \; ; \; g(x) \ll 0 \; ; \; x \in B(\bar{x}, \alpha). \qquad (P_\alpha)$$

By definition of a local solution, we may assume that $\alpha > 0$ is such that $\bar{x}$ is a global solution of (P_α). Of course (P_α) may have other global solutions, no matter how small α is. Let us denote the set of solutions of (P_α) as

$$S_\alpha := \{x \in \mathbb{R}^n \; ; \; f(x) = f(\bar{x}) \; ; \; g(x) \ll 0 \; ; \; x \in B(\bar{x}, \alpha)\}.$$

We say that ψ: $\mathbb{R}^n \to \mathbb{R}$ is an exact penalty function if, for $\alpha > 0$ small enough, S_α coincides with the argument of the minimum of ψ over $B(\bar{x}, \alpha)$, i.e.

$$\exists \alpha > 0 \; ; \; S_\alpha = \text{ argmin } \{\psi(x) \; ; \; x \in B(\bar{x}, \alpha)\}.$$

That is to say, it is equivalent, at least locally, to solve (P) or to minimize ψ. Note that this definition of exact penalty functions, in the case $\bar{x}$ is the unique solution of (P_α) for some $\alpha > 0$, amount to say that $\bar{x}$ is a strict local minimum of ψ. Note also that it may happen that $f(x) \neq \psi(x)$ even for a feasible x and this is indeed the case for the potential $\mathcal{L}_r(x, \lambda)$ defined in (1.2). We now state our main result, which is a condition on λ under which such a potential is an exact penalty function.

Theorem 1. *Let $\lambda \in \mathbb{R}^p$ be such that*

$$\text{for all } j \in J, \; 0 \leq \lambda_j \leq \bar{\lambda}_j \quad \text{and if} \quad \bar{\lambda}_j > 0, \; \text{then } \lambda_j < \bar{\lambda}_j. \qquad (2.3)$$

Then, for any $r > \|\lambda - \bar{\lambda}\|_*$, the potential $\mathcal{L}_r(x, \lambda)$ is an exact penalty function.

The proof is in two steps. The first step is to establish the result when $\lambda = 0$. Then the condition on the penalty parameter r reduces to $r > \|\bar{\lambda}\|_*$. The second step consists in writing a perturbed problem whose cost is the Lagrangian

$$\mathcal{L}(x) := f(x) + \lambda^t g(x)$$

and which has the same solution set S_α, the unique Lagrange multiplier associated with $\bar{x}$ being $\lambda - \bar{\lambda}$, so that the conclusion of the Theorem is given using step 1.

Our first task is to study the regularity of the feasible set and to build a restoration operator satisfying some specific properties.

3.- A Restoration Operator

By a restoration operator we mean a mapping that associates with each x, sufficiently close to $\bar{x}$, some $y \in \mathbb{R}^n$ satisfying the two conditions of feasibility, i.e., $g(y) \ll 0$, and proximity of y to x, i.e. $\|y - x\| = O(\|g(x)^\sharp\|)$. It is already known that such an operator exists, because as $\bar{\lambda}$ is unique, the point $\bar{x}$ satisfies the hypothesis of Mangasarian and Fromovitz (see [11]), and we may apply Robinson's result [14]. However, for the sequel we need an operator satisfying some additional properties.

Let $\bar{\mathcal{I}}, \bar{\mathcal{J}}$ be two subset of $\{1, \cdots, p\}$ with $\bar{\mathcal{I}} \cap \bar{\mathcal{J}} = \emptyset$. We say that $MF(\bar{\mathcal{I}}, \bar{\mathcal{J}})$ holds at $\bar{x}$ if

(i) $\{\nabla g_i(\bar{x})\}_{i \in \bar{\mathcal{I}}}$ is linearly independent
(ii) $\exists d \in \mathbb{R}^n \ ; \ g_i'(\bar{x})d = 0, \ i \in \bar{\mathcal{I}} \ ; \ g_i'(\bar{x})d < 0, \ i \in \bar{\mathcal{J}}$.

We note that the hypothesis of Mangasarian and Fromovitz is precisely $MF(I, J(\bar{x}))$ where $J(x)$ is the set of active inequality constraints at x:

$$J(x) := \{j \in J \ ; \ g_j(x) = 0\}.$$

Define the support of $\bar{\lambda}$ (for the inequality constraints) as $s(\bar{\lambda}) := \{j \in J \ ; \ \bar{\lambda}_j > 0\}$. Of course $s(\bar{\lambda}) \subset J(\bar{x})$. We fix $\mathcal{I}$ and $\mathcal{J}$ as

$$\mathcal{I} := I \cup s(\bar{\lambda}), \quad \mathcal{J} := J(\bar{x}) \backslash s(\bar{\lambda}).$$

From the uniqueness of the Lagrange multiplier we know by Kyparisis [10] that hypothesis $MF(\mathcal{I}, \mathcal{J})$ holds at $\bar{x}$. We now construct a restoration operator π: $B(\bar{x}, \alpha) \to \mathbb{R}^n$ that satisfies

$$\begin{aligned} &\text{(i)} && g(\pi(x)) \ll 0, \\ &\text{(ii)} && \|\pi(x) - x\| = O(\|g^\sharp(x)\|), \\ &\text{(iii)} && g_j(\pi(x)) = \min(g_j(x), 0); \ j \in s(\bar{\lambda}). \end{aligned} \tag{3.4}$$

The new requirement is point (iii). It says that, if an inequality constraint in the support of $s(\bar{\lambda})$ is not violated, the restoration will not change its value, and if the constraint is violated then the value of this constraint after restoration is 0.

Proposition 3.1. Whenever α is small enough, there exists an operator π that satisfies (3.4).

Proof. We consider a set parametrized by $z \in \mathbb{R}^{|s(\bar{\lambda})|}$:

$$D(z) := \{x \in \mathbb{R}^n; \ g_I(x) = 0; \ g_j(x) = z_j, \ j \in s(\bar{\lambda}); \ g_j(x) \le 0, \ j \in \mathcal{J}\}.$$

We know that $MF(\mathcal{I}, \mathcal{J})$ holds for $D(0)$ at $\bar{x}$. Robinson's result [14] implies the stability of the multifunction $D(z)$, in the following sense: $\exists \varepsilon > 0$,

$M > 0$ such that whenever $\|z\| \le \varepsilon$, $x \in B(\bar{x}, \alpha)$, there exists $y \in D(z)$ satisfying

$$\|y - x\| \le M(\|g_I(x)\| + \|g_{s(\bar{\lambda})}(x) - z\| + \|g_{\mathcal{J}}(x)^+\|). \tag{3.5}$$

Now, given $x \in B(\bar{x}, \alpha)$, take $z := \min(g_{s(\bar{\lambda})}(x), 0)$. Then $\|z\| \le \varepsilon$ for α small enough. Defining $\pi(x)$ as the element y in $D(z)$ that satisfies (3.5), we obtain the desired result. ■

4.- Proof of the Main Result

We first consider the case when $\lambda = 0$. Then our main result reduces to the following statement:

Proposition 4.1. The function $\theta_r(x)$ is an exact penalty function whenever $r > \|\bar{\lambda}\|_*$.

Proof. We use the restoration operator π of section 3. Let $\alpha_0 > 0$ be so small that $\bar{x}$ is solution of $(P)_{\alpha_0}$ and choose $\alpha \in (0, \alpha_0)$ such that π is well defined over $B(\bar{x}, \alpha)$. We set $y = \pi(x)$. We will prove that there exists $\alpha > 0$ such that if $x \in B(\bar{x}, \alpha)$, then

$$\begin{aligned} &y \in B(\bar{x}, \alpha_0) \text{ and } f(y) \le \theta_r(x) \text{ with } f(y) < \theta_r(x) \\ &\text{when } x \text{ is not feasible.} \end{aligned} \tag{4.6}$$

As $f(x) = \theta_r(x)$ whenever x is feasible, the conclusion follows. So let us prove (4.6). Given $x \in B(\bar{x}, \alpha)$, we have by (3.4 ii)

$$\|y - \bar{x}\| \le \|y - x\| + \|x - \bar{x}\| \le O(\|g^\sharp(x)\|) + \|x - \bar{x}\| = O(\|x - \bar{x}\|) = O(\alpha).$$

Reducing α if necessary, we obtain $y \in B(\bar{x}, \alpha_0)$. It remains to prove the second part of (4.6). We have, as f is of class C^1,

$$f(y) = f(x) + f'(\bar{x})(y - x) + \rho_1(x, y) \tag{4.7}$$

where by $\rho_i(x, y)$ we mean some functions such that $\rho_i(x, y)/\|y - x\| \to 0$ when $x \to \bar{x}$ and $y \to \bar{x}$. By the optimality system (1.1), we have

$$\begin{aligned} f'(\bar{x})(y - x) &= -\sum_{i=1}^{p} \bar{\lambda}_i g_i'(\bar{x})(y - x) = \sum_{i=1}^{p} \bar{\lambda}_i (g_i(x) - g_i(y)) + \rho_2(x, y) \\ &= \bar{\lambda}^t (g(x) - g(y)) + \rho_2(x, y) \\ &= \bar{\lambda}^t g(x)^\sharp + \bar{\lambda}^t (g(x) - g(x)^\sharp - g(y)) + \rho_2(x, y). \end{aligned}$$

Setting $\rho_3(x,y) := \rho_1(x,y) + \rho_2(x,y)$, we get with (4.7) and the definition of a dual norm

$$f(y) \le f(x) + \|\bar{\lambda}\|_* \|g(x)^\sharp\| + \bar{\lambda}^t(g(x) - g(x)^\sharp - g(y)) + \rho_3(x,y). \qquad (4.8)$$

By (3.4 iii) we have $g_j(x) - g_j(x)^\sharp = g_j(y), \quad j \in I \cup s(\bar{\lambda})$, so that

$$\bar{\lambda}_i(g_i(x) - g_i(x)^\sharp - g_i(y)) = 0, \ \forall i = 1, \cdots, p,$$

and finally with (4.8) $f(y) \le f(x) + \|\bar{\lambda}\|_* \|g(x)^\sharp\| + \rho_3(x,y)$. Given $r > \|\bar{\lambda}\|_*$, set $\varepsilon := r - \|\bar{\lambda}\|_*$. For α small enough, we have $\|y - x\| = O(\|g(x)^\sharp\|)$, hence $\|\rho_3(x,y)\| \le \frac{\varepsilon}{2}\|g(x)^\sharp\|$, so that

$$f(y) \le f(x) + \|\bar{\lambda}\|_* \|g(x)^\sharp\| + \frac{\varepsilon}{2}\|g(x)^\sharp\|,$$

which proves that (4.6) holds, with equality only if $\|g(x)^\sharp\| = 0$, i.e. x is feasible; in that case, $y = x$ so that the equality indeed always holds. ■

We recall that $\mathcal{L}(x,\lambda) := f(x) + \lambda^t g(x)$ is the ordinary Lagrangian. We now get the main result by considering the following perturbed problem and its local version

$$\min \mathcal{L}(x,\lambda) \ ; \ g(x) \ll 0. \qquad (P^\lambda)$$

$$\min \mathcal{L}(x,\lambda) \ ; \ g(x) \ll 0 \ ; \ x \in B(\bar{x},\alpha). \qquad (P^\lambda_\alpha)$$

We check in Proposition 4.2 below that, under the conditions of Theorem 1, if α is small enough, then (P_α) and (P^λ_α) have the same solutions. Theorem 1 follows by combining Propositions 4.1 and 4.2.

Proposition 4.2. If condition (2.3) holds, then for $\alpha > 0$ small enough, (P_α) and (P^λ_α) have the same solutions. In particular, $\bar{x}$ is a local solution of (P^λ) to which is associated the unique multiplier $\mu := \bar{\lambda} - \lambda$.

Proof. a) Let $\alpha_0 > 0$ be so small that $\bar{x}$ is a solution of (P_{α_0}), and choose $\alpha \in (0, \alpha_0)$. Fix λ satisfying (2.3). From the discussion in section 3, it follows that if α is small enough, there exists an operator σ that with x, feasible for (P^λ_α) associates y satisfying

$$g(y) \ll 0, \quad g_j(y) = 0, \ \forall j \in s(\bar{\lambda}), \quad y - x = O(\|g_{s(\bar{\lambda})}(x)\|). \qquad (4.9)$$

We will prove that, whenever α is small enough

$$\begin{array}{l} y \in B(\bar{x},\alpha_0) \text{ and } f(y) = \mathcal{L}(y,\lambda) \le \mathcal{L}(x,\lambda) \\ \text{with equality only if } y = x. \end{array} \qquad (4.10)$$

Indeed, for some $\tau \in (0,1)$ we have

$$\begin{aligned}\mathcal{L}(y,\lambda) - \mathcal{L}(x,\lambda) &= \mathcal{L}(y,\bar\lambda) - \mathcal{L}(x,\bar\lambda) + (\bar\lambda-\lambda)^t[g(x)-g(y)] \\ &= \mathcal{L}'_x(x+\tau(y-x),\bar\lambda)(y-x) + (\bar\lambda-\lambda)^t g(x) \\ &= [\mathcal{L}'_x(x+\tau(y-x),\bar\lambda) - \mathcal{L}'_x(\bar x,\bar\lambda)](y-x) + \\ &\quad + (\bar\lambda-\lambda)^t g(x) \\ &= \rho_4(x,y) + (\bar\lambda-\lambda)^t g(x),\end{aligned} \tag{4.11}$$

with $\|\rho_4(x,y)\|/\|x-y\| \to 0$ when $x \to \bar x$ and $y \to \bar y$. Fix $\varepsilon := \frac{1}{2}\min\{\bar\lambda_j - \lambda_j;\ j \in s(\lambda)\}$. By (2.3), we have $\varepsilon > 0$. Reducing α if necessary, as x is feasible for (P_α^λ) we obtain with the last relation in (4.9): $\rho_4(x,y) \le \varepsilon\|g_{s(\bar\lambda)}(x)\|_1$, where $\|y\|_1 := \sum_i |y_i|$. Then, as $g_j(x) \le 0$ for all $j \in s(\bar\lambda)$, we have with (4.11)

$$\mathcal{L}(y,\lambda) - \mathcal{L}(x,\lambda) \le \varepsilon\|g_{s(\bar\lambda)}(x)\|_1 + 2\varepsilon \sum_{s(\bar\lambda)} g_i(x) = -\varepsilon\|g_{s(\bar\lambda)}(x)\|_1,$$

i.e. $\mathcal{L}(y,\lambda) \le \mathcal{L}(x,\lambda)$, the equality being possible only if $g_{s(\bar\lambda)}(x) = 0$. Now

$$\|y-\bar x\| \le O(\|g_{s(\bar\lambda)}(x)\|) + \|x-\bar x\| = O(\|x-\bar x\|) = O(\alpha)$$

so that $\|y-\bar x\| < \alpha_0$ whenever α is small enough; (4.10) follows.

b) We have proved that a solution of (P_α^λ) is a solution to

$$\min f(x)\ ;\ g(x) \ll 0\ ;\ g_j(x) = 0, \quad j \in s(\lambda)\ ;\ x \in B(\bar x, \alpha_0). \tag{$\hat P_\alpha$}$$

We observe that, as $\bar\lambda$ is the unique Lagrange multiplier associated to $\bar x$, then reducing $\alpha > 0$ if necessary, it follows that to any solution $\hat x$ of (P_α) is associated at least a multiplier $\hat\lambda$ such that $\hat\lambda_j > 0$, $\forall j \in s(\bar\lambda)$, hence $g_{s(\bar\lambda)}(\hat x) = 0$, i.e. $\hat x$ is feasible for $(\hat P_\alpha)$. This proves that (P_α) and (P_α^λ) have the same set of solutions. In particular $\bar x$ is a solution of (P_α^λ), and it is easily checked that $\bar\lambda - \lambda$ is the unique Lagrange multiplier associated with $\bar x$. ■

5.- Relation with Perturbation Theory

The property of exact penalization is intimately related to lower estimates for the problem with perturbation in the right hand side:

$$\min f(x)\ ;\ g(x) \ll u. \tag{Q^u}$$

We define likewise the local perturbed problem

$$\min f(x)\ ;\ g(x) \ll u\ ;\ x \in B(\bar x, \alpha). \tag{Q_α^u}$$

This relation was first observed in Clarke [4], and developed by Burke [3], who obtained the following result (we give a proof for the sake of completeness).

Proposition 5.1. Assume that the norm used in θ_r satisfies the relation

$$z \ll u \Rightarrow \|z^\sharp\| \leq \|u^\sharp\|. \tag{5.12}$$

Let $\alpha > 0$ be such that $\bar{x}$ is a solution of (P_α). Then the penalty function $\theta_r(x)$ has a local minimum at $\bar{x}$ iff the function

$$u \to \inf(Q_\alpha^u) + r\|u^\sharp\| \tag{5.13}$$

has a local minimum at 0.

Proof. Relation (5.13) means that there exists $\beta > 0$ such that whenever $\|u\| \leq \beta$:

$$f(\bar{x}) \leq \min_{x \in B(\bar{x},\alpha)} \{f(x)\ ;\ g(x) \ll u\} + r\|u^\sharp\|,$$

i.e.

$$f(\bar{x}) = \min_{\substack{\|u\| \leq \beta \\ x \in B(\bar{x},\alpha)}} \{f(x) + r\|u^\sharp\|\ ;\ g(x) \ll u\}.$$

In particular, for $\alpha > 0$ small enough, $x \in B(\bar{x},\alpha)$ satisfies $\|g(x)^\sharp\| \leq \beta$, and we may take $u = g(x)^\sharp$. It follows that $f(\bar{x}) \leq f(x) + r\|g(x)^\sharp\|\ ;\ \forall x \in B(\bar{x},\alpha)$, i.e. $\theta_r(x)$ has a local minimum at $\bar{x}$.

Let us prove the converse implication. Given $\alpha > 0$ if $\|u\|$ is small enough, then Q_α^u is feasible, as a consequence of the results of [14] (note that as the multiplier is unique, the Mangasarian-Fromovitz qualification hypothesis holds at $\bar{x}$). Then Q_α^u has at least a solution x^u. Then using (5.12)

$$f(\bar{x}) \leq f(x^u) + r\|g(x^u)^\sharp\| = \min(Q^u)_\alpha + r\|g(x^u)^\sharp\| \leq \min(Q^u)_\alpha + r\|u^\sharp\|. \quad \blacksquare$$

We may likewise consider a perturbation of problem (P_α^λ) :

$$\min \mathcal{L}(x,\lambda)\ ;\ g(x) \ll u\ ;\ \|x - \bar{x}\| \leq \alpha. \tag{$Q_\alpha^{\lambda,u}$}$$

It is easily checked that, if λ satisfies (2.3) and α and $\|u\|$ are small enough, then (Q_α^u) and $(Q_\alpha^{\lambda,u})$ have the same solutions. Combining Theorem 1 and Proposition 5.1 we obtain the following lower estimate:

Theorem 2. *Assume that to $\bar{x}$, local solution of (P), is associated the unique multiplier $\bar{\lambda}$. Then, for all λ satisfying (2.3) and $r > \|\lambda - \bar{\lambda}\|_*$, the function $\min(Q_\alpha^u) + \lambda^t u + r\|u^\sharp\|$ has a local minimum at 0.*

We note that the lower estimate implies $\liminf_{t \searrow 0} \frac{1}{t}(\min(Q_\alpha^{tu}) - \min(P_\alpha)) \geq -\bar{\lambda}^t u$. This is a particular case of the results of Gauvin and Tolle [6].

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