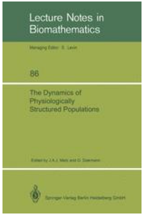
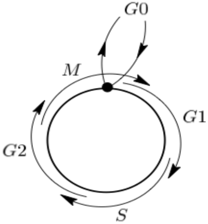
Structured eq. in biology

entropy and Monge-Kantorovich distance

Benoît Perthame





Outline

1. Motivations

2. Generalised relative entropy

3. Monge-Kantorovich distance and PDEs

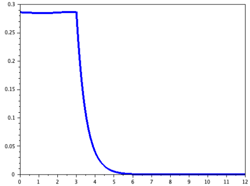
4. Sexual reproduction

The renewal equation

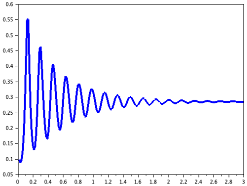
$$
\begin{cases}
\frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \quad t \geq 0,\ x \geq 0 \\
N(t) := n(t, x=0) = \int_0^\infty b(y) n(t,y) dy \\
n(t=0, x) = n^{init}(x)
\end{cases}
$$

$$b,\ d \in L^\infty_+(0,\infty).$$

- Very useful (demography, cell cycle, anomalous diffusions)
- Very standard (Feller, Kermack-McKendrick)
- Many extensions



Left : $b = d$



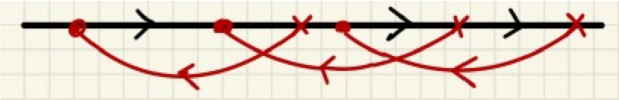
Right : $N(t)$

Growth-fragmentation/TCP

Other useful equation

$$\begin{cases} \frac{\partial n(t,x)}{\partial t} + \frac{\partial [g(x)n(t,x)]}{\partial x} + b(x)n(t,x) = k \int_x^{\infty} b(y)\kappa(x,y)n(t,y)dy \\ n(t, x=0) = 0, \qquad g(0) > 0 \end{cases}$$

- $b(y) =$ is the division rate of cells $(k = 2)$,
  of polymers $(k \geq 2)$,
  of messages of sizes $(k = 1)$,
- $\kappa(x,y) = 0$ for $x > y$ $(x =$ size after division$)$



Remark : The renewal equation : jumps are to 0

Growth-fragmentation/TCP

Other useful equation

$$\begin{cases} \frac{\partial n(t,x)}{\partial t} + \frac{\partial [g(x)n(t,x)]}{\partial x} + b(x)n(t,x) = k \int_x^\infty b(y)\kappa(x,y)n(t,y)dy \\ n(t,x=0)=0, \qquad g(0)>0 \end{cases}$$

- $b(y) =$ is the division rate of cells/polymers/messages of sizes y
- $\kappa(x,y)=0 \qquad$ for $y>x \qquad (x =$ size after division$)$

- $\frac{d}{dt}\int_0^\infty n(t,x)dx = (k-1)\int_0^\infty b(x)n(t,x)dx$
- $\frac{d}{dt}\int_0^\infty xn(t,x)dx = \int_0^\infty g(x)n(t,x)dx.$

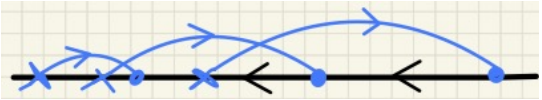
Assuming : $\int_0^y \kappa(x,y)dx = 1, \qquad k\int_0^y x\kappa(x,y)dx = y$

Many other structured equations

Immunity distribution by waning and boosting

$$\frac{\partial n(t,x)}{\partial t} - \frac{\partial [xn(t,x)]}{\partial x} + bI(t)n(t,x) = bI(t)\int_0^x \kappa(x,y)n(t,y)dy$$

- $n(t,x)$ = population density with immune level x
- $I(t)$ = infected population
- b = encounter rate



- 0. Diekmann, Rost-Barbarossa, Heffernan-Keeling

Many other structured equations

Age and size structured

$$\begin{cases} \frac{\partial n(t,x,z)}{\partial t} + \frac{\partial n(t,x,z)}{\partial x} + \frac{\partial [g(z)\, n]}{\partial z} + d(x,z)\, n(t,x,z) = 0, \quad x > 0,\ z > 0, \\ n(t,x,z=0) = 0, \\ n(t,x=0,z) = \int_{x=0}^{\infty} \int_{z'=z}^{\infty} d(x',z')\kappa(z,z')n(t,x',z')dx'dz \end{cases}$$

- Doumic, M. ; Hoffmann, M. ; Krell, N. ; Robert, L. et al (2015)
- D. Chafai, F. Malrieu, and K. Paroux (2010)
- J. Bertoin and A. R. Watson, (2020)
- J.-B. Bardet, A. Christen, A. Guillin, F. Malrieu, and P.-A. Zitt (2013)

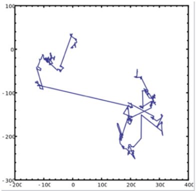
Many other structured equations

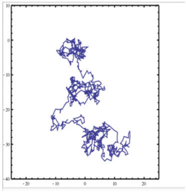
Age and space structured (subdiffusions)

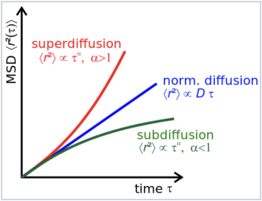
- Berry, H. ; Lepoutre, T. ; González, A. ; *Acta Appl. Math. (2016)*
- Calvez, V. ; Gabriel, P. ; Mateos G. ; *Asymptot. Anal. (2019)*
- Franck M. ; Goudon T. ; *KRM (2018)*, • Min Tang et al ; *Phys Rev. Res. (2022)*

$$\begin{cases} \frac{\partial n(t,x,z)}{\partial t} + \frac{\partial n(t,x,z)}{\partial x} + d(x)n(t,x,z) = 0, \quad x > 0,\ z \in \mathbb{R}^d \\ n(t,x=0,z) = \int_0^\infty \int_{\mathbb{R}^d} d(x)n(t,x,z-\varepsilon\eta)\kappa(\eta)dxd\eta \end{cases}$$

- Different limits as $\varepsilon \to 0$







$$D_t^\alpha \rho = \Delta\rho \qquad \text{subdiffusion eq.}$$

Many other structured equations

The multi-time renewal equation

$$\begin{cases} \frac{\partial}{\partial t} n(t, x_1, ..., x_N) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} n(t, X_N) + d(X_N) n(t, X_N) = 0 \\ n(t, x_1 = 0, x_2, ..., x_N) = \int_0^\infty dn(t, x_2, .., x_N, y) dy \\ n(t, x_1, x_2, ...x_i, ..., x_N) = 0, \qquad i = 2, ...N \end{cases}$$

- Used to evaluate efficiency of the Covid tracing Apps
- Spikes in neuroscience
- Non-Markovian processes

And systems of such eq.

Many other structured equations

They share the properties

- non necessarily conservative
- solutions are non-negative
- not necessarily compact
- not self-adjoint

Outline

1. Motivations

2. Generalised relative entropy

3. Monge-Kantorovich distance and PDEs

4. Sexual reproduction

Generalized relative entropy

All (linear) equations preserving positivity satisfy the GRE when there are

$$\mathcal{L}\varphi + \lambda_0 \varphi = 0, \quad \varphi > 0, \qquad \mathcal{L}^* \psi + \lambda_0 \psi = 0, \quad \psi > 0,$$

and

$$\int \varphi \psi = 1.$$

For the renewal equation it is e,nough that

$$\int b(x) e^{-\int_0^x d} dx > 1$$

or

$$\int b(x) e^{-\int_0^x d} dx = 1 \quad \text{and} \int e^{-\int_0^x d} < \infty$$

Generalized relative entropy

All (linear) equations preserving positivity satisfy the GRE

$$\mathcal{L}\varphi + \lambda_0 \varphi = 0, \quad \varphi > 0, \qquad \mathcal{L}^* \psi + \lambda_0 \psi = 0, \quad \psi > 0,$$

Let

$$\frac{\partial n(t)}{\partial t} + \mathcal{L} n(t,x) = 0.$$

GRE Principle. For all $H(\cdot)$ convex, $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\varphi(x)}$ satisfies

$$\frac{d}{dt} \int \psi(x) \varphi(x) H\big(u(t,x)\big) dx = -D_H(t) \leq 0$$

- $\psi \varphi H\big(u(t,x)\big)$ is a subsolution (P. Michel, S. Mischler, BP)
- **Loskot and Rudnicki (1991) :** abstract semi-group inequality
- The important information is in D_H

Generalized relative entropy

All (linear) equations preserving positivity satisfy the GRE

$$\mathcal{L}\varphi + \lambda_0 \varphi = 0, \quad \varphi > 0, \qquad \mathcal{L}^* \psi + \lambda_0 \psi = 0, \quad \psi > 0,$$

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$$\frac{d}{dt} \int \psi(x) \varphi(x) H\big(u(t,x)\big)\, dx = -D_H(t) \le 0$$

Why is it useful ?

- Various estimates as in Perron-Frobenius theory
- Convergence to steady state (Poincaré inequality)

Generalized relative entropy

$$\mathcal{L}\varphi + \lambda_0 \varphi = 0, \quad \varphi > 0, \qquad \mathcal{L}^* \psi + \lambda_0 \psi = 0, \quad \psi > 0,$$

$$\frac{\partial n(t)}{\partial t} + \mathcal{L} n(t,x) = 0.$$

GRE Principle. For all $H(\cdot)$ convex, $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\varphi(x)}$ satisfies

$$\frac{d}{dt} \int \psi(x) \varphi(x) H\big(u(t,x)\big)\, dx = -D_H(t) \le 0$$

- $\int \psi(x) n(t,x) dx = e^{\lambda_0 t} \int \psi(x) n^{init}(x) dx$ (Conservation law)

Take $H(u) = u$ and $H(u) = -u$

Generalized relative entropy

$$\mathcal{L}\varphi + \lambda_0\varphi = 0, \quad \varphi > 0, \qquad \mathcal{L}^*\psi + \lambda_0\psi = 0, \quad \psi > 0,$$

$$\frac{\partial n(t)}{\partial t} + \mathcal{L}n(t,x) = 0.$$

GRE Principle. For all $H(\cdot)$ convex, $u(t,x) = e^{-\lambda_0 t}\frac{n(t,x)}{\varphi(x)}$ satisfies

$$\frac{d}{dt}\int \psi(x)\varphi(x)H\big(u(t,x)\big)\,dx = -D_H(t) \le 0$$

- $\int \psi(x)n(t,x)dx = e^{\lambda_0 t}\int \psi(x)n^{init}(x)dx$ (Conservation law)
- $\int \psi(x)|n(t,x)|dx \le e^{\lambda_0 t}\int \psi(x)|n^{init}(x)|dx$ (L^1 contraction)

Take $H(u) = |u|$

Generalized relative entropy

$$\mathcal{L}\varphi + \lambda_0 \varphi = 0, \quad \varphi > 0, \qquad \mathcal{L}^* \psi + \lambda_0 \psi = 0, \quad \psi > 0,$$

$$\frac{\partial n(t)}{\partial t} + \mathcal{L} n(t,x) = 0.$$

GRE Principle. For all $H(\cdot)$ convex, $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\varphi(x)}$ satisfies

$$\frac{d}{dt} \int \psi(x) \varphi(x) H\big(u(t,x)\big)\, dx = -D_H(t) \le 0$$

- $\int \psi(x) n(t,x) dx = e^{\lambda_0 t} \int \psi(x) n^{init}(x) dx$ (Conservation law)
- $\int \psi(x) |n(t,x)| dx \le e^{\lambda_0 t} \int \psi(x) |n^{init}(x)| dx$ (L^1 contraction)
- $u^{init} \le C^{init} \Longrightarrow u(t,x) \le C^{init}$ (Max. principle)

Take $H(u) = (u - C^{init})_+^2$

Generalized relative entropy

$$\mathcal{L}\varphi + \lambda_0\varphi = 0, \quad \varphi > 0, \qquad \mathcal{L}^*\psi + \lambda_0\psi = 0, \quad \psi > 0,$$

$$\frac{\partial n(t)}{\partial t} + \mathcal{L}n(t,x) = 0.$$

GRE Principle. For all $H(\cdot)$ convex, $u(t,x) = e^{-\lambda_0 t}\frac{n(t,x)}{\varphi(x)}$ satisfies

$$\frac{d}{dt}\int \psi(x)\varphi(x)H\big(u(t,x)\big)\,dx = -D_H(t) \le 0$$

- $\int \psi(x)n(t,x)dx = e^{\lambda_0 t}\int \psi(x)n^{init}(x)dx$ (Conservation law)
- $\int \psi(x)|n(t,x)|dx \le e^{\lambda_0 t}\int \psi(x)|n^{init}(x)|dx$ (L^1 contraction))
- $u^{init} \le C^{init} \Longrightarrow u(t,x) \le C^{init}$ (Max. principle)
- $\partial_t \ln n(t,x) \le \max_x -\mathcal{L}n^{init}(x)$ (for coef. independent of time)

Because $\partial_t n$ satisfies the same equation

Generalized relative entropy

More generally, one can compare solutions

$$\frac{\partial n(t)}{\partial t} + \mathcal{L} n(t,x) = 0.$$

GRE Principle. For all $H(\cdot)$ convex, $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\Phi(t,x)}$ satisfies

$$\frac{d}{dt} \int \Psi(t,x)\Phi(x) H\big(u(t,x)\big)\, dx = -D_H(t) \leq 0$$

Whenever

$$\frac{\partial \Phi(t)}{\partial t} + \mathcal{L}\Phi(t,x) = 0$$

$$\frac{\partial \Psi(t)}{\partial t} + \mathcal{L}^* \Psi(t,x) = 0$$

Example. Periodic coeficients and λ_0 is replaced by the Floquet eigenvalue

Generalized relative entropy

Examples. $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\varphi(x)}$

Fokker-Planck eq.

$$\frac{\partial n(t,x)}{\partial t} - \Delta n + \mathrm{div}(n \nabla V) = 0$$
$$\lambda_0 = 0, \quad \psi = 1, \quad \varphi = e^V,$$

conservative pb. have just relative entropy

$$\frac{d}{dt} \int \frac{n(t,x)^2}{\varphi} dx = -D_2$$

$$D_H = \int \varphi H''(u) |\nabla u|^2$$

Generalized relative entropy

Examples. $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\varphi(x)}$

Fokker-Planck eq.

$$\frac{\partial n(t,x)}{\partial t} - \Delta n - \mathrm{div}(n \nabla V) = 0$$

$$\lambda_0 = 0, \quad \psi = 1, \quad \varphi = e^{-V},$$

conservative pb. have just relative entropy

$$\frac{d}{dt} \int \frac{n(t,x)^2}{\varphi} dx = -D_2, \qquad D_H = \int \varphi H''(u) |\nabla u|^2$$

The Poincaré inequality : When $D^2 V \geq \nu Id$, then

$$\nu \int u(x)^2 \varphi dx \leq \int |\nabla u|^2 \varphi dx, \qquad \forall u \ s.t. \int u \varphi dx = 0$$

Generalized relative entropy

Examples : Renewal eq. $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\varphi(x)}$

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \\ N(t) := n(t, x=0) = \int_0^\infty b(y) n(t,y) dy \end{cases}$$

$$\frac{\partial}{\partial x} \varphi(x) + [d(x) + \lambda_0] \varphi(x) = 0$$

$$-\frac{\partial}{\partial x} \frac{1}{\varphi(x)} + [d(x) + \lambda_0] \frac{1}{\varphi(x)} = 0$$

Note that the adjoint eq. is

$$-\frac{\partial}{\partial x} \psi(x) + [d(x) + \lambda_0] \psi(x) = \psi(0) b(x)$$

Generalized relative entropy

Examples : Renewal eq. $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\varphi(x)}$

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \\ N(t) := n(t, x=0) = \int_0^\infty b(y) n(t,y) dy \end{cases}$$

$$\frac{\partial}{\partial x}\varphi(x) + [d(x) + \lambda_0]\varphi(x) = 0$$

$$-\frac{\partial}{\partial x}\frac{1}{\varphi(x)} + [d(x) + \lambda_0]\frac{1}{\varphi(x)} = 0$$

$$\frac{\partial}{\partial t} u(t,x) + \frac{\partial}{\partial x} u(t,x) = 0$$

$$\frac{\partial}{\partial t}\varphi H(u(t,x)) + \frac{\partial}{\partial x}\varphi H(u(t,x)) + [d(x) + \lambda_0]\varphi H(u(t,x)) = 0$$

Generalized relative entropy

Examples : Renewal eq. $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\varphi(x)}$

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \\ N(t) := n(t, x=0) = \int_0^\infty b(y) n(t,y) dy \end{cases}$$

$$\frac{\partial}{\partial t} \varphi H(u(t,x)) + \frac{\partial}{\partial x} \varphi H(u(t,x)) + [d(x) + \lambda_0] \varphi H(u(t,x)) = 0$$

$$-\frac{\partial}{\partial x} \psi(x) + [d(x) + \lambda_0] \psi(x) = \psi(0) b(x)$$

$$\frac{\partial}{\partial t} \psi\varphi H(u(t,x)) + \frac{\partial}{\partial x} \psi\varphi H(u(t,x)) = -\psi(0) b(x) \varphi H(u(t,x))$$

Generalized relative entropy

Examples : Renewal eq. $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\varphi(x)}$

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \\ N(t) := n(t, x=0) = \int_0^\infty b(y) n(t,y) dy \end{cases}$$

$$\frac{\partial}{\partial t} \psi\varphi H(u(t,x)) + \frac{\partial}{\partial x} \psi\varphi H(u(t,x)) = -\psi(0) b(x) \varphi H(u(t,x))$$

$$\frac{d}{dt} \int_0^\infty \psi\varphi H(u(t,x)) = \psi(0) \big[\varphi(0) H(u(t,0)) - \int_0^\infty b(x) \varphi H(u(t,x))\big]$$

Normalise $\varphi(0) = 1 = \int_0^\infty b\varphi$ to simplify and

$$\frac{d}{dt} \int_0^\infty \psi\varphi H(u(t,x)) \quad = \psi(0) \big[H(\textstyle\int b\varphi u) - \int \underbrace{b\varphi}_{\text{proba.}} H(u)\big]$$

$$\leq 0 \qquad \text{(Jensen's inequality)}$$

Generalized relative entropy

Examples : Growth-fragmentation eq.

The spectral theory is not so easy (see M. Doumic, P. Gabriel)

$$\frac{\partial n(t,x)}{\partial t} + \frac{\partial [g(x)n(t,x)]}{\partial x} + b(x)n(t,x) = 2\int_x^\infty b(y)\kappa(x,y)n(t,y)dy$$

$$D_H = 2\iint \overbrace{\psi(x)\varphi(y)b(y)\kappa(x,y)}^{\text{possibly degenerate}}$$

$$\Big[H(u(y)) - H(u(x)) - H'(u(x))(u(y)-u(x))\Big]\,dxdy$$

Generalized relative entropy

A general consequence of GRE is

'Theorem' With a non-degeneracy condition*, it holds

$$\|e^{-\lambda_0 t} n(t,x) - \rho^{init}\varphi\| \to 0 \qquad \text{as } t \to \infty.$$

$$\rho^{init} := \frac{\int n^{init}(x)\psi(x)dx}{\int \varphi(x)\psi(x)dx}$$

* When the entropy dissipation vanishes and eq. holds, we should have $u = 1$

Generalized relative entropy

Spectral gap. Next step is to know if there is $\lambda_1 > 0$ such that

$$\|e^{-\lambda_0 t} n(t,x) - \rho^{init}\varphi\| \leq Ce^{-\lambda_1 t} \; ?$$

By linearity, we work with $u = \frac{ne^{-\lambda_0 t}}{\varphi} - \rho^{init}$

- Poincaré inequality : when $\int \psi\varphi u(x) = 0$

$$\lambda_1 \int \psi\varphi H(u)dx \leq D_H(u)$$

If it holds, then

$$\frac{d}{dt}\int \psi(x)\varphi(x)H\big(u(t,x)\big)dx = -D_H(t) \leq -\lambda_1 \int \psi\varphi H(u)dx$$

Generalized relative entropy

Spectral gap. Next step is to know if there is $\lambda_1 > 0$ such that

$$\|e^{-\lambda_0 t} n(t,x) - \rho^{init} \varphi\| \leq C e^{-\lambda_1 t} \ ?$$

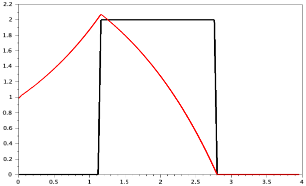
- Poincaré inequality : when $\int \psi(x)\varphi(x)u(x) = 0$

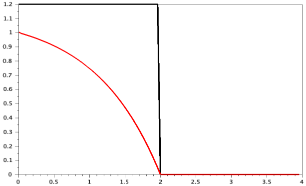
$$\lambda_1 \int \psi \varphi H(u) dx \leq D_H(u)$$

Theorem : Assume there is $\lambda_1 > 0$ such that

$$\lambda_1 \psi(x) \leq b(x)\psi(0)$$

then the Poincaré inequality holds


0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2 2.2
0 0.5 1 1.5 2 2.5 3 3.5 4


0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1 1.2
0 0.5 1 1.5 2 2.5 3 3.5 4

Generalized relative entropy

Theorem : Assume there is $\lambda_1 > 0$ such that

$$\lambda_1 \psi(x) \le b(x)\psi(0)$$

then the Poincaré inequality holds

$$\lambda_1 \int \psi\varphi H(u)dx \le D_H(u), \qquad \forall u \ s.t. \int \psi(x)\varphi(x)u(x) = 0$$

Proof : Set $\mu(x) = b(x)\varphi(x)$ (proba.) and $h(u) = |u|$

$$\int \mu(x)|u(t,x)|dx - \left|\int \mu(x)u(t,x)dx\right|$$

$$= \int \mu(x)|u(t,x)|dx - \left|\int \left[\mu(x) - \frac{\lambda_1}{\psi(0)}\varphi(x)\psi(x)\right] u(t,x)dx\right|$$

$$\ge \int \mu(x)|u(t,x)|dx - \int \underbrace{\left[\mu(x) - \frac{\lambda_1}{\psi(0)}\varphi(x)\psi(x)\right]}_{\ge 0} |u(t,x)|dx$$

$$= \frac{\lambda_1}{\psi(0)} \int |u(t,x)|\varphi(x)\psi(x)dx.$$

Generalized relative entropy

Spectral gap. Aim is to find $\lambda_1 > 0$ such that

$$\|e^{-\lambda_0 t} n(t,x) - \rho^{init} \varphi\| \leq C e^{-\lambda_1 t}$$

- Poincaré inequality : when $\int \psi \varphi u(x) = 0$

$$\lambda_1 \int \psi \varphi H(u) dx \leq D_H(u)$$

- Doeblin's method
- Method of integral equation

(Ryzhik-BP, Doumic, Hairer-Mattingly, Gabriel, Mischler, Cañizo-Yoldas, Laurencot, Salort...)

- **Very useful** : Allows to prove global stability for 'small' nonlinearities

Conclusion

In a model, all is perfect.

It is in reality that everything falls apart

Italo Calvino

Conclusion

For a non-degenerate equation as

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \quad x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty b(y) n(t,y) dy \end{cases}$$

with eigenelements

$$\mathcal{L}\varphi + \lambda_0 \varphi = 0, \qquad \mathcal{L}^* \psi + \lambda_0 \psi = 0, \quad \varphi, \ \psi > 0, \quad \int \varphi \psi = 1$$

GRE Principle : For all $H(\cdot)$ convex, $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\varphi(x)}$ satisfies

$$\frac{d}{dt} \int \psi(x) \varphi(x) H\big(u(t,x)\big) dx = -D_H(t) \leq 0$$

Conclusion

With eigenelements

$$\mathcal{L}\varphi + \lambda_0 \varphi = 0, \qquad \mathcal{L}^* \psi + \lambda_0 \psi = 0, \quad \varphi,\ \psi > 0, \quad \int \varphi \psi = 1$$

GRE Principle : For all $H(\cdot)$ convex, $u(t,x) = e^{-\lambda_0 t} \frac{n(t,x)}{\varphi(x)}$ satisfies

$$\frac{d}{dt} \int \psi(x)\varphi(x) H\big(u(t,x)\big)\, dx = -D_H(t) \leq 0$$

A priori estimates : $u(t,x) \leq C$

Large time convergence : $\|n(t)e^{-\lambda_0 t} - \rho^{ini}\varphi\| \to 0$

When $\varphi \notin L^1$

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \quad x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty d(y) n(t,y) dy \end{cases}$$

$$b(x) = d(x) = \frac{\alpha}{1+x}$$

$$\frac{\partial}{\partial x} \varphi(x) + d(x)\varphi(x) = 0,$$

$$\varphi(x) = \frac{1}{(1+x)^\alpha}, \qquad \lambda_0 = 0, \qquad \psi = 1$$

For $0 < \alpha < 1$

$$\int \varphi(x) dx = \infty$$

When $\varphi \notin L^1$

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \quad x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty d(x) n(t,y) dy \end{cases}$$

$$b(x) = d(x) = \frac{\alpha}{1+x}$$

$$\varphi(x) = \frac{1}{(1+x)^\alpha}, \qquad \lambda_0 = 0, \qquad \psi = 1$$

$$\int n(t,x) dx = 1$$

Theorem For $\alpha < 1$, we have

$$N(t) \approx O(t^{-1+\alpha})$$

Similar to the heat equation in $\mathbb{R}^d$, $n(t,x)$ spreads in $(0,\infty)$

When $\varphi \notin L^1$

$$\frac{\partial}{\partial t} n(t,x) - \Delta n = 0,$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} n^2(t,x)dx = -\int_{\mathbb{R}^d} |\nabla n|^2 dx$$

Nash inequality : for $\theta = \frac{2}{d+2}$

$$\int_{\mathbb{R}^d} n^2(t,x)dx \le C(d)\Big(\int_{\mathbb{R}^d} |\nabla n|^2 dx\Big)^{1-\theta}\Big(\int_{\mathbb{R}^d} n dx\Big)^{\theta}$$

Therefore, one concludes

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} n^2(t,x)dx \le -C(d,M)\Big(\int_{\mathbb{R}^d} n^2 dx\Big)^{\frac{1}{\theta}}$$

When $\varphi \notin L^1$

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \quad x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty d(x) n(t,y) dy \end{cases}$$

$$d(x) = \frac{\alpha}{1+x}, \qquad \varphi(x) = \frac{1}{(1+x)^\alpha}$$

Theorem : For $\alpha < 1$, we have $N(t) \approx O(t^{-1+\alpha})$

By the entropy method, with $\int_0^\infty d(x)\varphi(x) = 1$

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty \varphi\Big(\frac{n}{\varphi}\Big)^2 = -\int_0^\infty d(x)\varphi(x)\frac{n^2}{\varphi^2} + \Big(\int_0^\infty d(x)\varphi(x)\frac{n}{\varphi}\Big)^2$$

$$\leq -\nu \left(\int_0^\infty \varphi\Big(\frac{n}{\varphi}\Big)^2\right)^q \qquad ???$$

When $\varphi \notin L^1$

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \quad x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty d(x) n(t,y) dy \end{cases}$$

$$d(x) = \frac{\alpha}{1+x}, \qquad \varphi(x) = \frac{1}{(1+x)^\alpha}$$

Theorem : For $\alpha < 1$, we have $N(t) \approx O(t^{-1+\alpha})$

Proof : Laplace transform : $D(x) = \int_0^x d$

$$\frac{\partial}{\partial x} \widehat{n}(s,x) + [s + d(x)] \widehat{n}(t,x) = \delta(s) n^0(x)$$

$$\widehat{n}(s,x) = \widehat{N}(s) e^{-(s+D(x))} + \text{Initial}(s)$$

$$\widehat{N}(s) = \widehat{N}(s) \int_0^\infty d(x) e^{-(s+D(x))} + \text{Initial}(s)$$

$$\int_0^t \frac{N(\tau)}{(t-\tau)^\alpha} d\tau = \text{Initial}(t)$$

When $\varphi \notin L^1$

$$\begin{cases} \varepsilon^\beta \frac{\partial}{\partial t} n(t,x,z) + \frac{\partial}{\partial x} n(t,x,z) + d(x) n(t,x,z) = 0, \quad x \geq 0,\ z \in \mathbb{R}^d \\ N(t) := n(t, x=0, z) = \int_0^\infty d(x) n(t,y,z-\varepsilon\eta)\kappa(\eta) dy d\eta \end{cases}$$

$$d(x) = \frac{\alpha}{1+x}$$

Theorem (BP, Min Tang) as $\varepsilon \to 0$, we have

(i) For $\alpha > 1,\ \beta = 2,\quad \rho(t,z) = \int n(t,x,z)dx$ satisfies the heat equation

(ii) For $\alpha < 1,\ \alpha\beta = 2,\quad \rho(t,z) = \int n(t,x,z)dx$ satisfies the *subdiffusion* eq.

$$\frac{\partial}{\partial t} \int_0^t \frac{\rho(s,z)}{(s-t)^\alpha} ds = \Delta\rho(t,z) + \text{Initial}$$

Infinite times renewal equation

The multi-time renewal equation

$$\begin{cases} \frac{\partial}{\partial t} n(t, x_1, ..., x_N) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} n(t, X_N) + d(X_N) n(t, X_N) = 0 \\ n(t, x_1 = 0, x_2, ..., x_N) = \int_0^\infty dn(t, x_2, .., x_N, y) dy \end{cases}$$

- Used to evaluate efficiency of the Covid tracing Apps
- Spikes in neuroscience
- Non-Markovian processes

What happens when $N \to \infty$?

Meaning of the boundary condition

Infinite times renewal equation

$$\begin{cases} \frac{\partial}{\partial t} n_N(t, x_1, ..., x_N) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} n_N(t, X_N) + d_N(X_N) n_N(t, X_N) = 0 \\ n_N(t, x_1 = 0, x_2, ..., x_N) = \int_0^\infty d_N n_N(t, x_2, .., x_N, y) dy \end{cases}$$

Define and assume

- $X_N = (x_1, ..., x_N)$
- $d_N(X_N) = \sum_{i=1}^N r_i(x_1, ..., x_i), \qquad \sum_{i=1}^{\infty} \|r_i\|_\infty < \infty$
- $n_N^{(K)}(X_K) = \int n_N(X_N) dx_{k+1} ... dx_N$

Infinite times renewal equation

$$\begin{cases} \frac{\partial}{\partial t} n_N(t, x_1, ..., x_N) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} n_N(t, X_N) + d_N(X_N) n_N(t, X_N) = 0 \\ n_N(t, x_1 = 0, x_2, ..., x_N) = \int_0^{\infty} d_N n_N(t, x_2, .., x_N, y) dy \end{cases}$$

- $X_N = (x_1, ..., x_N)$
- $d_N(X_N) = \sum_{i=1}^{N} r_i(x_1, ..., x_i), \qquad \sum_{i=1}^{\infty} \|r_i\|_\infty < \infty$
- $n_N^{(K)}(X_K) = \int n_N(X_N) dx_{k+1} ... dx_N$

The marginals satisfy

$$\frac{\partial}{\partial t} n_N^{(K)}(t, x_1, ..., x_K) + \sum_{i=1}^{K} \frac{\partial}{\partial x_i} n_N^{(K)}(t, X_K) + d_K(X_K) n_N^{(K)}(t, X_K) = \varepsilon_N^{(K)} \leq \sum_{i=K}^{\infty} \|r_i\|_\infty < \infty$$

Infinite times renewal equation

$$\frac{\partial}{\partial t} n_N^{(K)}(t, x_1, ..., x_K) + \sum_{i=1}^{K} \frac{\partial}{\partial x_i} n_N^{(K)}(t, X_K) + d_K(X_K) n_N^{(K)}(t, X_K) = \varepsilon_N^{(K)} \leq \sum_{i=K}^{\infty} \|r_i\|_\infty < \infty$$

One can pass to the limit because one has a Cauchy sequence

$$\|n_M^{(K)} - n_N^{(K)}(t)\|_1 \leq \sum_{i=M}^{N} \|r_i\|_\infty < \infty \to 0$$

One gets

$$\frac{\partial}{\partial t} n_\infty^{(K)}(t, x_1, ..., x_K) + \sum_{i=1}^{K} \frac{\partial}{\partial x_i} n_\infty^{(K)}(t, X_K) + d_K(X_K) n_\infty^{(K)}(t, X_K) = \varepsilon^{(K)} \leq \sum_{i=K}^{\infty} \|r_i\|_\infty < \infty$$

Infinite times renewal equation

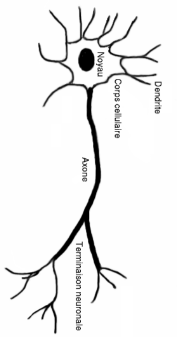
$$\begin{cases} \frac{\partial}{\partial t} n_\infty^{(K)}(t, x_1, ..., x_K) + \sum_{i=1}^{K} \frac{\partial}{\partial x_i} n_\infty^{(K)}(t, X_K) + d_K(X_K) n_\infty^{(K)}(t, X_K) = \varepsilon^{(K)} \to 0 \quad as\ K \to \infty \\ n_\infty^{(K)}(t, x_1 = 0, x_2, ..., x_K) = \int_0^\infty n_\infty^{(K)}(t, x_2, ..., x_K, y) dy \end{cases}$$

Theorem : For a 'consistent' initial data, the solution of this hierarchy is unique.

The nonlinear renewal equation

For instance : time elapsed eq. in neuroscience with K. Pakadman, D. Salort, C. Rieutord

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x, N(t)) n(t,x) = 0, \quad x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty d(y, N(t)) n(t,y) dy \end{cases}$$



x = time elapsed since the last discharge
$N(t)$ = activity of the network

$$\frac{\partial d(x,N)}{\partial N} > 0 \qquad \text{(Excitatory)}$$

$$\frac{\partial d(x,N)}{\partial N} < 0 \qquad \text{(Inhibitory)}$$

What should the solution do in these circumstances ?

The nonlinear renewal equation

In general nonlinearities appear in the form

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x, I_1(t)) n(t,x) = 0, \quad x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty b(y, I_2(t)) n(t,y) dy \\ I_j(t) = \int q_j(x) n(t,x) dx, \qquad j = 1,\ 2 \end{cases}$$

What should the solution do in these circumstances ?

The nonlinear renewal equation

In general nonlinearities appear in the form

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x, I_1(t))n(t,x) = 0, \quad x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty b(y, I_2(t))n(t,y)dy \\ I_j(t) = \int q(_j x)n(t,x)dx, \qquad j = 1,\ 2 \end{cases}$$

'Theorem' When there is a spectral gap and nonlinearity is small then

(i) there is a unique steady state $\overline{n}(x)$

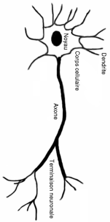
(ii) $\quad \|n(t,x) - \overline{n}(x)\|_{L^1(\psi)} \to 0$

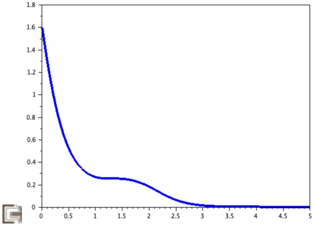
Easy for self-adjoint operators.

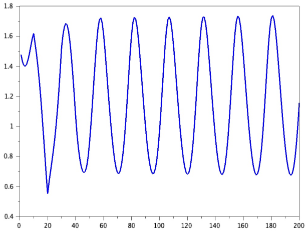
The nonlinear renewal equation

For large nonlinearities periodic solutions may appear

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \quad x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty b(y, I(t)) n(t,y) dy \\ I(t) = \int q(x) n(t,x) dx \end{cases}$$







Periodic solution : Left $n(t,x)$. Right : $N(t)$

The nonlinear renewal equation

For large nonlinearities periodic solutions may appear

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \quad x \geq 0 \\ N(t) := n(t, x=0) = \displaystyle\int_0^\infty b(y, I(t)) n(t,y) dy \\ I(t) = \int q(x) n(t,x) dx \end{cases}$$

Are there examples where global relaxation appears ?

The Kermack-McKendrick renewal eq.

The SI model of epidemiology was proposed in 1927 as :

$$\begin{cases} \frac{d}{dt}S(t) = B - \mu_S S(t) - I(t)S(t) \\ I(t) := \int_0^\infty \beta(s) n_I(t,s) ds \\ \frac{\partial}{\partial t} n_I(t,s) + \frac{\partial}{\partial s} n_I(t,s) + (\mu_I + \gamma(s)) n_I(t,s) = 0 \\ n_I(t, s=0) = I(t)S(t) \end{cases}$$

Theorem (Magal, McCluskey, Webb, 2010). Define

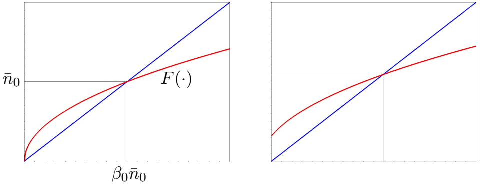
$$\mathcal{E}(t) = \int_0^\infty \psi(s)\bar{n}_I(s)\Big[\frac{n_I(t,s)}{\bar{n}_I(s)} - \ln\frac{n_I(t,s)}{\bar{n}_I(s)}\Big] ds + \bar{S}\ln S(t) - S(t).$$

Then, we have

$$\frac{d}{dt}\mathcal{E}(t) \le -D(t) \le 0, \qquad D(t) = \frac{\mu_S}{S(t)}(\bar{S} - S(t))^2.$$

GRE : concave nonlinear eq.

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x) n(t,x) = 0, \quad t \geq 0,\ x \geq 0, \\ n(t,0) = F\big(\int_0^\infty b(x) n(t,x) dx \big), \quad 0 < \beta_0 := \int_0^\infty b(x) e^{-D(x)} dx \end{cases}$$



Theorem (P. Michel) For this nonlinear equation

- There is a uniqueness steady state $\bar{n}(x)$
- $n^{init} \leq C_+ \bar{n}(x)$, then $\forall t \geq 0$, $n(t,x) \leq C_+ \bar{n}(x)$,
- $n^{init} \geq C_- \bar{n}(x)$, then $\forall t \geq 0$, $n(t,x) \geq C_- \bar{n}(x)$,
- $\int_0^\infty |n(t) - \bar{n}_0| \psi \leq e^{-at} \int_0^\infty |n^{init} - \bar{n}_0| \psi.$

See also Calsina, Diekmann, Farkas

The nonlinear renewal equation

Inhibitory time elapsed eq. in neuroscience with D. Salort, C. Rieutord

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x, N(t)) n(t,x) = 0, \quad x \geq 0 \\ N(t) := n(t, x=0) = \int_0^\infty d(y, N(t)) n(t,y) dy \end{cases}$$

Theorem Assume a non-degeneracy condition and the inhibitory condition

$$\frac{\partial d(x,N)}{\partial N} < 0.$$

Then there is L^1 non-expansion and $\|n(t,x) - \overline{n}(x)\|_{L^1(\psi)} \to 0$

Outline

1. Motivations
2. Generalised relative entropy
3. Monge-Kantorovich distance and PDEs
4. Sexual reproduction

Monge-Kantorovich distance

Restriction : Conservative problems (probability measures)

Ackleh, Carrillo, Gwiazda, Gabriel, Datong Zhou,...

Fournier, Locherbach, Monmarché, Guillin, Zitt, Eberle..

With Nicolas Fournier

Monge-Kantorovich distance

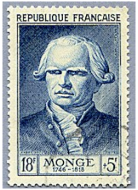
Motivations :

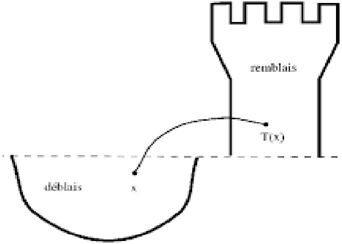
- measure solutions are possible
- weak topology vs strong topology (entropy)
- weak topology is useful (long time behaviour)
- approximation by particles ($N \to \infty$)
- Used in data science

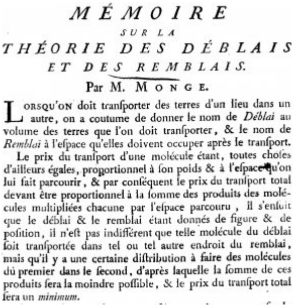
Theorem : For all the problems of yesterday, there is a choice of a such that

$$d_{MK,a}(n_1(t), n_2(t)) \leq d_{MK,a}(n_1(0), n_2(0))$$

Monge transport problem







MÉMOIRE SUR LA THÉORIE DES DÉBLAIS ET DES REMBLAIS.

Par M. MONGE.

Lorsqu'on doit tranſporter des terres d'un lieu dans un autre, on a coutume de donner le nom de *Déblai* au volume des terres que l'on doit tranſporter, & le nom de *Remblai* à l'eſpace qu'elles doivent occuper après le tranſport.

Le prix du tranſport d'une molécule étant, toutes choſes d'ailleurs égales, proportionnel à ſon poids & à l'eſpace qu'on lui fait parcourir, & par conſéquent le prix du tranſport total devant être proportionnel à la ſomme des produits des molécules multipliées chacune par l'eſpace parcouru, il s'enſuit que le déblai & le remblai étant donnés de figure & de poſition, il n'eſt pas indifférent que telle molécule du déblai ſoit tranſportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine diſtribution à faire des molécules du premier dans le ſecond, d'après laquelle la ſomme de ces produits ſera la moindre poſſible, & le prix du tranſport total ſera un *minimum*.

- G. Monge formulated the 'excavation and embankments' problem in 1781
- Minimize the transportation cost

Monge transport problem

- Ω open subset of $\mathbb{R}^d$
- Cost function $c(x,y) \geq 0,\ x,y \in \Omega, \qquad c(x,x) = 0$
- $n_1,\ n_2$ probability measures on Ω

• **G. Monge (1781) :** find $T : \Omega \to \Omega$ such that

$$T_\# n_1 = n_2 \qquad \text{(push forward)}$$

$$c(x,y) = |x - y|$$

$$\min_T \int_\Omega c(x, T(x)) n_1(x) dx$$

Monge transport probleme

- Ω open subset of $\mathbb{R}^d$
- Cost function $c(x,y) \geq 0,\ x,y \in \Omega, \qquad c(x,x) = 0$
- $n_1,\ n_2$ probability measures on Ω

• **G. Monge (1781) :** $\qquad T : \Omega \to \Omega, \qquad T_{\#} n_1 = n_2$

$$\min_T \int_\Omega c(x,\ T(x)) n_1(x) dx$$

$T_{\#} n_1 = n_2$ means 'incompressibility', for all measurable function u

$$\int u(T(x)) n_1(dx) = \int u(y) n_2(dy) = \int u(T(x)) n_2(T(x)) \big|\mathrm{Det}\, DT(x)\big|$$

$$\mathrm{Det}\, DT(x) = \frac{n_1(x)}{n_2(T(x))}, \qquad y = T(x)$$

if this makes sense

Monge transport problem

- Ω open subset of $\mathbb{R}^d$
- Cost function $c(x, y) \geq 0,\ x, y \in \Omega, \qquad c(x, x) = 0$
- $n_1,\ n_2$ probability measures on Ω

• **G. Monge (1781) :** $\qquad T : \Omega \to \Omega, \qquad T_{\#} n_1 = n_2$

$$c(x, y) = |x - y|$$

$$\min_T \int_\Omega c(x, T(x)) n_1(x) dx$$

G. Monge makes two important observations

Monge transport problem

- Ω open subset of $\mathbb{R}^d$
- Cost function $c(x,y) \geq 0,\ x,y \in \Omega, \qquad c(x,x) = 0$
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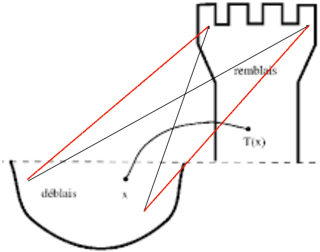
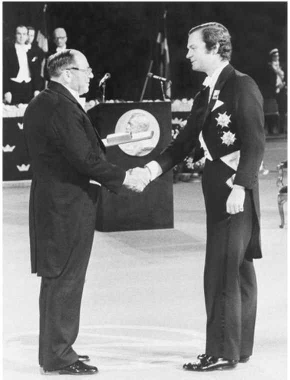


l'équation de la ſurface à laquelle toutes les routes doivent être normales, ſera

$$\left.\begin{array}{l}\frac{1}{3}[(Z-\zeta')^3-(Z'-\zeta')^3-(Z''-\zeta')^3+(Z'''-\zeta')^3]\left[\frac{dd\zeta'}{dx'^2}\frac{dd\zeta'}{dy'^2}-\left(\frac{dd\zeta'}{dx'dy'}\right)^2\right] \\ -\frac{1}{2}[(Z-\zeta')^2-(Z'-\zeta')^2-(Z''-\zeta')^2+(Z'''-\zeta')^2]\left\{\left[1+\left(\frac{d\zeta'}{dy'}\right)^2\right]\frac{dd\zeta'}{dx'^2}\right. \\ \left.-2\frac{d\zeta'}{dx'}\frac{d\zeta'}{dy'}\frac{dd\zeta'}{dx'dy'}+\left[1+\left(\frac{d\zeta'}{dx'}\right)^2\right]\frac{dd\zeta'}{dy'^2}\right\} \\ +(Z-Z'-Z''+Z''')\left[1+\left(\frac{d\zeta'}{dx'}\right)^2+\left(\frac{d\zeta'}{dy'}\right)^2\right]\end{array}\right\} = 0.$$

L. Kantorovich (1912-1986)



- L. Kantorovich was an economist (Nobel Pr. 1975)
- He was interested in economy (mines-factories)
- Kantorovich published his first paper in 1939
- not aware of Monge's paper
- In 1946, the world marked the bicentenary of Gaspard Monge (1746-1818)

Monge-Kantorovich distance

- Ω open subset of $\mathbb{R}^d$
- Cost function $c(x, y) \geq 0$, $x, y \in \Omega$, $\qquad c(0, 0) = 0$
- n_1, n_2 probability measures on Ω

- **G. Monge :** $\qquad T : \Omega \to \Omega, \qquad T_\# n_1 = n_2$

$$\min_T \int_\Omega c(x, T(x)) n_1(x) dx$$

has a solution only when n_1, n_2 are 'smooth' enough

- **L. Kantorovich :**

$$d_c(n_1, n_2) := \min_{v(\cdot,\cdot)} \int_\Omega c(x, y) v(x, y) dx dy$$

the *min* taken for probabily measures $v(\cdot, \cdot)$ with marginales

$$\int_\Omega v(x, y) dy = n_1(x), \qquad \int_\Omega v(x, y) dx = n_2(y)$$

Monge-Kantorovich distance

- n_1, n_2 probability measures
- cost $c(x,y)$

- **G. Monge :** $T : \Omega \to \Omega, \qquad T_{\#} n_1 = n_2$

$$\min_T \int_{\mathbb{R}^d} c(x, T(x)) n_1(x) dx$$

- **L. Kantorovich :**

$$d_c(n_1, n_2) := \min_v \int_{\mathbb{R}^d} c(x,y) v(x,y) dxdy$$

$$\int_\Omega v(x,y) dy = n_1(x), \qquad \int_\Omega v(x,y) dx = n_2(y)$$

- Monge is equivalent to choose $v(x,y) = n_1(x)\delta(y = T(x))$
- $d_c(n_1, n_2) = \mathbb{E}[c(X,Y)]$ with $\mathcal{L}X = n_1$, $\mathcal{L}Y = n_2$

Monge-Kantorovich distance

- n_1, n_2 probability measures
- cost $c(x, y)$

- **G. Monge :** $T : \Omega \to \Omega, \qquad T_{\#} n_1 = n_2$

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$$\int_\Omega v(x, y) dy = n_1(x), \qquad \int_\Omega v(x, y) dx = n_2(y)$$

- Monge is equivalent to choose $v(x, y) = n_1(x) \delta(y = T(x))$

Indeed with such a T, we have $\forall u : \Omega \to \Omega$

$$\int u(T(x)) n_1(x) dx = \int u(y) n_1(x) \delta(y - T(x)) = \int u(y) v(x, y) dxdy$$

Monge-Kantorovich distance

- n_1, n_2 probability measures
- cost $c(x,y)$

- **G. Monge :** $T:\Omega\to\Omega, \qquad T_\# n_1 = n_2$

$$\min_T \int_{\mathbb{R}^d} c(x, T(x)) n_1(x) dx$$

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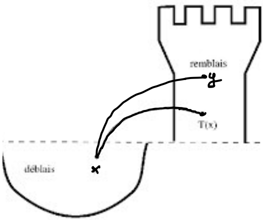
- Monge is equivalent to choose $v(x,y) = n_1(x)\delta(y = T(x))$
-



Kantorovich allows to 'cut big stones in small pieces'

Monge-Kantorovich distance

- n_1, n_2 probability measures
- cost $c(x, y)$

- **G. Monge :** $T : \Omega \to \Omega, \qquad T_{\#} n_1 = n_2$

$$\min_T \int_\Omega c(x, T(x)) n_1(x) dx$$

- **L. Kantorovich :**

$$d_c(n_1, n_2) := \min_{v(\cdot,\cdot)} \int_\Omega c(x, y) v(x, y) dxdy$$

$$\int_\Omega v(x, y) dy = n_1(x), \qquad \int_\Omega v(x, y) dx = n_2(y)$$

- Monge is equivalent to choose $v(x, y) = n_1(x)\delta(y = T(x))$
- Kantorovich can be seen as $v(x, y) = n_1(x) \sum a_i \delta(y = T_i(x))$

Monge-Kantorovich distance

- n_1, n_2 probability measures
- cost $c(x,y)$

- **G. Monge :** $T : \Omega \to \Omega, \qquad T_{\#}n_1 = n_2$

$$\min_T \int_\Omega c(x, T(x)) n_1(x) dx$$

- **L. Kantorovich :**

$$d_c(n_1, n_2) := \min_{v(\cdot,\cdot)} \int_\Omega c(x,y) v(x,y) dxdy$$

$$\int_\Omega v(x,y) dy = n_1(x), \qquad \int_\Omega v(x,y) dx = n_2(y)$$

- When $c(x,y)$ is a distance, then d_c is a distance on $\mathcal{P}(\Omega)$
- For $c(x,y) = |x-y|^p$, then $d_p^{\frac{1}{p}}$ is a distance (Wasserstein distance) on $\mathcal{P}^p(\Omega)$

Monge-Kantorovich distance

- **G. Monge :** $T : \Omega \to \Omega, \qquad T_\# n_1 = n_2$

$$d_c(n_1, n_2) := \min_T \int_\Omega c(x, T(x)) n_1(x) dx$$

- **L. Kantorovich :** $d_c(n_1, n_2) := \min_\nu \int_\Omega c(x,y)\nu(x,y)dxdy$

$$\int_\Omega \nu(x,y)dy = n_1(x), \qquad \int_\Omega \nu(x,y)dx = n_2(y)$$

- **Y. Brenier :** For $\Omega = \mathbb{R}^d$, $c(x,y) = |x-y|^2$,

n_i 'smooth', T is optimal if and only if

$$T(x) = \nabla\phi(x) \qquad \text{with} \qquad \phi : \mathbb{R}^d \to \mathbb{R} \;\; \text{convex}$$

$$\det\, D^2\phi(x) = \frac{n_1(x)}{n_2(T(x))} \qquad \text{Monge-Ampere eq.}$$

(Evans-Gangbo, Cafarelli-Feldman-McCann, Trudinger-Wang, Ambrosio, Villani, Figalli...)

Monge-Kantorovich problem

Two other contributions for completeness

Kantorovich duality

- **Kantorovich dual problem :**

$$d_c(n_1, n_2) = \sup_{\Phi,\Psi} \left\{ \int_{\mathbb{R}^d} \Phi(x) n_1(dx) + \int_{\mathbb{R}^d} \Psi(y) n_2(dy) \right\}$$

with the Kantorovich potentials Φ, Ψ such that

$$\Phi(x) + \Psi(y) \leq c(x, y)$$

- The truck problem... Φ or Ψ can be negative

Benamou-Brenier

Monge-Kantorovich is a linear optimisation problem !

Quadratic is better

- **Benamou-Brenier (2000)** For $c(x,y) = \frac{|x-y|^2}{2}$

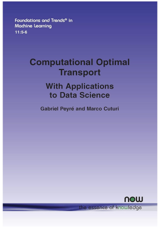
$$d_2(n_1, n_2) := \min \left\{ \int_0^1 \int_\Omega |v(s,x)|^2 \mu(s,x) dxds \right.$$

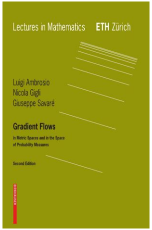
with the constraint

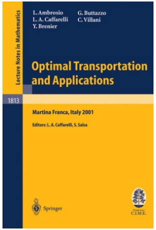
$$\begin{cases} \partial_s \mu + \mathrm{div}(v\mu) = 0, \\ \mu(0,x) = n_1(x), \ \mu(1,x) = n_2(x) \end{cases}$$

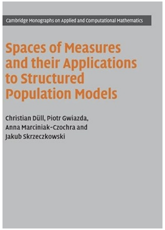
Important computational tool

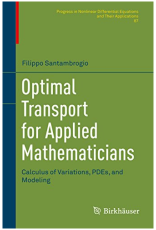
Monge-Kantorovich problem

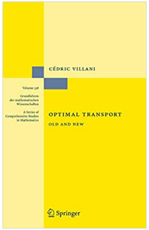












Global coupling method : Fourier eq.

$$\partial_t n - \Delta n = 0, \qquad x \in \mathbb{R}^d, \ t \geq 0$$

Theorem : For all costs $c(x-y)$,

$$d_c(n_1(t), n_2(t)) \leq d_c(n_1^{init}, n_2^{init})$$

Proof : Consider v solution of

$$\frac{\partial v}{\partial t} - \Delta_x v - \Delta_y v - 2\nabla_x \cdot \nabla_y v = 0, \quad x, y \in \mathbb{R}^d, \ t \geq 0$$

with a compatible initial data

$$\int v^{init}(x,y)dy = n_1^{init}(x) \qquad \int v^{init}(x,y)dx = n_2^{init}(y)$$

Step 1. $v \geq 0$ because $\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$ is nonnegative

Global coupling method : Fourier eq.

$$\partial_t n - \Delta n = 0, \qquad x \in \mathbb{R}^d, \;\; t \geq 0$$

Theorem : For costs $c(x-y)$, we have

$$d_c(n_1(t), n_2(t)) \leq d_c(n_1^{init}, n_2^{init})$$

Proof : Consider v solution of

$$\frac{\partial v}{\partial t} - \Delta_x v - \Delta_y v - 2\nabla_x \cdot \nabla_y v = 0, \quad x, y \in \mathbb{R}^d, \;\; t \geq 0$$

$$\int v^{init}(x,y)dy = n_1^{init}(x) \qquad \int v^{init}(x,y)dx = n_2^{init}(y)$$

Step 2. Marginals are correct. Integrate in y :

$$\frac{\partial v_1(x,t)}{\partial t} - \Delta_x v_1 = 0$$

Global coupling method : Fourier eq.

$$\partial_t n - \Delta n = 0, \qquad x \in \mathbb{R}^d,\ t \geq 0$$

Theorem : For costs $c(x-y)$, we have

$$d_c(n_1(t), n_2(t)) \leq d_c(n_1^{init}, n_2^{init})$$

Proof : Consider v solution of

$$\frac{\partial v}{\partial t} - \Delta_x v - \Delta_y v - 2\nabla_x \cdot \nabla_y v = 0, \quad x, y \in \mathbb{R}^d,\ t \geq 0$$

Step 3. The distance diminishes

$$\frac{d}{dt} \int c(x-y) v(x,y,t) dxdy =$$
$$\int v(x,y,t)\Big(\Delta_x c(x-y) + \Delta_y c(x-y) + 2\nabla_x \nabla_y c(x-y|)\Big) dxdy$$
$$= \int v(x,y,t)\Big(\Delta_{x+y} c(x-y)\Big) dxdy = 0$$

Global coupling method : Fourier eq.

$$\partial_t n - \Delta n = 0, \qquad x \in \mathbb{R}^d, \; t \geq 0$$

Theorem : For costs $c(x-y)$, we have

$$d_c(n_1(t), n_2(t)) \leq d_c(n_1^{init}, n_2^{init})$$

Step 4. Conclusion. For all initial coupling v^{init}

$$d_c(n_1(t), n_2(t)) \leq \int c(x-y)v(x,y,t)dxdy$$

$$= \int c(x-y)v^{init}(x,y)dxdy$$

$$\approx d_c(n_1^{init}, n_2^{init})$$

Global coupling method : Fourier eq.

$$\partial_t n - \Delta n = 0, \qquad x \in \mathbb{R}^d, \; t \geq 0$$

Theorem : For costs $c(x-y)$, we have

$$d_c(n_1(t), n_2(t)) \leq d_c(n_1^{init}, n_2^{init})$$

Not completely convincing because

- The solution is a convolution of initial data
- Large choice of possible $c(x,y)$
- Question : $\partial_t n - \Delta(A(x)n) = 0$?

Global coupling : Scattering

- **Kinetic scattering** Particles moving vith velocity v, density $n(t, x, v)$

$$\partial_t n + v.\nabla_x n + n = \int n(t, x, V(v; h)) \det D_v V \, d\mu(h)$$

Assume that jumps reduce distance

$$\int |V^{-1}(W, h) - V^{-1}(W', h)|^p d\mu(h) \le L|W - W'|, \qquad \forall h$$

Theorem : For $L < 1$, this equation is non-expansive for the distance

$$c(x, v; y, w) = \alpha|x - y| + |v - w|$$

α small enough compared to $1 - L$.

Global coupling : Scattering

Proof :

$$\partial_t F + v.\nabla_x F + w.\nabla_y F + F(x,v,y,w,t) =$$

$$\int F(x, V(v,h), y, V(w,h), t) \det(D_v V(v,h)) \det(D_v V(w,h)) d\mu(h)$$

Step 1. $F \geq 0$

Step 2. Marginales are correct

Step 3.

$$\frac{d}{dt} \int [\alpha|x-y| + |v-w|]F \leq \alpha \int |v-w|F - (1-L) \int |v-w|F$$

Global coupling : Aggregation eq.

$$\partial_t n(t,x) + \mathrm{div}(n \nabla K \star n) = 0$$

$$K \star n = \int_{\mathbb{R}^d} K(x-y) n(t,y) dy.$$

It is important for particle systems.
A measure solution is

$$n(t,x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - X_i(t))$$

with

$$\frac{d}{dt} X_i(t) = \frac{1}{N} \sum_{j=1}^{N} K(X_i(t) - X_j(t))$$

Global coupling : Aggregation eq.

$$\partial_t n(t,x) + \mathrm{div}(n \nabla K \star n) = 0$$

Theorem : When K is even and concave, it is non-expansive for $\mathbf{d_2}$

Proof : Consider the coupling

$$\partial_t v(t,x,y) + \mathrm{div}_x(v \nabla_x K \star v_1(t,x)) + \mathrm{div}_y(v \nabla_y K \star v_2(t,y)) = 0$$

with $v^{init} \geq 0$ having the marginales $n_1^{init}(x)$, $n_2^{init}(y)$ and

$$v_1(t,x) = \int v(t,x,y)dy, \qquad v_2(t,y) = \int v(t,x,y)dx$$

Step 1. Obviously $v(t) \geq 0$

Global coupling : Aggregation eq.

$$\partial_t n(t,x) + \mathrm{div}(n \nabla K \star n) = 0$$

Theorem : When K is even and concave, it is non-expansive for $\mathbf{d_2}$

Proof : Consider the coupling

$$\partial_t v(t,x,y) + \mathrm{div}_x(v \nabla_x K \star v_1(t,x)) + \mathrm{div}_y(v \nabla_y K \star v_2(t,y)) = 0$$

with $v^{init} \geq 0$ having the marginales $n_1^{init}(x)$, $n_2^{init}(y)$ and

$$v_1(t,x) = \int v(t,x,y)dy, \qquad v_2(t,y) = \int v(t,x,y)dx$$

Step 2. $v(t)$ has the correct marginales, because integrating in y for instance

$$\partial_t v_1(t,x) + \mathrm{div}_x(v_1 \nabla_x K \star v_1(t,x)) = 0$$

and thus $\quad v_1 = n_1$

Global coupling : Aggregation eq.

$$\partial_t n(t,x) + \mathrm{div}(n \nabla K \star n) = 0$$

Theorem : When K is even and concave, it is non-expansive for $\mathbf{d_2}$

$$\partial_t v(t,x,y) + \mathrm{div}_x(v \nabla_x K \star v_1(t,x)) + \mathrm{div}_y(v \nabla_y K \star v_2(t,y)) = 0$$

Step 3. One computes

$$\frac{d}{dt} \int \frac{|x-y|^2}{2} v dx dy = \int v \, \Big(x-y, \nabla K \star v_1(t,x) - \nabla K \star v_2(t,y)\Big)$$

$$= \int v \; v' \; (x-y, \nabla K(x-x') - \nabla K(y-y'))$$

exchange x, x' ; y, y'

$$= \frac{1}{2} \int v \; v' \; (x-x'-y+y', \nabla K(x-x') - \nabla K(y-y'))$$

$$\leq 0$$

Structured equation

Theorem (N. Fournier, BP)

- The Renewal and Growth-Fragmentation equations, and all others, are non-expansive for the cost

$$c(x-y) = \min(|x-y|, a), \qquad a \text{ related to Lipschitz bounds on coef.}$$

- For renewal it is a strict contraction and $d_c(n(t), \bar{n}) \to 0$

History

Fournier and Locherbach (neural networks)

Chafai, Malrieu, Paroux, Guillin, Zitt... (TCP connections)

Structured equation

Renewal equation as an example.

$$\begin{cases} \frac{\partial n(x,t)}{\partial t} + \frac{\partial [g(x)n]}{\partial x} + d(x)n = b(x)N(t), \qquad t \geq 0,\ x \geq 0, \\ n(x=0,t) = 0, \qquad N(t) = \int_0^\infty d(x)n(x,t)dx. \end{cases}$$

$$g' \leq 0, \qquad g(0) \geq 0, \qquad \int_0^\infty b = 1$$

$$0 < a < 1 \qquad \text{and} \qquad a = \inf_{|x-y|<1} \frac{|x-y| \max(d(x), d(y))}{|d(x) - d(y)|}$$

Example : $d(x) = \alpha + \beta x^p,\ p \geq 1.$

$$c(x-y) = \min(|x-y|, a)$$

$$d_c(n_1(t), n_2(t)) \leq d_c(n_1^{init}, n_2^{init})$$

Structured equation

$$\frac{\partial v}{\partial t} + \frac{\partial g(x) v}{\partial x} + \frac{\partial g(y) v}{\partial y} + \max(d(x), d(y)) v$$
$$= b(x)\delta(x-y) \int \min(d(x'), d(y'))\, v(dx', dy', t)$$
$$+ b(x) \int \left(d(x') - d(y)\right)_+ v(dx', y, t)$$
$$+ b(y) \int \left(d(y') - d(x)\right)_+ v(x, dy', t)$$

with an initial data v^{init} whose marginals are n_1^{init} and n_2^{init}.

Step 1. $v \geq 0$

Step 2. Marginales are correct (more tricky to see)

Structured equation

$$\frac{\partial v}{\partial t} + \frac{\partial g(x)v}{\partial x} + \frac{\partial g(y)v}{\partial y} + \max(d(x), d(y))v$$
$$= b(x)\delta(x-y)\int \min(d(x'), d(y'))\, v(dx', dy', t)$$
$$+ b(x)\int \left(d(x') - d(y)\right)_+ v(dx', y, t) + \ldots$$

Step 3.

$$\frac{d}{dt}\int c(x-y)v + \int c(x,y)\max(d(x), d(y))v = 0$$
$$+ \int c(x,y)b(x)\left(d(x') - d(y)\right)_+ v(dx', y, t) + \ldots$$

one needs

$$\max(d(x), d(y))c(x,y) \geq \int c(z,y)b(z)dz(d(x) - d(y))_+$$
$$+ \int c(x,z)b(z)dz\left(d(y) - d(x)\right)_+$$

Outline

1. Motivations
2. Generalised relative entropy
3. Monge-Kantorovich distance and PDEs
4. Sexual reproduction

Sexual reproduction

$$\partial_t n(x,t) + n(x,t) = \int_{\mathbb{R}^{2d}} K(x; x', x'_*) n(x',t) n(x'_*,t) dx' dx'_*,$$

$t \geq 0, \; x \in \mathbb{R}^d$ with

$$\int_{\mathbb{R}^d} K(x; x', x'_*) dx = 1$$

to preserve total number $\int n(x,t)dx = 1$

For instance, with $h(\cdot)$ a probability measure

$$K(x; x', x'_*) = \int_0^1 \delta_{x'\sigma + x'_*(1-\sigma)}(x) h(\sigma) d\sigma$$

Magal, Raoul, Mirrahimi, Calvez-Garnier-Patout, Poyato, Degond, Frouvelle, Schmeiser-Kanzler

Sexual reproduction

Theorem For $c(x,y) = |x-y|^p$, $1 \le p < \infty$, we have

$$d_c(n_1(t), n_2(t)) \le d_c(n_1^{init}, n_2^{init})$$

for n_1^{init}, n_2^{init} probability measures.

Proof by coupling

$$\partial_t v(x,y,t) + v(x,y,t) = \\ = \int \overline{K}(x,y;x',x'_*,y',y'_*) v(x',y') v(x'_*,y'_*) dx' dy' dx'_* dy'_*$$

with

$$\overline{K}(x,y;x',x'_*,y',y'_*) = \int_0^1 h(\sigma) \delta_{\sigma x' + (1-\sigma) x'_*}(x) \delta_{\sigma y' + (1-\sigma) y'_*}(y) d\sigma.$$

Global coupling : Boltzmann Eq.

Homogeneous Boltzmann Eq.

$$\begin{cases} \partial_t f(v,t) = \int_{\mathbb{R}^3\times\mathbb{S}^2} [f(v',t)f(v'_*,t) - f(v,t)f(v_*,t)]B(\theta)dv_*d\sigma \\ v' = \frac{1}{2}(v+v_*) + \frac{1}{2}|v-v_*|\sigma, \qquad v'_* = \frac{1}{2}(v+v_*) - \frac{1}{2}|v-v_*|\sigma \\ \cos(\theta) = \frac{v-v_*}{|v-v_*|}\cdot\frac{v'-v'_*}{|v'-v'_*|}, \qquad \int_0^\pi B(\theta)d\theta = 1 \end{cases}$$

Theorem (Tanaka,1978).

The Boltzmann equation is non-expansive for d_2.

Coupled homogeneous Boltzmann Eq. : $F(v,w,t)$

$$\frac{d}{dt}\int_{(\mathbb{R}^3)^2} \Psi(v,w)F(v,w,t)dvdw$$

$$= \int_{(\mathbb{R}^3)^4}\int_0^\pi\int_0^{2\pi} \Big[\Psi(v',w') + \Psi(v'_*,w'_*) - \Psi(v_*,w_*) - \Psi(v,w)\Big]$$

$$B(\theta)F(v,w,t)F(v_*,w_*,t)dvdwdv_*dw_*d\theta d\varphi$$

Global coupling : Boltzmann Eq.

Coupled homogeneous Boltzmann Eq. : $F(v, w, t)$

$$\frac{d}{dt}\int_{(\mathbb{R}^3)^2} \Psi(v, w)F(v, w, t)dvdw$$
$$= \int_{(\mathbb{R}^3)^4}\int_0^{\pi}\int_0^{2\pi} \Big[\Psi(v', w') + \Psi(v'_*, w'_*) - \Psi(v_*, w_*) - \Psi(v, w)\Big]$$
$$B(\theta)F(v, w, t)F(v_*, w_*, t)dvdwdv_*dw_*d\theta d\varphi$$

$$v' = \frac{1}{2}(v + v_*) + \frac{1}{2}|v - v_*|\sigma, \quad v'_* = \frac{1}{2}(v + v_*) - \frac{1}{2}|v - v_*|\sigma$$
$$w' = \frac{1}{2}(w + w_*) + \frac{1}{2}|w - w_*|\omega, \quad w'_* = \frac{1}{2}(w + w_*) - \frac{1}{2}|w - w_*|\omega$$

How to couple σ and ω ?

Global coupling : Boltzmann Eq.

and the sphere is parametrized as

$$\sigma = \cos(\theta)\frac{v - v_*}{|v - v_*|} + \sin(\theta)[l\cos(\varphi) + l_1\sin(\varphi)]$$

$$\omega = \cos(\theta)\frac{w - w_*}{|w - w_*|} + \sin(\theta)[l\cos(\varphi) + l_2\sin(\varphi)],$$

where $l = \frac{(v-v_*)\wedge(w-w_*)}{|(v-v_*)\wedge(w-w_*)|}$,

and l_1, l_2 are chosen so that

$$(\frac{v - v_*}{|v - v_*|}, l, l_1), \qquad (\frac{w - w_*}{|w - w_*|}, l, l_2)$$

are two direct orthonormal bases

Coupling method

- **Theorem (Otto)** The porous media equation is non-expansive for $\mathbf{d_2}$

$$\frac{\partial n}{\partial t} - \Delta A(n) = 0$$

I recommand the proof by Bolley-Carrillo

No proof known by 'global coupling'

Conclusion

Thanks to :

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N. Fournier, L. Kanzler

THANK YOU