Non irreducible infinite-dimensional SIS model

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Outline

1. Infinite-dimensional SIS model
2. The matrix case
3. Atoms in infinite dimension
4. Characterization of equilibria

Outline

Framework

Heterogeneous SIS model (temporary infection + no immunity on recovery) with a *trait* (or *feature*) set Ω such that:

- $(\Omega, \mathcal{F}, \mu)$: measured space.
- μ is a finite measure giving the population trait distribution.
- Individuals with the same trait have the same behavior.
- Individuals have the same trait over time.
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Examples:

- $\Omega = \{\mathrm{M}, \mathrm{W}\}$ (for STIs).
- $\Omega \subset \mathbb{R}^d$ (position).
- $\Omega \subset \mathbb{R}_+$ (age).
- Any cartesian product of the above examples.

Transmission equation

(Deterministic) Infinite-dimensional SIS model:

$$\partial_t u(t,x) = (1 - u(t,x)) \int_\Omega k(x,y) u(t,y) \mu(\mathrm{d}y) - \gamma(x) u(t,x)$$

with:

- $u(t,x)$: proportion of infected individuals at time t among individuals with trait x.
- $k(x,y)$: infection rate from individuals with trait y to individuals with trait x.
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For an initial condition $h \in L^\infty$ with $0 \leqslant h \leqslant 1$, the ODE has a unique global solution on $\mathbb{R}_+$, it satisfies $\forall t \in \mathbb{R}_+, \forall x \in \Omega, 0 \leqslant u(t,x) \leqslant 1$.

Equilibria

Let $g \in L^\infty$ with $0 \leqslant g \leqslant 1$. g is an *equilibrium* when

$$0 = (1 - g(x)) \int_\Omega k(x, y) g(y) \mu(\mathrm{d}y) - \gamma(x) g(x).$$

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- $g = \mathbb{0}$: disease-free equilibrium.
- $g \neq \mathbb{0}$: endemic equilibrium.

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Theorem (Delmas, Dronnier, Zitt (2021))

Under some integrability assumptions on k and γ, we have:

- *If $R_0 \leqslant 1$:*
  - *$g = \mathbb{0}$ is the only equilibrium.*
  - *$u(t,x) \to g = \mathbb{0}$.*
- *If $R_0 > 1$, there exists (at least) an endemic equilibrium.*
- *If $R_0 > 1$ and k is irreducible:*
  - *There exists a unique endemic equilibrium g^*.*
  - *Unless $u(0) = \mathbb{0}$, $u(t,x) \to g^*$.*

Some questions

- What is an irreducible kernel ?
- What if k is not irreducible ? Uniqueness ? Convergence ?

Outline

Graph associated to a matrix

In this section, Ω is a finite set.

Transmission graph associated to k (who may infect who ?):

- Vertices: Ω
- Edges: An edge from j to i when $k_{i,j} > 0$.

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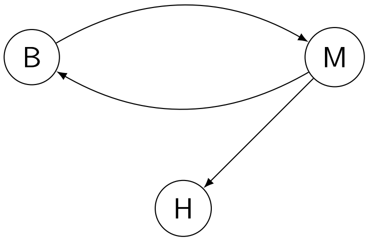


Figure: West Nile Virus (Bowman et al. (2005))

Irreducibility

We say that k is *irreducible* if, for every two different vertices, there is a path from one of them to the other (= strongly connected graph).

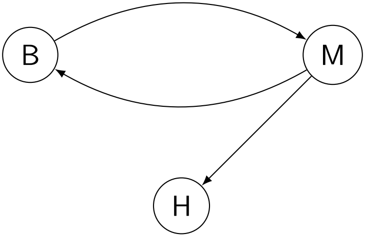
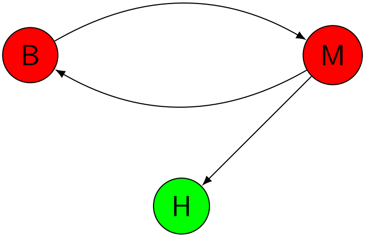


Figure: West Nile Virus (Bowman et al. (2005))

Atom

For $A \subset \Omega$, let $k_A = (k(x,y))_{x,y \in A}$.

The *atoms* of k are the maximal subsets A of Ω such the graph of k_A is irreducible (= strongly connected components).



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Solution: We use the density of edges leaving a set.

We say that $A \subset \Omega$ measurable is *invariant* when

$$\int_{A^c \times A} k(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) = 0.$$

Intuitively, A does not infect A^c.

Irreducible kernel

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We say that k is *irreducible* when its only invariant sets are $\varnothing$ and Ω.

Atoms

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The *atoms* of k are its maximal irreducible sets. Intuitively, the atoms are the maximal sets such that "anyone may infect anyone".

Atoms and R_0

Remider: We have $R_0 = \rho\left((k(x,y)/\gamma(y))_{x,y\in\Omega}\right)$.

For $A \subset \Omega$ measurable, let $R_0(A) = \rho\left((k(x,y)/\gamma(y))_{x,y\in A}\right)$ be its *intrinsic* basic reproduction number.

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Proposition (Schwartz (1961))

Under some integrability assumptions on k and γ:

$$R_0 = \max_{A \text{ atom}} R_0(A).$$

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Proposition (Schwartz (1961))

Under some integrability assumptions on k and γ:

$$R_0 = \max_{A \text{ atom}} R_0(A).$$

R_0 does not depend on the value of k between atoms.

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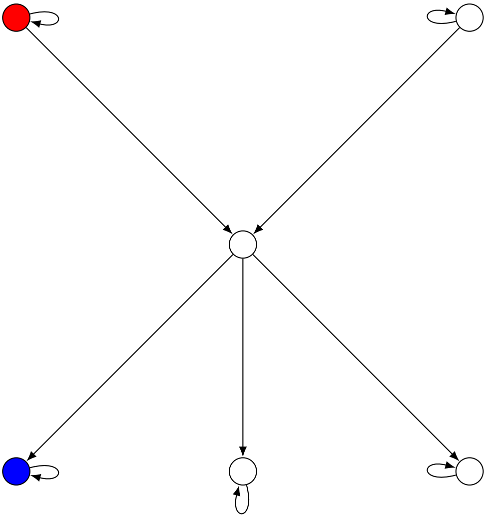
Order relation on atoms



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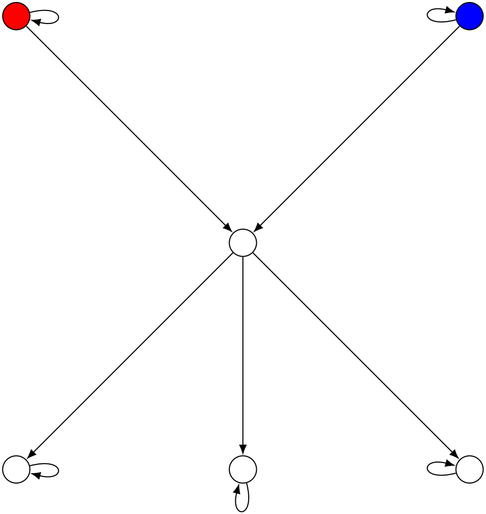
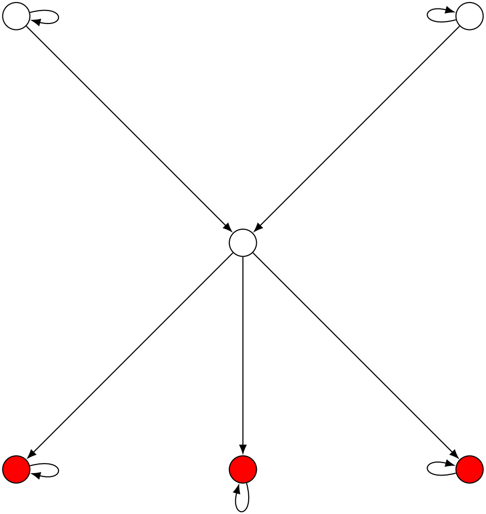


Antichains of atoms

$\{A_1, \dots, A_n\}$ antichain of atoms when atoms are not comparable for $\preccurlyeq$.



Characterization of equilibria

- *Supercritical atom*: $R_0(A) > 1$.
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Consequence:

- When $R_0 \leqslant 1$, $\mathbb{0}$ is the only equilibrium.
- When $R_0 > 1$, there exists (at least) an endemic equilibrium.

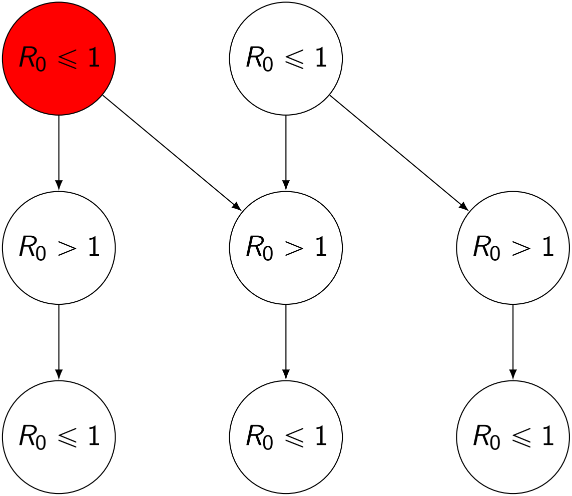
Convergence of solutions

Theorem (Delmas, L., Zitt (2024))

Under some integrability assumptions on k and γ, for any initial condition $h \in L^\infty$ with $0 \leqslant h \leqslant 1$, $u(t)$ converges to an equilibrium.

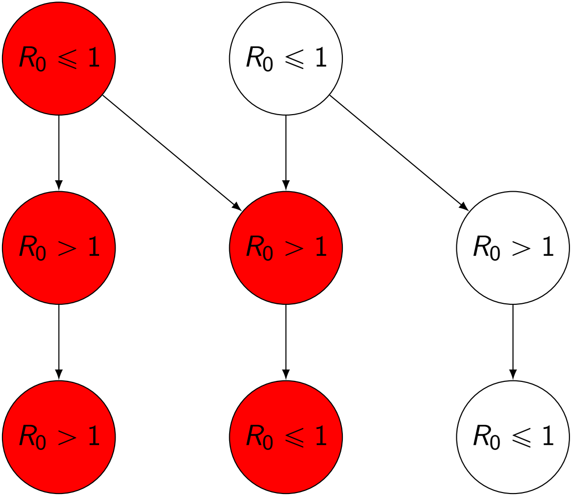
Convergence of solutions

$t = 0$



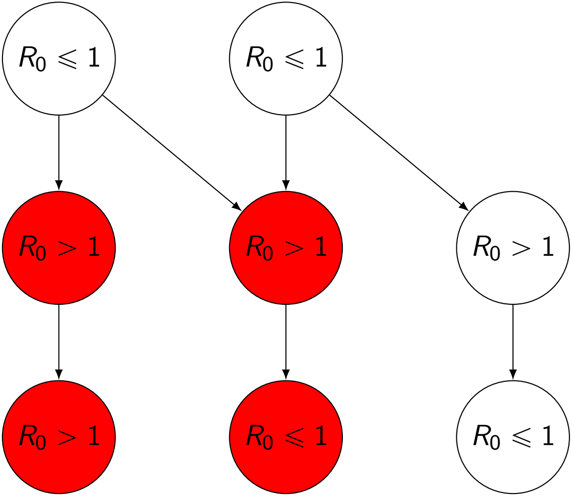
Convergence of solutions

$t > 0$



Convergence of solutions

$t \to +\infty$



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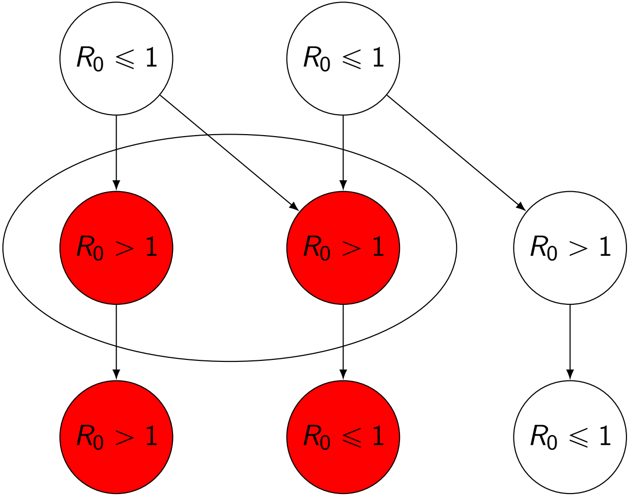

Thank you for your attention!