Propagation of exchangeability and moment duality

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Exchangeability

A random vector $(X_1, ..., X_n)$ is *exchangeable* if

$$(X_1, ..., X_n) \overset{\mathcal{D}}{=} (X_{\pi(1)}, ..., X_{\pi(n)}) \text{ for } \pi \in \mathcal{S}_n$$

Example : $\mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(X_1 = 0, X_2 = 0) = \frac{1}{2}$.
$X_1, X_2 \sim \mathcal{B}(\frac{1}{2})$ but not independent !

$\rightarrow$ Exchangeability allows to get rid of independence

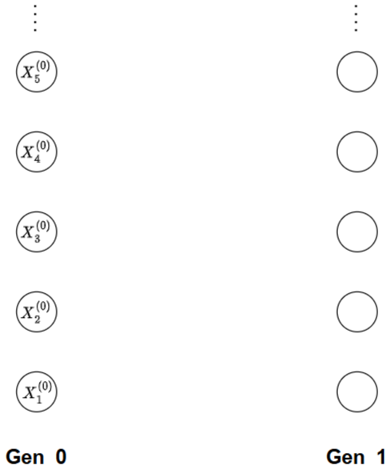
Propagation of exchangeability

Model : initial types

Infinite countable population with types in $\{0,1\}$ initially given by $(X_i^{(0)})_{i\in\mathbb{N}}$ exchangeable.

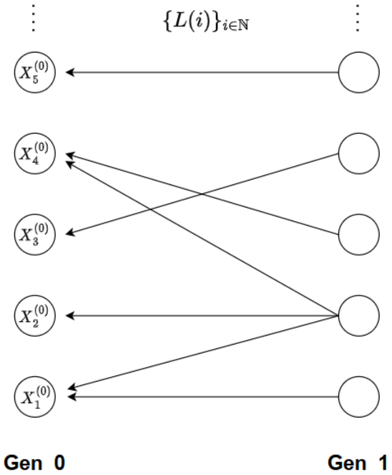
Model : discrete generations

Discrete non-overlapping generations.



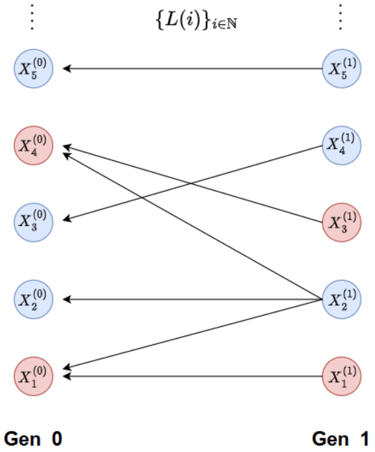
Model : ancestors picking

Choice of parents given by $\{L(i)\}_{i\in\mathbb{N}}$, with $L(i)$ potential ancestors of individual i.



Model : type inheritance

For $i \in \mathbb{N}$, set $X_i^{(1)} = \inf_{j \in L(i)} X_j^{(0)}$.



How to propagate exchangeability?

Definition 1: Propagation of exchangeability

We say that the random function L *propagates exchangeability* if whenever $(X_i^{(0)})_{i\in\mathbb{N}}$ is exchangeable, also $(X_i^{(1)})_{i\in\mathbb{N}}$ is exchangeable.

Examples in finite population :

- Wright-Fisher model $\rightarrow$ i.i.d. choice of parents
- Cannings model $\rightarrow$ exchangeable choice of parents

Theorem

Set $L : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ with $L(S) = \bigcup_{i \in S} L(i)$.

Definition 2: Forgetfulness

We say that L is *forgetful* if $|L(S)| \overset{d}{=} |L(S')|$ for all samples $S, S' \subset \mathbb{N}$ such that $|S| = |S'|$.

Theorem 1

L propagates exchangeability if and only if L is forgetful

Examples

- **Discrete Λ-lookdown :** Select a subgroup $G \in \mathcal{P}(\mathbb{N})$ and set $R := \inf G$. If $i \in G$, $L(i) = \{R\}$, else $L(i)$ is the smallest *free* individual.

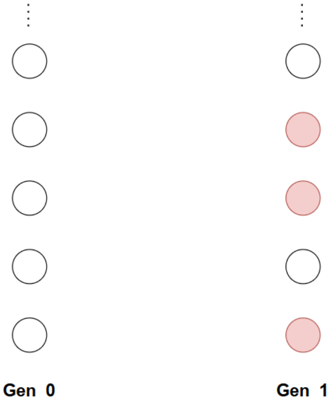
- **Branching :** $\{\xi_i\}_{i\in\mathbb{N}}$ i.i.d. $\mathbb{N}_0$-valued random variables. Define $\{L(i)\}_{i\in\mathbb{N}}$ by the rule: $j \in L(i)$ if and only if

$$\sum_{l=1}^{i-1} \xi_l < j \leq \sum_{l=1}^{i} \xi_l$$

.

Example 1 : discrete Λ-lookdown

Select a subgroup $G \in \mathcal{P}(\mathbb{N})$ and set $R := \inf G$.



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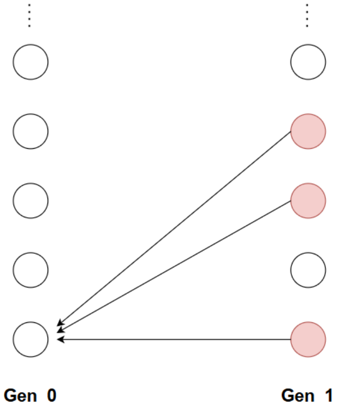

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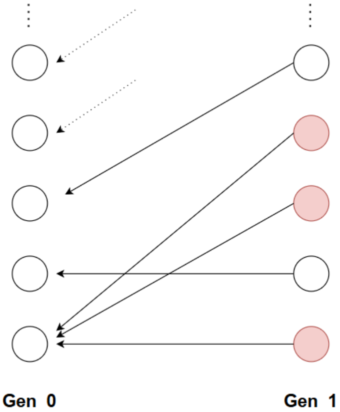
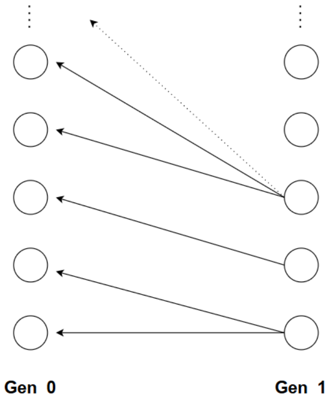


Example 2 : branching

Set $j \in L(i)$ if and only if $\sum_{l=1}^{i-1} \xi_l < j \leq \sum_{l=1}^{i} \xi_l$ where $\{\xi_i\}_{i \in \mathbb{N}}$ i.i.d. in $\mathbb{N}_0$.

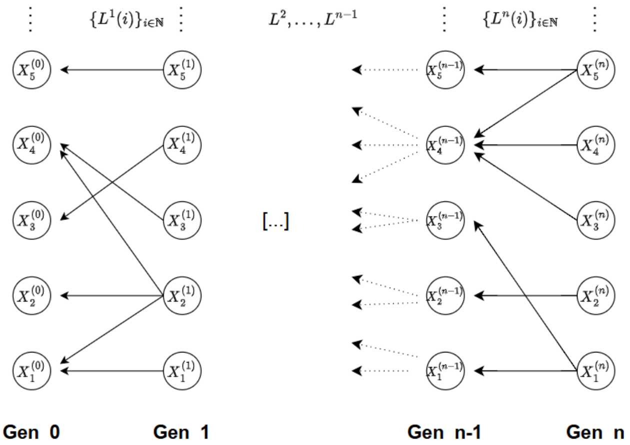
$\xi_1 = 2, \xi_2 = 1, \xi_3 = 3, \dots$



Moment duality

Forward process

→ Construct a $\{0,1\}^{\mathbb{N}}$-valued discrete process started at $(X_i^{(0)})_{i\in\mathbb{N}}$ with $\{L^{(k)}\}_{k\in\mathbb{N}}$ i.i.d.



Forward process

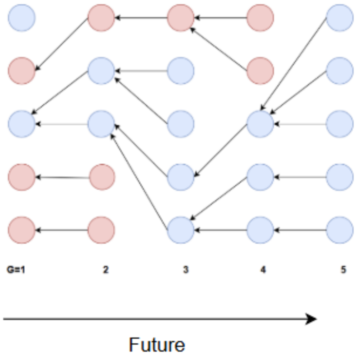
If L is forgetful, then for every $k \in \mathbb{N}$, the sequence $(X_i^{(k)})_{i\in\mathbb{N}}$ is exchangeable.

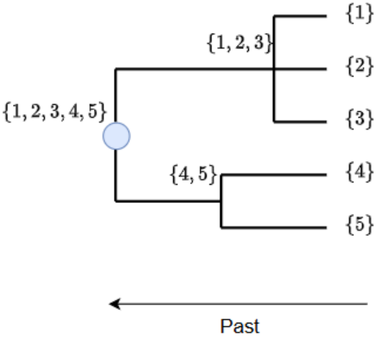
By De Finetti's theorem, for every $k \in \mathbb{N}$, there exists Θ_k random variable in $[0,1]$ s.t. :

1. $\Theta_k \overset{a.s.}{=} \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} X_j^{(k)}$
2. Conditionally on Θ_k, the $X_i^{(k)}$ are i.i.d with law $\mathcal{B}(\Theta_k)$

$\rightarrow$ The process $(\Theta_k)_{k\geq 0}$ is Markov by construction.

Duality





Backward process and moment duality

Definition 3: Backward process

Define the $\mathbb{N}$-valued Markov process $(A_k)_{k\geq 0}$ as the block-counting process associated with L :

$$\text{for } n, m \in \mathbb{N},\ \mathbb{P}(A_1 = m | A_0 = n) = \mathbb{P}(|L(\{1, ..., n\})| = m)$$

Theorem 2

Assuming L is forgetful, the forward and backward processes are moment dual, i.e.

$$\mathbb{E}_x\left[\Theta_k^n\right] = \mathbb{E}_n\left[x^{A_k}\right]$$

with $\Theta_0 = x, A_0 = n$.

An application

Construction of lookdown models

Combine Theorems 1&2: Construct a discrete forward process dual to a known backward process induced by the appropriate L function.

Lookdown models (Donnelly & Kurtz 1999): Countable representation of famous diffusions.

$\rightarrow$ Kingman coalescent : already exists but easier proof.
$\rightarrow$ Ξ-lookdown with selection : new construction (to our knowledge)

Questions

1. Beyond the $\{0, 1\}$ case ?
2. Inventory of possible L functions ?
3. Links with graph theory ?

THANK YOU !

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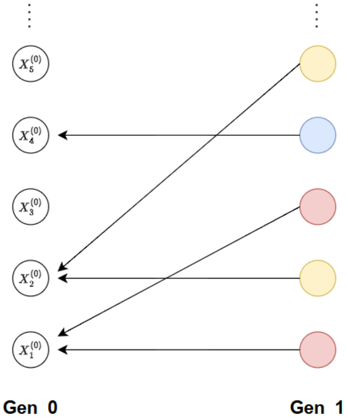
Example : Ξ-lookdown

Divide $\mathbb{N}$ into groups $G_1, G_2, G_3 \ldots$ Then if $i \in G_j$, set $L(i) = \{\inf G_j\}$



Approximation Poisson