A pathwise construction of Birth-Death-Swap systems leading to an averaging result in the presence of two timescales

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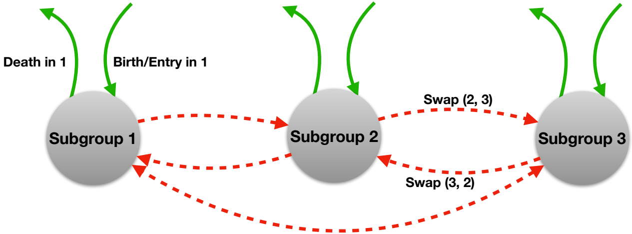
Systèmes de Birth-Death-Swap

Dynamique stochastique de population structuré en sous-groupes discrets (lieu d'habitation, habitudes alimentaires, stratégie...):

- **Population**: $Z_t = (Z_t^1, ..., Z_t^p) \in \mathbb{N}^p$.
- Composition de la population modifiée par les évènements démographiques et les swap (changement de caractéristiques).



- **Population agrégée** $Z_t^\natural = \sum_{i=1}^p Z_t^i$ = taille de la population.

Cadre classique: Processus de naissance et mort multi-types Makoviens.

1. Pas de changements de caractéristiques.
2. Intensités des évènements démographiques ne dépendent que de l'état de la population.

Généralisation : Systèmes de Birth-Death-Swap (BDS).

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   - ↳ Applications: écologie (Auger et al.), biologie (Billiard et al. (2017)), botnets en interaction (Song et al. (2010)).
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   - ↳ Prendre en compte la variabilité de l'environnement au cours du temps ⇒ Intensités stochastiques :

$$\mathrm{P}(\text{ ev de type } \gamma \in]t, t + \mathrm{d}t] | \mathcal{G}_t) = \mu^\gamma(\omega, t, Z_t)\mathrm{d}t.$$

Theoretical motivations

Questions

1. General conditions for existence of BDS systems?
2. How does heterogeneity/swap events impact the global population dynamics?

   ↳ Study of the aggregated population in the presence of two timescales (fast swap events).

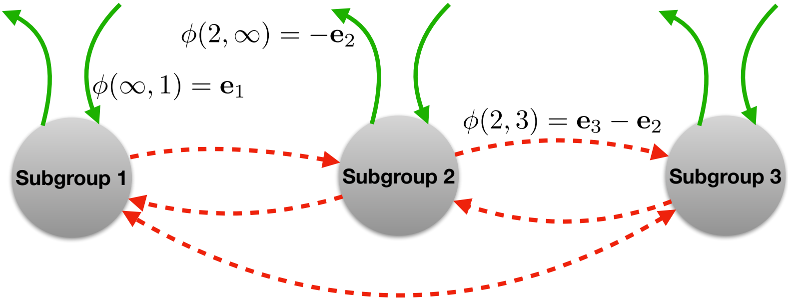
Stochastic intensities: classical results cannot be applied.

↳ Development of new tools for the study of BDS systems:

- Realization of point processes by strong domination.
- General averaging result by stable convergence.

Outline

Birth, Death and Swap events



Each event is characterized by a particular jump in the population:

- **Jumps function** $\phi : \mathcal{J}$ (set of all events) $\longrightarrow \mathbb{Z}^p$.
  - Swap event at t: $\Delta Z_t = \phi(i,j) = \mathbf{e}_j - \mathbf{e}_i$.
  - Birth event: $\Delta Z_t = \phi(\infty, i) = \mathbf{e}_i$, Death: $\Delta Z_t = \phi(i, \infty) = -\mathbf{e}_i$.

Jumps counting system

- Each type of event (birth, death, swap) $\gamma \in \mathcal{J}$ is associated with:

$$N_t^\gamma = \sum_{0<s\leqslant t} \mathbb{1}_{\{\Delta Z_s = \phi(\gamma)\}} \quad (1)$$

- **Assumption**:
  - $\forall \gamma \in \mathcal{J}$, N^γ has the $\mathcal{G}_t$-intensity $\boldsymbol{\mu}(\omega, t, Z_{t^-})$:

$$\mathrm{P}(N_{t+dt}^\gamma - N_t^\gamma = 1 | \mathcal{G}_t) \simeq \mu^\gamma(t, Z_t)dt,$$

  - $\boldsymbol{\mu}(t,z) = (\mu^\gamma(t,z))_{\gamma\in\mathcal{J}}$ a predictable functional.
- **Vector notation**: the multivariate counting process $\mathbf{N} = (N^\gamma)_{\gamma\in\mathcal{J}}$ has the $\mathcal{G}_t$-multivariate intensity $\boldsymbol{\mu}(t, Z_{t^-})$.

Birth Death Swap system (BDSs)

Examples:

- Birth intensity functional: $\mu^{b,i}(\omega, t, z) = b_t^i(\omega) z^i + \underbrace{\lambda^i(t, Y_t)}_{\text{immigration rate}}$
- Death intensity functional: $\mu^{d,i}(\omega, t, z) = d_t^i(\omega) z^i + \sum_{j \neq i} \underbrace{c(z^i, z^j)}_{\text{competition}}$

Population decomposition:

- Population process can be expressed as a linear function of **N**:

$$Z_t = Z_0 + \sum_{\gamma \in \mathcal{J}} \phi(\gamma) N_t^{\gamma} = Z_0 + \phi \odot \mathbf{N}_t.$$

Outline

Existence and uniqueness of BDS equation

- Driving $p(p+1)$ independent Poisson measures $\mathbf{Q} = (Q^\gamma)_{\gamma \in \mathcal{J}}$.
- **BDS multivariate differential system**:

$$\mathbf{N}_t = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leqslant \boldsymbol{\mu}(s, Z_{s-})\}} \mathbf{Q}(\mathrm{d}s, \mathrm{d}\theta), \quad Z_t = Z_0 + \phi \odot \mathbf{N}_t. \tag{2}$$

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- **Idea**: control birth part $\mathbf{N}^b$ of $\mathbf{N}$:

$$\boldsymbol{\mu}^b(\omega, t, z) \leqslant k_t \mathbf{g}(z^{\natural}), \tag{3}$$

with $(k_t) \in \mathcal{P}(\mathcal{G}_t)$ locally bounded and $\mathbf{g}$ verifying $\sum_{n \geqslant 1} \frac{1}{\sum g^i(n)} = \infty$.

Proposition

There exists a unique well-defined solution $\mathbf{N}$ *of* (2), **strongly dominated** *by a multivariate counting process* $\mathbf{G}$*:* $\mathbf{G} - \mathbf{N}$ *is a multivariate counting process.*

Outline

BDS systems with fast swap events

Hyp: intensity of swap events $\sim O(\frac{1}{\epsilon}) >> $ demographic events $\sim O(1)$.

The BDS system now depends on a small parameter ϵ:

$$Z_t^\epsilon = Z_0 + \phi^s \odot \mathbf{N}_t^{s,\epsilon} + \mathbf{N}_t^{b,\epsilon} - \mathbf{N}_t^{d,\epsilon}, \tag{4}$$

$$d\mathbf{N}_t^{s,\epsilon} = \mathbf{Q}^s(\mathrm{d}t, [0, \frac{1}{\epsilon}\boldsymbol{\mu}^s(t, Z_{t-}^\epsilon)]), \quad d\mathbf{N}_t^{\mathrm{dem},\epsilon} = \mathbf{Q}^{\mathrm{dem}}(\mathrm{d}t, [0, \boldsymbol{\mu}^{\mathrm{dem}}(t, Z_{t-}^\epsilon)]).$$

- $\mathbf{N}^{s,\epsilon}$ is a "fast" counting system of intensity functional $\frac{1}{\epsilon}\boldsymbol{\mu}^s(t, z)$: explosion when $\epsilon \to 0$.
- **But** $\mathbf{N}^{\mathrm{dem},\epsilon}$ only depends on ϵ through Z^ϵ and is strongly dominated by a multivariate counting process which doesn't depend on ϵ

$$\forall \epsilon > 0, \quad \mathbf{N}^{\mathrm{dem},\epsilon} \prec \mathbf{G}^{\mathrm{dem}}.$$

Demographic processes and tightness

- **Aggregated process**:

$$Z_t^{\natural,\epsilon} = Z_0^{\natural} + N_t^{b,\natural,\epsilon} - N_t^{d,\natural,\epsilon} = F(Z_0, \mathbf{N}_t^{\mathrm{dem},\epsilon})$$

- **Strong domination** $\Rightarrow$ $(\mathbf{N}^{\mathrm{dem},\epsilon})$ is tight in $\mathcal{A}^{2p} \subset D(\mathbb{R}^+, \mathbb{N}^{2p})$.

Identification of limit points of $(\mathbf{N}^{\mathrm{dem},\epsilon})$

- $\mathcal{G}_t$-local martingale $\mathbf{N}_t^{\mathrm{dem},\epsilon} - \int_0^t \boldsymbol{\mu}^{\mathrm{dem}}(\omega, s, Z_{s-}^{\epsilon})\mathrm{d}s$.
- Deterministic intensity functional (Markov framework) $\Rightarrow$ Averaging result of Kurtz (1992).
- Here: $\boldsymbol{\mu}^{\mathrm{dem}}(\omega, t, z)$. Need convergence of random functionals preserving martingale properties $\Rightarrow$ **Stable convergence**

↳ Averaging result for stable limits of $\mathbf{N}^{\mathrm{dem},\epsilon}$.

Application: deterministic swap intensities

Particular case: deterministic swap intensity function $\boldsymbol{\mu}^s(z)$.

$$Z_t^\epsilon = Z_0 + \phi^s \odot \mathbf{N}_t^{s,\epsilon} + \mathbf{N}_t^{b,\epsilon} - \mathbf{N}_t^{d,\epsilon},$$

$$d\mathbf{N}_t^{s,\epsilon} = \mathbf{Q}^s(dt,]0, \frac{1}{\epsilon}\boldsymbol{\mu}^s(Z_{t-}^\epsilon)]), \quad d\mathbf{N}_t^{\mathrm{dem},\epsilon} = \mathbf{Q}^{\mathrm{dem}}(dt,]0, \boldsymbol{\mu}^{\mathrm{dem}}(\omega, t, Z_{t-}^\epsilon)]).$$

- **Pure swap processes** X of $\mathcal{G}_t$-intensity $\boldsymbol{\mu}^s$:
  - Population with NO demographic events.
  - Constant size: $X_0^\natural = d \Rightarrow X_t = d$ ($X \in \mathcal{U}_d$, populations of size d).
- Deterministic intensity $\boldsymbol{\mu}^s \Rightarrow X$ continuous time Markov chain.

Assumption: $\forall d \in \mathbb{N}$, the swap process restricted to $\mathcal{U}_d$ admits a unique stationary distribution $(\pi(d, \mathrm{d}x))_{x \in \mathcal{U}_d}$.

Convergence of the demographic system

Hyp: pure swap on $\mathcal{U}_d$ admits a unique stationary distribution $(\pi(d, \mathrm{d}x))_{x \in \mathcal{U}_d}$.

- Aggregated process $(Z^{\epsilon,\natural})$ birth and death intensities:

$$\mu^{b,\natural}(t, Z_t) = \sum_{i=1}^{p} \mu^{b,i}(t, Z_t), \quad \mu^{d,\natural}(t, Z_t) = \sum_{i=1}^{p} \mu^{d,i}(t, Z_t)$$

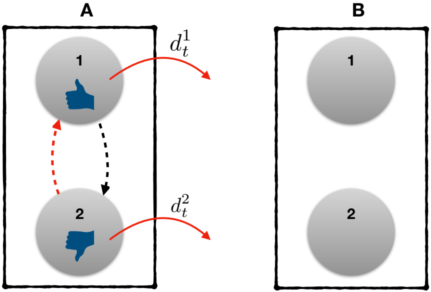
Theorem

The **aggregated processes** $Z^{\epsilon,\natural}$ *converge to the true Birth-Death process* $\bar{Z}^{\natural}$ *of intensity:*

$$\lambda^{b}(t, \bar{Z}^{\natural}_t) = \int_{\mathcal{U}_{\bar{Z}^{\natural}_t}} \mu^{b,\natural}(t, z)\pi(\bar{Z}^{\natural}_t, \mathrm{d}z), \quad \lambda^{d}(t, \bar{Z}^{\natural}_t) = \int_{\mathcal{U}_{\bar{Z}^{\natural}_t}} \mu^{d,\natural}(t, z)\pi(\bar{Z}^{\natural}_t, \mathrm{d}z).$$

- **Averaging effect**: aggregated intensities depend non-linearly of the number of individuals in the population.

A toy example (I)

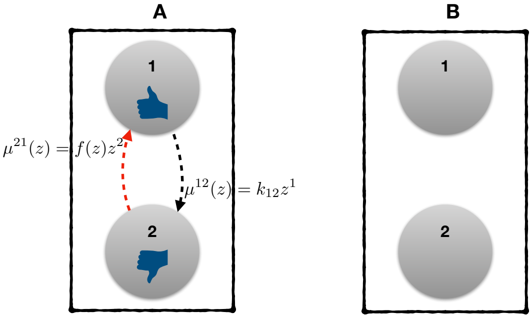


- Linear death functionals: $\mu^{d,i}(t, Z_t) = d_t^i Z_t^i$, $d_t^1 \leqslant d_t^2$

  (Aggregated death intensity) $\mu^{d,\natural}(t, Z_t) = d_t^1 Z_t^1 + d_t^2 Z_t^2$.

- If $Z_t^\natural = n$, individual death rate is $\frac{\mu^{d,\natural}(t,Z_t)}{n}$.

A toy example (II)



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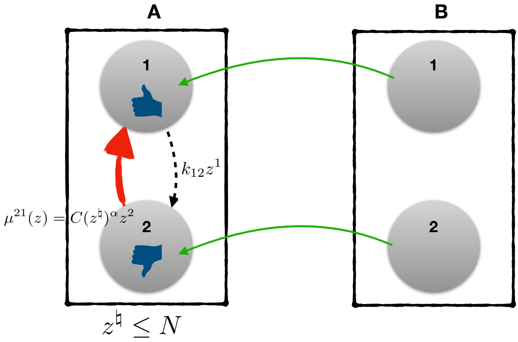

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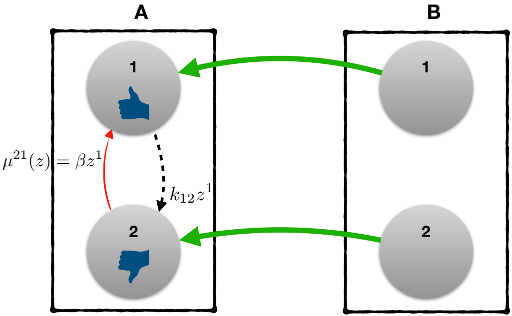

$z^\natural > N$

A toy example (II)

- Individual death rate in the limit aggregated population:
$$d(t, \bar{Z}_t^\natural) = \frac{1}{\bar{Z}_t^\natural} \lambda^d(t, \bar{Z}_t^\natural) = \frac{1}{\bar{Z}_t^\natural}(d_t^1 \pi(n, z^1) + d_t^2 \pi(n, z^2)).$$
- Aggregated death rate depend non-linearly on the population size n:
  - Small population ($n \leqslant N$):
  $$d(t, n) = d_t^1 \frac{C n^\alpha}{k_{12} + C n^\alpha} + d_t^2 \frac{k_{12}}{k_{12} + C n^\alpha}$$
  - Large population ($n > N$) :
  $$d(t, n) = d_t^1 \frac{\beta}{\beta + k_{12}} + d_t^2 \frac{k_{12}}{k_{12} + \beta}$$

Thank You For Your Attention