Uniqueness & Non-Uniqueness for the Mean Field Control of Fisheries

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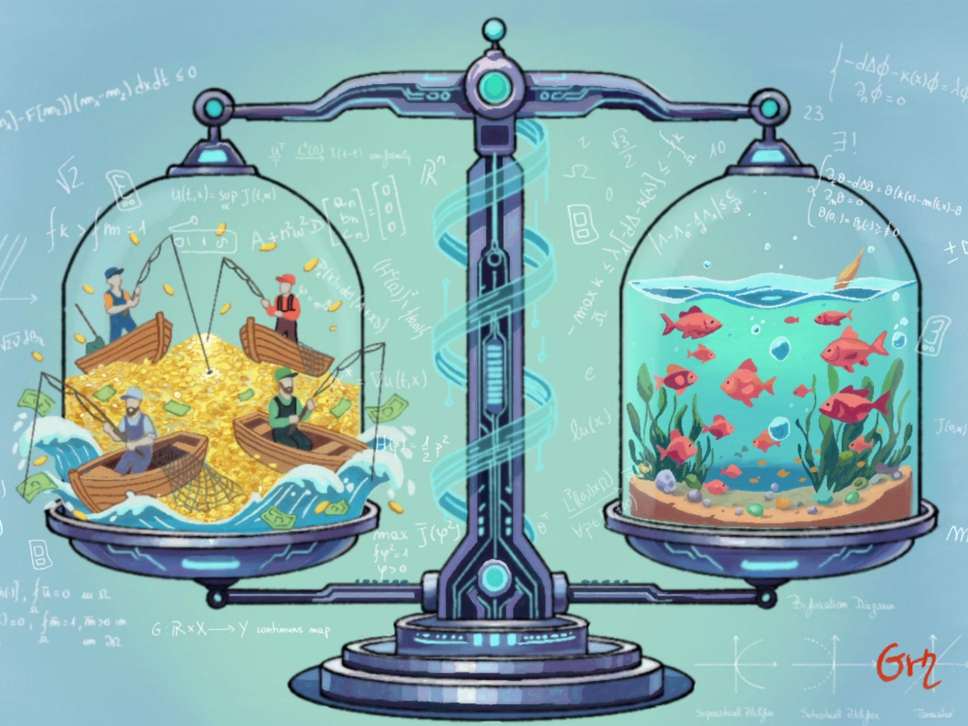
u(t,x) = sup J(t,x)
∇u(t,x)
H(p) = ½ p²
ln(x)
max J(φ²)
G : ℝ×X ⟶ Y continuous map
Bifurcation Diagram
Grh

Goal in a coordinated scenario

Assume a central planner, who selects and imposes the strategies of the players. No objections, no uncertainness due to the anticipation of others' strategies.

The goal is

$$\sup_{\alpha \in L^2((0;T)\times\Omega)} J^T_{\text{coop}}(\alpha), \quad J^T_{\text{coop}}(\alpha) := \int_0^T \int_\Omega \left(\theta\, m - \frac{1}{2}|\alpha|^2\, m \right) dx\, dt \tag{1}$$

where

$$\begin{cases} \partial_t \theta - d\Delta\theta - \theta(k(x) - m - \theta) = 0, \ \theta > 0 & \text{in } (0,T)\times\Omega, \\ \partial_t m - \nu\Delta m + \nabla\cdot(m\,\alpha) = 0, \ \fint_\Omega m = 1 & \text{in } (0,T)\times\Omega, \\ \partial_n \theta = \partial_n m = 0 & \text{on } \partial\Omega, \\ \theta(t=0,\cdot) = \theta_0(\cdot) \geq 0, \not\equiv 0, \quad m(t=0,\cdot) = m_0(\cdot) & \text{in } \Omega, \end{cases}$$

$d, \nu > 0$ diffusivity and volatility coefficients. θ density of fish, m density of fishermen.

Mean Field Control System [6]

From calculus of variations, the best strategy is $\alpha^*(t,x) = \nabla u(t,x)$ and we can write the following MFC system

$$\begin{cases} -\partial_t u - \nu\Delta u - \frac{1}{2}|\nabla u|^2 = \theta\left(1-\eta\right) & \text{in } (0,T)\times\Omega, \\ \partial_t m - \nu\Delta m + \nabla\cdot(m\nabla u) = 0,\ \fint_\Omega m = 1 & \text{in } (0,T)\times\Omega, \\ \partial_t\theta - d\Delta\theta - \theta\left(k(x) - m - \theta\right) = 0,\ \theta > 0 & \text{in } (0,T)\times\Omega, \\ -\partial_t\eta - d\Delta\eta - \eta\left(k(x) - m - 2\theta\right) = m,\ \eta > 0 & \text{in } (0,T)\times\Omega, \\ \partial_n u = \partial_n m = \partial_n\theta = \partial_n\eta = 0 & \text{on } \partial\Omega, \\ u(T) = 0,\ m(0) = m_0,\ \theta(0) = \theta_0,\ \eta(T) = 0 & \text{in } \Omega. \end{cases} \quad \text{(MFC)}$$

As $T\to+\infty$, there should hold

$$u_T(t,x) \sim \bar\lambda + \bar u(x), m_T \sim \bar m, \theta_T \sim \bar\theta, \eta_T \sim \bar\eta,$$

where $(\bar\lambda, \bar u, \bar\eta, \bar m, \bar\theta)$ solves the so-called “ergodic system”.

[6] Z. Kobeissi, I. Mazari-Fouquer, and D. Ruiz-Balet. “Mean-field games for harvesting problems: Uniqueness, long-time behaviour and weak KAM theory”. (2024); “The tragedy of the commons: a Mean-Field Game approach to the reversal of travelling waves”. (2024).

Main model in $\Omega := (0;\ell)$

It will be convenient to use the Hopf-Cole transform $\bar{\varphi} = e^{\frac{\bar{u}}{2\nu}}$ to get rid of one of the equations in the ergodic system. Solving the equation on $\bar{m}$ we obtain $\bar{\varphi}^2 = \bar{m}$, and the ergodic system reduces to

$$\begin{cases} -2\nu^2\bar{\varphi}'' - \bar{\theta}(1-\bar{\eta})\bar{\varphi} = -\bar{\lambda}\,\bar{\varphi},\ \fint_0^\ell \bar{\varphi}^2 = 1,\ \bar{\varphi} > 0 & \text{in } (0;\ell), \\ -d\bar{\theta}'' - \bar{\theta}(k - \bar{\varphi}^2 - \bar{\theta}) = 0, \quad \bar{\theta} > 0 & \text{in } (0;\ell), \\ -d\bar{\eta}'' - \bar{\eta}(k - \bar{\varphi}^2 - 2\bar{\theta}) = \bar{\varphi}^2, \quad \bar{\eta} > 0 & \text{in } (0;\ell). \end{cases} \tag{S}$$

Assume $k > 1$, we can observe that

$$\bar{\varphi} \equiv 1, \quad \bar{\lambda} = \bar{\theta}(1-\bar{\eta}) = (k-1)\left(1 - \frac{1}{k-1}\right) = k-2,$$
$$\bar{\theta} = k-1,$$
$$\bar{\eta} = \frac{1}{k-1}.$$

A particular solution of (S) is $\left(\bar{\theta}, \bar{\eta}, \bar{\lambda}, \bar{\varphi}\right) = \left(k-1, \frac{1}{1-k}, k-2, 1\right)$.

Results: Uniqueness & Non-Uniqueness

Theorem 1 (Uniqueness of solution for k large. G.L., I.Mazari-Fouquer, G.Nadin)

For any $d, \nu, \ell > 0$ there exists $k_0 > 2$ such that for any $k \geq k_0$ the system (S) admits a unique solution.

The strategy consists in finding regimes where the Lasry-Lions monotonicity condition holds.

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Theorem 2 (Multiple solutions for (S). G.L., I.Mazari-Fouquer, G.Nadin)

For any $d, \nu > 0$, there exists $k^* \in (1;2)$ such that, for any $k_1 \in (1;k^*)$, there exists $\ell(k_1) > 0$ such that for any $\epsilon > 0$ there exists $k \in (k_1 - \epsilon; k_1 + \epsilon)$ for which system (S) admits a $\ell(k_1)-$periodic non-constant solution.

This result is proved using bifurcation theory.

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Corollary 3

For any $d, \nu > 0$, there exist $\ell(k_1) > 0$ and $k_0, k_1 > 1$ such that, for $k = k_1$, (S) has at least two solutions, while for any $k \geq k_0$ (S) has a unique solution.

Some comments and heuristics on Theorem 2

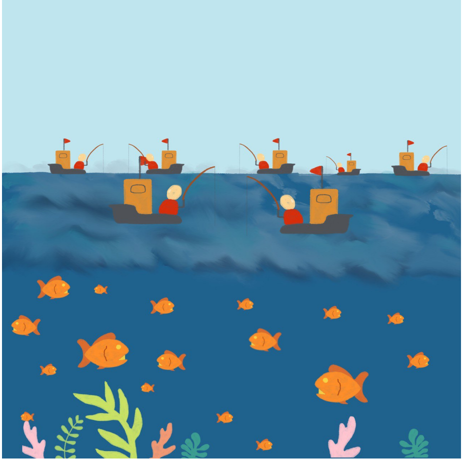

Figure: Effect of overfishing in cooperative scenario: if $k < 2$, $\bar{\lambda} = \bar{\theta}(1 - \bar{\eta}) = k - 2 < 0$. We have an unbalanced ecosystem: $1 - \bar{\eta} < 0$.

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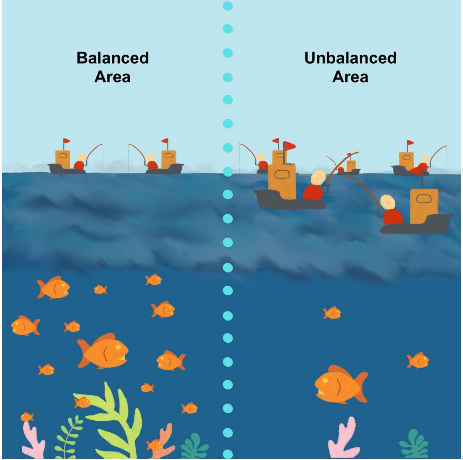

Figure: Effect of overfishing in cooperative scenario: $1-\bar{\eta}>0$ on the left zone, $1-\bar{\eta}<0$ on the right.

How to build multiple solutions mathematically?

- The **Lasry-Lions monotonicity condition** is only sufficient for the uniqueness of solution, but not necessary.
- To prove the existence of **multiple solutions**, we have to explicitly build a family of non-constant functions that still satisfy (S).
- Strategy: we consider an oscillation of the constant solution $\left(\bar{\theta},\bar{\eta},\bar{\lambda},\bar{\varphi}\right)=\left(k-1,\frac{1}{1-k},k-2,1\right)$ of (S).
- Use the Crandall-Rabinowitz theorem on an appropriate function.

Some remarks on numerical simulations

- Numerical simulations suggest a (global) pitchfork shape.
- We prove that the bifurcation is not transcritical.
- Numerical simulations show that the static gain $\bar{J} = \int_0^\ell \bar{\theta}\bar{\varphi}^2 - 2\nu^2 \left|\bar{\varphi}'\right|^2$ is strictly greater for the oscillatory solutions rather than the constant one.

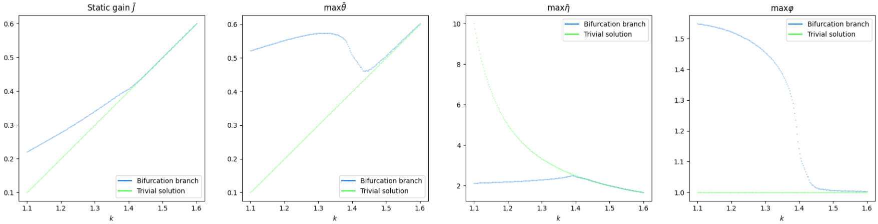

Figure: Characterization of the bifurcation branch.

2D numerical simulations

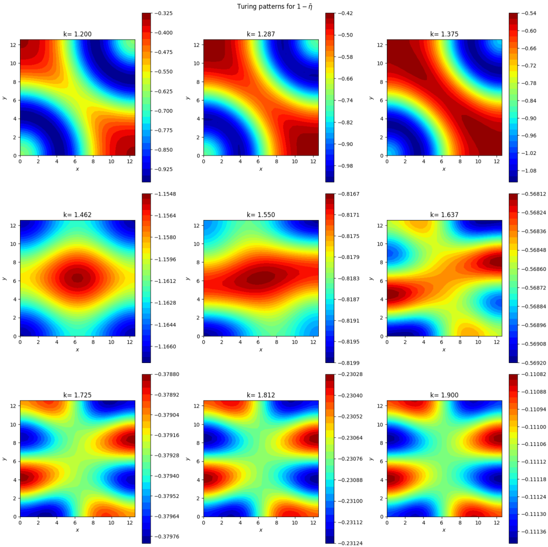

Figure: Turing patterns for $1-\bar{\eta}$ in $(0;4\pi)^2$, $d=2\nu^2$, $k\in(1.2;1.9)$.

Conclusion & Perspectives

- Too few resources: unbalanced ecosystem and **multiple oscillating** solutions occur.
- Enough resources: balanced ecosystem and the **unique** solution is the constant one.
- For constant k, in contrast with the competitive model (Grégoire's lecture), we can obtain multiple solutions that seem to improve the static gain.
- It is expected that the original problem converges to the static (S) as $T \to +\infty$, but this is still open even more when uniqueness does not hold.


Thank you!

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