Asymptotic Analysis of a Discrete Model with Heavy-Tailed Mutation kernel in Evolutionary Dynamics

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Plan

Biological framework

We study populations where individuals are characterized by a quantitative trait affecting reproduction and survival.
The trait distribution evolves through:

- Heredity: transmission of traits to offspring,
- Mutation: introducing variability,
- Selection: favoring individuals with higher survival or reproduction.

Asexual populations (cells, bacteria)

Usual biological assumptions:

- large populations
- small mutation steps
- long time scale

The main goal:

- Predict the long term evolutionary dynamics.

A classical model of eco-evolutionary dynamics

Continuum of alleles model (Kimura, 1965)

$$\begin{cases} \partial_t n(t,x) = \underbrace{R(x, I(t))n(t,x)}_{\text{selection \& competition}} + \underbrace{\int_{\mathbb{R}^d} G(y-x)p(y)n(t,y)dy}_{\text{mutation}} \\ I(t) = \int_{\mathbb{R}^d} n(t,y)dy, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d. \end{cases}$$

Derived from an individual-based model (Champagnat, Ferrière and Méléard-2008).

Scalings

Small mutational effects:

$$G(y)dy \to G(\frac{y}{\varepsilon})\frac{dy}{\varepsilon^d}.$$

To capture the effects of mutations we also rescale in time:

$$t \to \frac{t}{\varepsilon}.$$

Hopf-Cole transformation:

$$n_\varepsilon(t,x) = e^{\frac{u_\varepsilon(t,x)}{\varepsilon}}.$$

Then

$$\begin{cases} \partial_t u_\varepsilon(t,x) = R(x, I_\varepsilon(t)) + \int_{\mathbb{R}^d} p(x+\varepsilon y)G(y)e^{\frac{u_\varepsilon(t,x+\varepsilon y)-u_\varepsilon(t,x)}{\varepsilon}}dy \\ I_\varepsilon(t) = \int_{R^d} n_\varepsilon(t,y)dy, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \end{cases}$$

How to characterize the phenotypic density

Theorem (Barles, Mirrahimi and Perthame-2009)

As $\varepsilon \to 0$, $(u_\varepsilon)_\varepsilon$ *converges to a viscosity solution to the constrained Hamilton-Jacobi equation*

$$\begin{cases} \partial_t u(t,x) = R(x, I(t)) + p(x) \int_{\mathbb{R}^d} G(y) e^{\nabla u(t,x).y} dy, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \\ \max_{x \in \mathbb{R}^d} u(t,x) = 0, \quad \forall t > 0. \end{cases}$$

Moreover, $(n_\varepsilon)_\varepsilon$ *converges in* $L^\infty(w * (0, +\infty), \mathcal{M}^1(\mathbb{R}))$ *to a measure* n *which satisfies almost everywhere* $t > 0$,

$$supp\ n(t,.) \subset \{x \in \mathbb{R}^d / u(t,x) = 0\}.$$

Probabilistic point of view

A direct derivation of this Hamilton-Jacobi equation in a specific case was established by Champagnat et al.[1]

[1] N. Champagnat et al. "Filling the gap between individual-based evolutionary models and Hamilton-Jacobi equations". In: *Journal de l'École polytechnique — Mathématiques* Tome 10 (2023), pp. 1247–1275.

Plan

The model

- Large population: parameterized by a capacity carrying parameter $K \to +\infty$.
- The mutation kernel has exponential decay:

$$G(y) = f(y)e^{-|y|}.$$

$\implies$ take into account large mutation jumps with a high rate.
We consider a discrete version of the selection-mutation model in $\mathcal{X}_K = \{i\delta_K, i \in \mathbb{Z}\}$, where $\delta_K \to 0$ is the step of discretization.

$$\begin{cases} \frac{d}{dt} n_i^K(t) = R(i\delta_K, I^K(t)) n_i^K(t) + \sum_{j \in \mathbb{Z}} p(j\delta_K) h_K G((j-i)h_K) n_j^K(t), \\ I^K(t) = \sum_{i \in \mathbb{Z}} \delta_K n_i^K(t \log K), \\ n_i^K(0) = n^{K,0}(i\delta_K), \end{cases}$$

where $h_K = \delta_K \log K$.

Scalings

Hopf-Cole transformation at logarithmic scale

$$u_i^K(t) = \frac{\log n_i^K(t\log K)}{\log K}, \quad n_i^K(t\log K) = K^{u_i^K(t)}$$

then

$$\begin{cases} \frac{d}{dt}u_i^K(t) = R(i\delta_K, I^K(t)) + \sum_{l\in\mathbb{Z}} p((l+i)\delta_K)h_K G(lh_K)e^{\log K(u_{l+i}^K(t)-u_i^K(t))}, \\ I^K(t) = \sum_{i\in\mathbb{Z}} \delta_K n_i^K(t\log K), \\ u_i^K(0) = u^{K,0}(i\delta_K). \end{cases}$$

We introduce the following linear interpolation; for all $x \in \mathbb{R}$, let i such that $x \in [i\delta_K, (i+1)\delta_K)$

$$\widetilde{u}^K(t,x) = u_i^K(t)(1 - \frac{x}{\delta_K} + i) + u_{i+1}^K(t)(\frac{x}{\delta_K} - i).$$

Main result

Theorem (.J-2025)

As $K \to +\infty$, a subsequence of $(\widetilde{u}^K)_K$ converges locally uniformly to a continuous viscosity solution to the Hamilton-Jacobi equation:

$$\begin{cases} \min(\partial_t u - R(x, I(t)) - p(x) \int_{\mathbb{R}} G(y) e^{\nabla u . y} dy, 1 - |\nabla u|) = 0, \\ \max_{x \in \mathbb{R}} u(t,x) = 0, \quad \forall t > 0, \\ u(0,.) = u^0(.). \end{cases}$$

Moreover, a subsequence of $(\widetilde{n}^K)_K$ converges in $L^\infty(w * (0,+\infty), \mathcal{M}^1(\mathbb{R}))$ to a measure n which satisfies almost everywhere $t > 0$,

$$supp\ n(t,.) \subset \{x \in \mathbb{R} / u(t,x) = 0\}.$$

To prove the convergence, we use the **semi-relaxed limits**:

$$\overline{u}(t,x) := \limsup_{\substack{K\to+\infty \\ (s,y)\to(t,x)}} \widetilde{u}^K(s,y) \quad \text{et} \quad \underline{u}(t,x) := \liminf_{\substack{K\to+\infty \\ (s,y)\to(t,x)}} \widetilde{u}^K(s,y).$$

The classical method:

- $\overline{u}$ **is a viscosity subsolution** of the HJ equation.
- $\underline{u}$ **is a viscosity supersolution** of the HJ equation.
- **A strong comparison principle** in the class of discontinuous viscosity solutions:

  $$\overline{u} \leq \underline{u},$$

  and hence $\overline{u} = \underline{u}$ which implies the convergence of $(\widetilde{u}^K)_K$ to $u = \overline{u} = \underline{u}$.

Difficulties

Difficulties in our case:

- The Hamiltonian can take infinite values.
- I is only BV and potentially discontinuous.
- The equation is given only on a grid.

Strategy of the proof

What we do:

- We prove a Lipschitz estimates in space.
- We prove that $\underline{u}$ is a viscosity supersolution to the HJ.
- We show that $\underline{u}$ has nice properties.
- We modify it and regularize it and use it as a test function for $\overline{u}$ to obtain a contradiction with the fact that $\sup \overline{u} - \underline{u} > 0$. We conclude that $\overline{u} = \underline{u}$ which means that $(\widetilde{u}^K)_K$ converges.
- We use the properties of $\underline{u}$ to prove that $\overline{u}$ is a viscosity subsolution to the HJ.

Merci de votre attention