Reaction-diffusion equations in population dynamics

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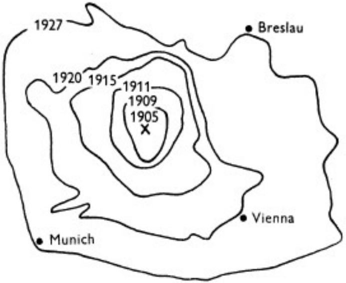
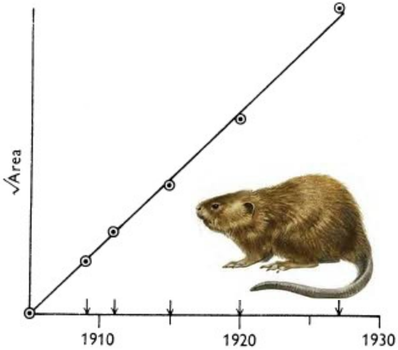
Muskrat invasion in Estern Europe





[Skellam 1951]

Biological invasion: sustainable increase of the area occupied by a population

The Fisher-KPP reaction-diffusion equation

$$\partial_t u - D\Delta u = \mu u\left(1 - \frac{u}{K}\right)$$

[Fisher 38, Kolmogorov-Petrovsky-Piskunov 37]

$u = u(t,x)$: population density at time t and space location x

$\partial_t u$: partial derivative of u with respect to t

$\Delta u := \sum_{i=1}^{d} \partial_{x_i x_i} u$: Laplacian of u in x
If the spatial dimension $d = 1$, then $\Delta u = \partial_{xx} u$

The Fisher-KPP reaction-diffusion equation

$$\underbrace{\partial_t u - D\Delta u}_{Brownian\ diffusion} = u \ \underbrace{\mu\left(1-\frac{u}{K}\right)}_{Growth\ rate}$$

$u = u(t,x)$: population density at time t and space location x

$D > 0$: diffusion coefficient

$\mu u\left(1-\frac{u}{K}\right)$: logistic growth term, intraspecific competition for limited ressources

$\mu > 0$: intrinsic growth rate at small density

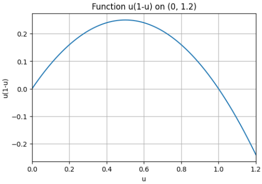
$K > 0$: carrying capacity
We could normalize $K = 1$ by letting $\widetilde{u} := u/K$.

The logistic growth rate

Assume that the population density $u = u(t)$ only depends on t:

$$\frac{du}{dt} = \mu u\left(1 - u\right)$$

2 steady states : $\{\ 0\ ;\ 1\ \}$
- 1 is globally attractive
- 0 is unstable



The unstability of 0 is unrealistic in some regimes ⇒ Allee effect.

The spreading property

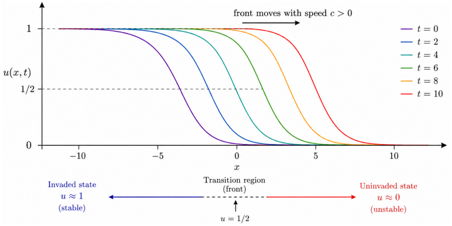
$$\partial_t u - D\Delta u = \mu u\left(1-u\right)$$

Theorem (Fisher 38, KPP 37, Aronson-Weinberger 78)

Take $u_0 = u(0,\cdot) \not\equiv 0$ *compactly supported with* $0 \le u_0 \le 1$*, then:*

$$\begin{cases} \lim_{t\to+\infty} u(t,x) = 1 & \textit{unif. in } |x| \le ct \quad \textit{if } 0 \le c < 2\sqrt{D\mu}, \\ \lim_{t\to+\infty} u(t,x) = 0 & \textit{unif. in } |x| \ge ct \quad \textit{if } c > 2\sqrt{D\mu}, \end{cases}$$

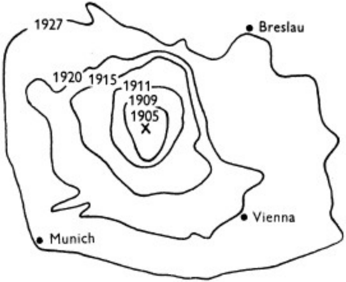
$c_* = 2\sqrt{D\mu}$ *is called the* ***spreading speed****.*

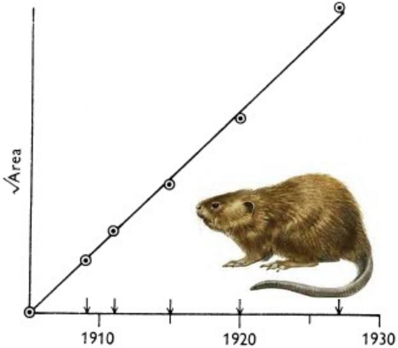


In dimension 2, the surface occupied by the population at time t is approximately $\pi(c_* t)^2 = 4\pi D\mu t^2$.

Muskrat invasion in Estern Europe

In dimension 2, the surface occupied by the population at time t is approximately $\pi(w^* t)^2 = 4\pi D\mu t^2$.





[Skellam 1951]

Same approach in other invasion phenomena

Confirmed with further measurements in [Andow-Kareiva-Levin-Okubo 1990], who used this model in other frameworks, with estimates of order of magnitude 10. The difficulty lies in estimating the diffusion coefficient D.

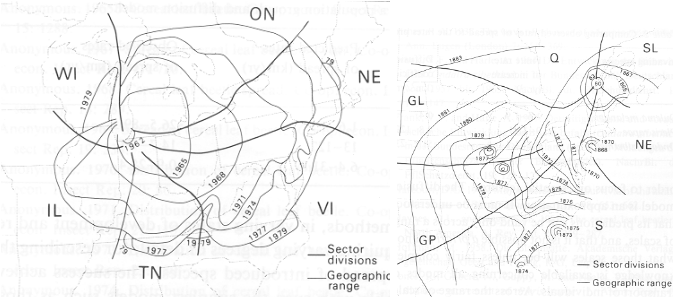


Figure: Left: Spread of cereal leaf beetle in Michigan. Right: Spread of small cabbage white butterfly in Quebec.

Heterogeneity of the environment should play a role

Reaction-diffusion equations in other frameworks

$$\underbrace{\partial_t u - D\Delta u}_{Brownian\ diffusion} = \underbrace{f(u)}_{Reaction\ term}$$

Simple model, used in

- **Ecology**: Refined models using nonlocal dispersal kernels [Lewis-Pacala 00] or Allee effects.
- **Archaeology**: Successfully used in [Ammerman - Cavalli-Sforza 84] to explain the spread of agriculture in Europe.
- **Genetics**: x is not a space but a trait variable that undergoes small mutations [Champagnat, Diekmann, Ferrière, Jabin, Méléard, Mirrahimi, Mischler, Perthame, Raoul...] → integral terms.
- **Combustion:** u is the temperature, $f(u) = 0$ if $0 \leq u \leq \theta$ where θ is an ignition temperature [Zel'dovich - Frank-Kamenetskii 38, Kanel' 61].
- **Epidemiology**: See after.
- etc.

Reaction diffusion systems

N species. $u_i = u_i(t,x)$: population density of species i

$$\partial_t u_i - D_i \Delta u_i = F_i(u_1, ... u_N)$$

Example: Lotka-Volterra

$$F_i(u_1, ..., u_N) = u_i(\mu_i + \alpha_{i,1} u_1 + ... + \alpha_{i,N} u_N)$$

where $\alpha_{i,j}$, for $i \neq j$, could be all positive (mutualistic), all negative (competitive), some might be negative, others positive (prey-predator).

Very difficult to investigate in general. Lack of comparison principle

Some general results exist for

- **mutualistic systems**: $\alpha_{i,j} > 0$ if $i \neq j$
- **gradient systems**: $F_i(u_1, ..., u_N) = \partial_{u_i} V(u_1, ..., u_N)$ for some potential V (physics models, or $V = f_1(u_1) + f_2(u_2) + \alpha u_1^2 u_2^2$).

Remark: If $N = 2$ (two species), competitive models could be turned into a mutualistic model. Used in [Okubo-Maini-Williamson-Murray 89] about Grey squirrel invasion in Britain.

The probabilistic approach: branching Brownian motion

"Consider the following process:

- *a particle starts at $x = 0$ and moves according to a Brownian motion.*
- *After a random time distributed as an exponential with parameter μ, the particle splits into two particles.*
- *These new particles move as independent Brownian motions and also split after independent exponential lifetimes, etc. "*

[J. Berestycki's lecture '14]

Let $u(t,x)$ the probability that at time t, there is at least on particle at the right of position x. One can prove that

$$\partial_t u - \frac{1}{2}\partial_{xx} u = \mu u(1-u) \text{ in } (0,\infty)\times\mathbb{R}, \quad u(0,x) = 1_{x\le 0}.$$

[Mc Kean 1975]

This observation has been used to find the next term in the location of the invading front [Bramson 1983]:

$$c^* t - \frac{3}{2^{3/2}\sqrt{\mu}} \ln t \quad \text{with } c^* = \sqrt{2\mu}.$$

Lecture outline

- I. Persistence and extinction in heterogeneous Fisher-KPP models, influence of fragmentation
- II. Influence of fragmentation on other relevant quantities in Fisher-KPP models
  - 1. Spreading speeds in Fisher-KPP models
  - 2. Optimal shape of the habitat in multidimensional media
  - 3. Optimal shape of the habitat for the total population at equilibrium
- III. Mean-field models for fishery management
- IV. Allee effect and invasion blocking

I. Persistence and extinction in heterogeneous Fisher-KPP models

Influence of fragmentation

The heterogeneous Fisher-KPP equation

$$\partial_t u - D\Delta u = u\left(\mu(x) - u\right)$$

$\mu(x)$: intrinsic growth rate at small density for individuals located in x

$\mu(x) > 0$: favourable area / habitat
$\mu(x) < 0$: unfavourable area

We could address heterogeneity in D as well.

Questions

1. Could we find a criterion for persistence/extinction of the population, i.e existence and attractiveness of a positive steady state ?
2. Could we quantify the influence of the "fragmentation of the habitat" on persistence using this model?

More general Fisher-KPP reaction terms

$$\partial_t u - \Delta u = f(x,u)$$

$f(x,u)$: growth rate, depends on the space variable x

$f(x,0) = 0$: no new individuals if $u = 0$

$\mu(x) := f'_u(x,0)$: growth rate per capita at small density ($f(x,u) \simeq \mu(x)u$ if u is small)

$\mu(x) > 0$: favourable area / habitat
$\mu(x) < 0$: unfavourable area

Hypotheses:

- $u \mapsto f(x,u)/u$ decreasing (intraspecific competition)
- $\exists M > 0 \mid \forall x,\ f(x,M) \leq 0$ (saturation)

Example: logistic growth rate $f(x,u) = u\big(\mu(x) - u\big)$

The SIR model in epidemiology

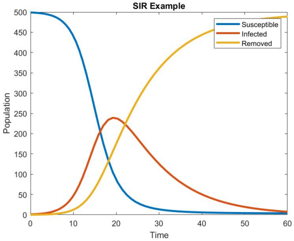
Introduced in [Ross 16, Kermack-McKendrick 27]

$$\begin{cases} \partial_t S &= -\beta S I \\ \partial_t I &= \beta S I - \gamma I \\ \partial_t R &= \gamma I \end{cases}$$

S: susceptible, I: infected, R: recovered
β: average number of transmissions per person per time unit
$1/\gamma$ = average infectious period
$\mathscr{R}_0 := \beta/\gamma$: basic reproduction number

Properties:

- $S + I + R = cst$
- if $\mathscr{R}_0 > 1$: outbreak of the disease, if $\mathscr{R}_0 < 1$: extinction



The SIR diffusive model

$$\begin{cases} \partial_t S & & = & -\beta(x) S I \\ \partial_t I & -D\Delta I & = & \beta(x) S I - \gamma I \\ \partial_t R & & = & \gamma I \end{cases}$$

with $S(t=0,x) = S_0(x), I(t=0,x) = I_0(x), R(t=0,x) = 0.$

From the 1st line, we get:

$$S(t,x) = S_0(x) e^{-\beta(x)\int_0^t I(s)ds}.$$

Let multiply the 2nd line by γ and integrate from 0 to t:

$$\gamma I(t,x) - \gamma I_0(x) - D\gamma\Delta\Big(\int_0^t I(s,x)ds\Big)$$
$$= \gamma \int_0^t \Big(\beta(x) S_0(x) e^{-\beta(x)\int_0^s I(\cdot,x)} I(s,x) - \gamma I(s,x)\Big) ds.$$

Hence, as $\partial_t R = \gamma I$:

$$\begin{aligned} \partial_t R - D\Delta R &= \gamma I_0(x) + \gamma \int_0^t \Big(S_0(x) e^{-\beta\int_0^s I(\cdot,x)} \beta I(s,x) - \gamma I(s,x)\Big) ds \\ &= \gamma I_0(x) + \gamma S_0(x)\big(1 - e^{-\frac{\beta(x)}{\gamma}R}\big) - \gamma R. \end{aligned}$$

[Diekmann '77]

The SIR diffusive model

We have thus proved that the diffusive SIR equation

$$\begin{cases} \partial_t S & & = & -\beta(x) S I \\ \partial_t I & -D\Delta I & = & \beta(x) S I - \gamma I \\ \partial_t R & & = & \gamma I \end{cases}$$

with $S(t=0,x) = S_0(x), I(t=0,x) = I_0(x), R(t=0,x) = 0$.

reduces to the equivalent Fisher-KPP equation

$$\begin{aligned} \partial_t R - D\Delta R &= I_0(x) + \gamma S_0(x)\big(1 - e^{-\frac{\beta(x)}{\gamma}R}\big) - \gamma R \\ &= I_0(x) + f(x,R) \end{aligned}$$

with $f(x,0) = 0$, $f(x,M) \leq 0$ for M large, and $r > 0 \mapsto f(x,r)/r$ decreasing.

[Ducrot-Giletti '14, Ducasse '20]

The heterogeneous Fisher-KPP equation: heuristics

$$\partial_t u - D\Delta u = u\left(\mu(x) - u\right) \text{ in } \Omega, \quad \partial_n u = 0 \text{ on } \partial\Omega,$$

Ω open, bounded and smooth

When u is close to the steady state 0, the equation looks like

$$\partial_t u \simeq D\Delta u + \mu(x) u =: L_0 u.$$

L_0 is the linearized operator at $u = 0$.
$\lambda_1(\mu) :=$ **Neumann principal eigenvalue** of L_0 (Krein-Rutman theory):

$$\begin{cases} L_0\phi := D\Delta_x \phi + \mu(x)\phi = \lambda_1(\mu)\phi & \text{in } \Omega \\ \partial_n \phi = 0 \quad \text{on } \partial\Omega, \\ \phi > 0. \end{cases}$$

The eigenfunction ϕ is uniquely defined up to multiplication by a positive constant.

We expect $u(t,x) \simeq \phi(x) e^{\lambda_1(\mu) t}$ when $u(t,x)$ is small. Hence:

- 0 is a **stable** steady state if $\lambda_1(\mu) < 0$,
- 0 is an **unstable** steady state if $\lambda_1(\mu) > 0$.

The heterogeneous Fisher-KPP equation: persistence criterion

$$\partial_t u - D\Delta u = u\left(\mu(x) - u\right) \text{ in } \Omega, \quad \partial_n u = 0 \text{ on } \partial\Omega,$$

$\lambda_1(\mu) :=$ Neumann principal eigenvalue of L_0 (Krein-Rutman theory):

$$\begin{cases} D\Delta_x \phi + \mu(x)\phi = \lambda_1(\mu)\phi, \quad \phi > 0 \quad \text{in } \Omega, \\ \partial_n \phi = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Theorem

Ω *bounded smooth open set,* $\mu \in L^\infty(\Omega)$

- *If $\lambda_1(\mu) > 0$,* $\exists!$ *positive steady state. It is attractive for bounded initial data $u_0 \geq 0 \not\equiv 0$.* **(persistence)**
- *If $\lambda_1(\mu) \leq 0$, $u \equiv 0$ is globally attractive.* **(extinction)**

Sketch of proof when $\lambda_1(\mu) < 0$.

[Cantrell-Cosner 91, Berestycki-Hamel-Roques 05]

Some dependence results between the coefficients and the principal eigenvalue

$$\partial_t u - D\Delta u = u\left(\mu(x) - u\right) \text{ in } \Omega, \quad \partial_n u = 0 \text{ on } \partial\Omega$$

$$\begin{cases} D\Delta_x \phi + \mu(x)\phi = \lambda_1(\mu)\phi, \quad \phi > 0 \quad \text{in } \Omega, \\ \partial_n \phi = 0 \quad \text{on } \partial\Omega. \end{cases}$$

$\lambda_1(\mu) > 0$: persistence, $\lambda_1(\mu) \leq 0$: extinction

Proposition

1. $\lambda_1(\fint_\Omega \mu) = \fint_\Omega \mu \leq \lambda_1(\mu)$
2. *if* $\fint_\Omega \mu \geq 0$ *and* $\mu \not\equiv 0$, *then* $B \mapsto \lambda_1(B\mu)$ *is increasing*

Interpretation: There might be persistence for the habitat associated with μ (resp. $B_2\mu$) but extinction for its average $\fint \mu$ (resp. with $B_1\mu$ with $B_2 > B_1$). **Example:** $\mu(x) = B\sin x$.

⇒ **Heterogeneity of the habitat is favourable to persistence.**

[Berestycki-Hamel-Roques 05]

Fragmentation of the habitat

Habitat loss ⇒ emergence of discontinuities (*fragmentation*) in an organism's preferred environment (*habitat*).

Causes:

- Natural: geological processes, climate change.
- Human: agriculture, urban areas.

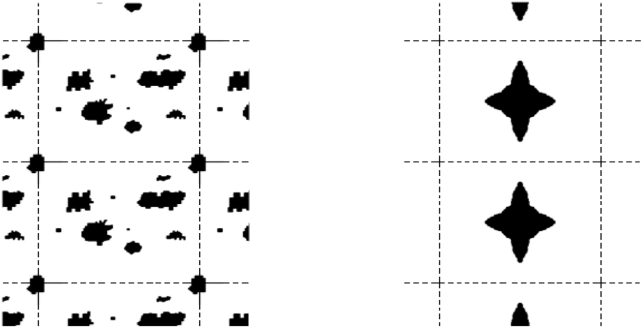
Effects: One of the main cause of *extinction* of species

- increased competition in remaining habitats
- size effects
- impossible immigration and rescue effects

Characterization of the “fragmentation“? Optimization of conservation strategies?

Fragmentation of the habitat

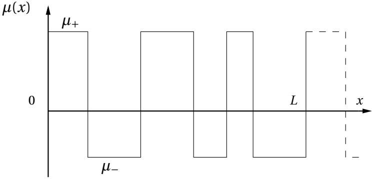
Habitat loss ⇒ emergence of discontinuities (*fragmentation*) in the habitat.

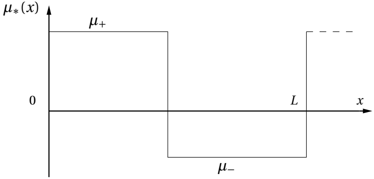


[Berestycki-Hamel-Roques 05]

The patch model

$$\mu(x) = \begin{cases} \mu^+ & \text{if} \quad x \in A \quad \text{“habitat”} \\ \mu^- & \text{if} \quad x \notin A \end{cases} \qquad \text{with } m^+ > m^-$$





a fragmented habitat μ the less fragmented habitat $\mu^\star$

The length of A is prescribed: it is the same in these two examples.

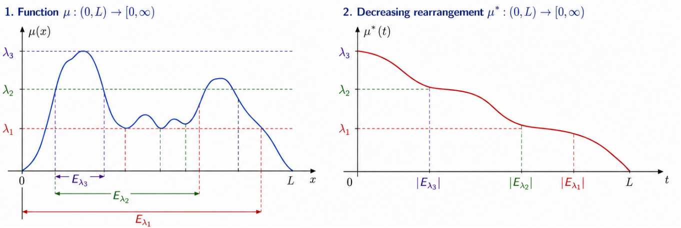
Definition of the Schwarz rearrangement

Definition

μ measurable bounded on $(0, L)$: $\exists!$ measurable $\mu^\star$, called the **decreasing rearrangement** *of μ,*

- *with the same distribution function,*
- *nonincreasing on $(0, L)$.*

Indeed, one has $\mu^\star(t) := \inf\big\{\lambda > 0, meas\{x \in (0, L), \mu(x) > \lambda\} \leq t\big\}$.



A continuous function μ and its Schwarz rearrangement $\mu^\star$.

Effect of the fragmentation on the persistence

$\lambda_1(\mu) > 0$: persistence
$\lambda_1(\mu) \leq 0$: extinction

Proposition

$$\lambda_1(\mu^\star) \geq \lambda_1(\mu)$$

Interpretation: There might be extinction for the habitat associated with μ but persistence for its rearrangement $\mu^\star$.

⇒ **Fragmentation of the habitat is unfavourable to persistence**

[Berestycki-Hamel-Roques 05]

A Faber-Krahn inequality

Proposition

$$\lambda_1(\mu^\star) \geq \lambda_1(\mu)$$

Proof in 1d. We recall that $\lambda_1(\mu)$ is defined by

$$\begin{cases} L_0\phi := D\phi'' + \mu(x)\phi = \lambda_1(\mu)\phi & \text{in } (0,L), \\ \phi'(0) = \phi'(L) = 0, \\ \phi > 0 \text{ in } (0,L). \end{cases}$$

$L_0\phi$ is self-adjoint, thus $\lambda_1(\mu)$ is a *Rayleigh quotient*:

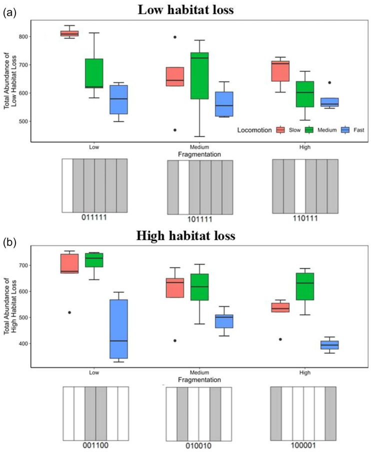
$$\lambda_1(\mu) = \max_{\alpha\in\mathscr{C}^1_{per}} \frac{< L_0\alpha, \alpha >_{L^2}}{< \alpha, \alpha >_{L^2}} = \max_{\alpha\in\mathscr{C}^1_{per}} \frac{1}{\int_0^L \alpha^2} \int_0^L (-D\alpha'^2 + \mu(x)\alpha^2).$$

Conclusion with two classical properties of rearrangements:

$$\int_0^L \mu^\star (\alpha^\star)^2 \geq \int_0^L \mu\alpha^2 \quad \text{(Hardy-Littlewood ineq.)}$$

$$\int_0^L (\alpha^\star)'^2 \leq \int_0^L \alpha'^2 \quad \text{(Polya-Szego ineq.)}$$

Experiments



Plates are covered with *E. Coli* except on white stripes. The total abundance of *C. Elegans* is measured.

"The main effect of fragmentation was also significant (p = 0.0257, F = 4.098). The slow-moving strain had significantly greater abundance under low levels of fragmentation than the slow-moving strain under high fragmentation (p = 0.0270, t = 3.1342)."

[Jacobs-Salmaniw-Lam-Zhai-Wang-Zhang '24]

Phenotypically-structured populations

Assume that $x \in \Omega \subset \mathbb{R}^d$ represents a phenotypical trait, that mutates randomly at rate D, and consider the following model

$$\partial_t u - D\Delta u = u\left(\mu(x) - \rho_u(t)\right) \text{ in } \Omega, \quad \partial_n u = 0 \text{ on } \partial\Omega$$

where $\rho_u(t) := \int_\Omega u(t,x)dx$. This nonlocal term arises because the individuals with trait x compete with all other individuals, not only the ones with trait x, in order to get resources.

[Champagnat-Ferrière-Méléard '07, Desvillettes-Jabin-Mischler-Raoul '08, Barles-Mirrahimi-Perthame '09, etc.]

One can prove that

- $u(t,x) \to 0$ as $t \to +\infty$ if $\lambda_1(\mu) > 0$,
- $u(t,x) \to \phi(x)$ as $t \to +\infty$ if $\lambda_1(\mu) \leq 0$,

$$\text{where } \begin{cases} D\Delta_x \phi + \mu(x)\phi = \lambda_1(\mu)\phi, \quad \phi > 0 & \text{in } \Omega, \\ \partial_n \phi = 0 \quad \text{on } \partial\Omega, \end{cases}$$

and ϕ is normalized by $\int_\Omega \phi = \lambda_1(\mu)$.
Hence, $\lim_{t\to+\infty} \int_\Omega u(t,\cdot) = \lambda_1(\mu)_+$ and thus an environment μ that maximizes $\lambda_1(\mu)$ maximizes the total population at equilibrium.

Conclusions

1. Could we find a criterion for persistence/extinction of the population, i.e existence and attractiveness of a positive steady state ?
   $\Rightarrow$ Yes. Persistence is characterized through the positivity of the principal eigenvalue of the linearized operator near $u \simeq 0$. This is realistic only if the Allee effect is negligible.
2. Could we quantify the influence of the "fragmentation of the habitat" on persistence using this model?
   $\Rightarrow$ The decreasing rearrangement $\mu^\star$ could be associated with the less fragmented habitat, and it maximizes the principal eigenvalue, which determines the persistence.

Next question: Consider two habitats in which the species persists. What is the "best" habitat for this species?

II. Influence of fragmentation on other relevant quantities in Fisher-KPP models

Questions

1. When the population persists, could we find notions quantifying this persistence?
2. Could we quantify the influence of the "fragmentation of the habitat" on these quantities?

II.1 Spreading speeds in Fisher-KPP models

The spreading property in homogeneous media

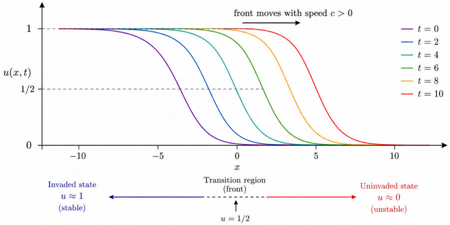
$$\partial_t u - D\Delta u = \mu u\left(1-u\right)$$

Theorem (Fisher 38, KPP 37, Aronson-Weinberger 78)

Take $u_0 = u(0,\cdot) \not\equiv 0$ *compactly supported with* $0 \leq u_0 \leq 1$*, then:*

$$\begin{cases} \lim_{t\to+\infty} u(t,x) = 1 & \text{unif. in } |x| \leq ct \quad \text{if } 0 \leq c < 2\sqrt{D\mu}, \\ \lim_{t\to+\infty} u(t,x) = 0 & \text{unif. in } |x| \geq ct \quad \text{if } c > 2\sqrt{D\mu}, \end{cases}$$

$c_* = 2\sqrt{D\mu}$ *is called the* **spreading speed***.*



In dimension 2, the surface occupied by the population at time t is approximately $\pi(c_* t)^2 = 4\pi D\mu t^2$.

The spreading property in periodic media

$$\partial_t u - d\partial_{xx} u = u\left(\mu(x) - u\right), \quad x \in \mathbb{R}$$

μ periodic in x: $\mu(x+L) = \mu(x)$.

Persistence criterion: $\lambda_1(\mu) > 0$. Let $\bar{u}$ the unique positive steady state:

$$-d\bar{u}'' = \bar{u}\big(\mu(x) - \bar{u}\big), \quad x \in \mathbb{R}$$

Theorem

(1d) For all $u_0 \not\equiv 0$ initial datum with compact support,

$$\begin{array}{lll} \lim_{t\to+\infty} u(t,x) = \bar{u}(x) & \textit{unif. } |x| \le ct & \textit{if } 0 \le c < c(\mu), \\ \lim_{t\to+\infty} u(t,x) = 0 & \textit{unif. } |x| \ge ct & \textit{if } c > c(\mu), \end{array}$$

where

$$c(\mu) = \min_{p>0} \frac{k_p(\mu)}{p}.$$

[Freidlin-Gartner 79, Weinberger 04, Berestycki-Hamel-N. 08]

Numerical illustrations

periodic

almost periodic

random

The spreading property in periodic media

Theorem

(1d) For all $u_0 \not\equiv 0$ initial datum with compact support,

$$\begin{array}{lll} \lim_{t\to+\infty} u(t,x) = \bar{u}(x) & \text{unif. } |x| \leq ct & \text{if } 0 \leq c < c(\mu), \\ \lim_{t\to+\infty} u(t,x) = 0 & \text{unif. } |x| \geq ct & \text{if } c > c(\mu), \end{array}$$

where

$$c(\mu) = \min_{p>0} \frac{k_p(\mu)}{p}$$

and $\bar{u}(x)$ is the unique positive steady state.

$$L_p\phi := e^{px} L_0\big(e^{-px}\phi\big) = \phi'' - 2p\phi' + (p^2 + \mu(x))\phi$$

$k_p(\mu)$ = periodic principal eigenvalue of L_p:

$$\begin{cases} L_p\phi = k_p(\mu)\phi, \\ \phi > 0, \\ \phi \text{ periodic.} \end{cases}$$

Example: if k constant, then $\phi = 1$, $k_p = \mu + p^2$, $c(\mu) = \min_{p>0} \frac{\mu+p^2}{p} = 2\sqrt{\mu}$.

Fragmentation slows down the invasion

Definition

μ measurable bounded on $(0,L)$: $\exists!$ measurable $\mu^\star$, called the **decreasing rearrangement** *of μ,*

- *with the same distribution function,*
- *nonincreasing on $(0,L)$.*

Theorem

$$c(\mu) \leq c(\mu^\star)$$

[N. 09]

- **The less fragmented habitat yields the largest invasion speed.**
- Solves an open problem from [Shigesada-Kawasaki-Teramoto 86].
- Limits of the model: due to Allee effect, very large unfavourable areas should block the invasion [Lewis-Keener 00, Berestycki-Rodriguez-Ryzhik '13, N.-Strugarek-Vauchelet '18], see also [Ding-Hamel-Zhao '13] for a study of bistable pulsating traveling waves in periodic media with large/small periods.

Heuristics for the spreading speed characterization

Where does the formula $c(\mu) = \min_{p>0} \frac{k_p(\mu)}{p}$ come from?

$$\begin{cases} L_p\phi = \phi'' - 2p\phi' + (p^2 + \mu(x))\phi = k_p(\mu)\phi \text{ in } \mathbb{R}, \\ \min_{(0,L)} \phi = 1, \quad \phi \text{ periodic.} \end{cases}$$

The invasion is "pulled" by what happens far ahead of the front

$$\partial_t u - \partial_{xx} u = u\Big(\mu(x) - u\Big) \simeq \mu(x)u, \text{ where } u \simeq 0$$

Assume $u_0(x) = 0$ if $x \geq 0$, then

- $\bar{u}(t,x) := \|u_0\|_\infty \phi(x) e^{-p(x-c_p t)}$ solution of the linearized eq. if $c_p := k_p/p$,
- $\bar{u}(0,x) \geq u_0(x)$ for all $x \in \mathbb{R}$.
- As $u\big(\mu(x) - u\big) \leq \mu(x)u$, it follows from the comparison principle that $u(t,x) \leq \bar{u}(t,x)$.

Thus, if $c > c(\mu)$, taking p such that $c(\mu) < k_p/p = c_p < c$, one gets

$$\forall x \geq ct,\ u(t,x) \leq \bar{u}(t,x) = \|u_0\|_\infty \phi(x) e^{-p(x-c_p t)} \leq \|u_0\|_\infty \|\phi\|_\infty e^{p(c_p - c)t} \to 0$$

as $t \to +\infty$. The difficult part is to address the cv to $\bar{u}(x)$ when $|x| \leq ct$ with $c < c(\mu)$.

Reformulation of the problem

$$L_p\phi = \phi'' - 2p\phi' + (p^2 + \mu(x))\phi$$

$k_p(\mu)$ = periodic principal eigenvalue of L_p:

$$\begin{cases} L_p\phi = k_p(\mu)\phi, \\ \phi > 0, \\ \phi \text{ periodic.} \end{cases}$$

$$c(\mu) = \min_{p>0} \frac{k_p(\mu)}{p}$$

$\Rightarrow$ If $k_p(\mu^\star) \geq k_p(\mu)\ \forall p$, then $c(\mu^\star) \geq c(\mu)$.

Reformulation of the problem Prove that:

$$k_p(\mu^\star) \geq k_p(\mu)$$

where $\mu^\star$ is the Schwarz rearrangement of μ.

Comparison with the case $p = 0$

When $p = 0$, $k_0(\mu) = \lambda_1(\mu)$ and we have proved before that

$$k_0(\mu^\star) \geq k_0(\mu)$$

[Berestycki-Hamel-Roques 05]

Issues when $p \neq 0$:

- No Rayleigh quotient since L_p is not symmetric.

$$L_p\phi = \phi'' - 2p\phi' + (p^2 + \mu(x))\phi.$$

- Rearrangement properties are integral ones.
- Very few litterature on the rearrangement of non-symmetric operators [Alvino-Trombetti-Lions 90-91, Hamel-Nadirashvili-Russ 05-07]

→ Find an integral characterization of $k_p(\mu)$.

An integral characterization of $k_p(\mu)$

Proposition

$$k_p(\mu) = \max_{\alpha\in\mathscr{C}^1_{per}} \frac{1}{\int_0^L \alpha^2}\Big(\int_0^L \big(-\alpha'^2+\mu(x)\alpha^2\big) + p^2 \frac{L^2}{\int_0^L \frac{1}{\alpha^2}}\Big)$$

[N. 09]

Corollary

$k_p(\mu^\star) \geq k_p(\mu)$ *for all* p *and thus* $c(\mu^\star) \geq c(\mu)$.

Proof. Follows from:

$$\int_0^L \mu^\star(\alpha^\star)^2 \geq \int_0^L \mu\alpha^2 \quad \text{and} \quad \int_0^L (\alpha^\star)'^2 \leq \int_0^L \alpha'^2,$$

and $\int_0^L \frac{1}{\alpha^2} = \int_0^L \frac{1}{(\alpha^\star)^2}$ since the rearrangement preserves the distribution function. □

Other dependence results

$$\partial_t u - \nabla\cdot\big(D(x)\nabla u\big) + q(x)\cdot\nabla u = u\,\Big(\mu(x) - u\Big), \quad x\in\mathbb{R}^N$$

$D(x)$ diffusion matrix field, $q(x)$ advection term

Spreading speed $c(D,q,\mu)$ under hypotheses

- $B \mapsto c(BD, 0, \mu_0)$ increasing if μ_0 cst [Ber.-Hamel-Nadirashvili 10].
- $B \mapsto c(D, 0, B\mu)$ increasing w.r.t $B > 0$ if $⨍\mu \geq 0$ cst [Ber.-Hamel-Roques '05].
- $B \mapsto c(I_N, Bq, \mu_0)$: complex, depends on the flow of q [Audoly-Berestycki-Pomeau 01, Ryzhik-Kiselev 01, Ryzhik-Zlatos 07].
- Extensions to random stationary ergodic heterogeneities [Shigesada-Kawasaki-Teramoto 86, Nolen-Xin 05-07-09, N. 15].

II.2. Optimal shape of the habitat in multidimensional media

Shape of the optimal habitat in multidimensional media

$$\partial_t u - D\Delta u = u\left(\mu(x) - u\right), \quad x \in C, \quad \partial_n u = 0 \text{ on } \partial C$$

$C := \Pi_{i=1}^{N}(-L_i/2, L_i/2)$: cube

$$\mu(x) = \begin{cases} \mu^+ & \text{if} \quad x \in H \quad \text{“habitat”} \\ \mu^- & \text{if} \quad x \in C \backslash H \end{cases} \qquad \text{with } \mu^+ > \mu^-$$

What is the habitat $H \subset C$, with $|H|$ prescribed, that is the best for species persistence?

$\lambda_1(\mu)$ periodic principal eigenvalue of $L_0\phi := \Delta\phi + \mu(x)\phi$.

$$\text{Persistence} \quad \Leftrightarrow \quad \lambda_1(\mu) > 0$$

Reformulation: What is the habitat $H \subset C$, with $|H|$ prescribed, that maximizes $\lambda_1(\mu)$?

Shape of the optimal habitat in multidimensional media

$$\mu(x) = \begin{cases} \mu^+ & \text{if} \quad x \in H \quad \text{“habitat”} \\ \mu^- & \text{if} \quad x \in C \backslash H \end{cases} \qquad \text{with } \mu^+ > \mu^-$$

$\lambda_1(\mu)$ periodic principal eigenvalue of $L_0\phi := \Delta\phi + \mu(x)\phi$.

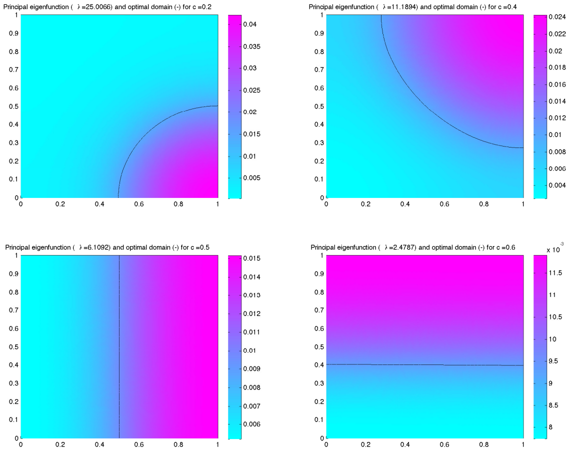
What is the habitat $H \subset C$, with $|H|$ prescribed, that maximizes $\lambda_1(\mu)$?

1. An optimal H is monotonic w.r.t x_i for all i [Berestycki-Hamel-Roques 05]

2. [Roques-Hamel 07] :
    - the maximizer H^* exists
    - it depends on $\mu^+ - \mu^-$ and $|H|$
    - it looks like stripes, quarters of balls or complementary of quarters of balls.

3. Stripes are local maximizers if $|H|$ small, local minimizers if $|H|$ large [Kao-Lou-Yanagida 07]

Numerical simulations


Principal eigenfunction (λ =25.0066) and optimal domain (-) for c =0.2

Principal eigenfunction (λ =11.1894) and optimal domain (-) for c =0.4

Principal eigenfunction (λ =6.1092) and optimal domain (-) for c =0.5

Principal eigenfunction (λ =2.4787) and optimal domain (-) for c =0.6

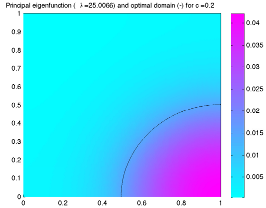
[Lamboley-Laurain-N.-Privat ’16]

Balls are not optimal

Theorem

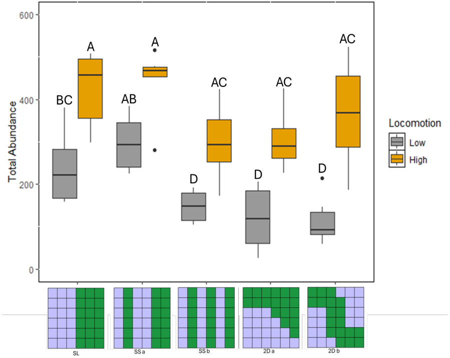
If the boundary of the optimal set is analytic, it does not contain any piece of sphere.

[Lamboley-Laurain-N.-Privat '16]



Principal eigenfunction (λ =25.0066) and optimal domain (-) for c =0.2

Experiments



Green represents areas with resources, blue represents areas with toxicants. The total abundance of *C. Elegans* is measured.

"We found that slow movers, which have low ability to escape or navigate toxic matrix, suffer strongly as patches become more fragmented (SSa → SSb → 2D variants) and decline significantly as fragmentation increases."

[Baragary - Mazari-Fouquer - N. - Hessler - Zhang submitted]

II.3. Optimal shape of the habitat for the total population at equilibrium

Maximization of the total population at equilibrium

Assume the population persists. What is the habitat that maximizes the total population at equilibrium?

Let $\int_\Omega \mu > 0$. Then $\lambda_1(\mu) 0$ for all $D > 0$ and ∃! positive steady state $\bar{u}$, which is globally attractive

$$\begin{cases} D\Delta\bar{u} + (\mu(x) - \bar{u})\,\bar{u} = 0 & x \in \Omega \\ \frac{\partial\bar{u}}{\partial\nu} = 0 & x \in \partial\Omega \end{cases}$$

[Cantrell-Cosner 91, Berestycki-Hamel-Roques 05]

$$G_D(\mu) := \int_\Omega \bar{u}[\mu] \quad \text{is the total population size at equilibrium}$$

$$\mathscr{B} := \left\{\mu,\ 0 \le \mu \le \kappa\,, \fint_\Omega \mu = \mu_0\right\} \quad \text{admissible growth rates}$$

Mathematical formulation: what is the growth rate $\mu \in \mathscr{B}$ that maximizes G_D? [Lou 08]

⚠G_D is not convex nor energetic w.r.t μ.

Bang-bang property of the maximizers

$$G_D(\mu) := \int_\Omega \bar{u}[\mu] \quad \text{is the total population size at equilibrium}$$

$$\mathscr{B} := \left\{ \mu \in L^\infty(\Omega), 0 \leq \mu \leq \kappa, \fint_\Omega \mu = \mu_0 \right\}$$

Theorem

Let $\overline{\mu}_D$ be a maximizer of G_D. Then there exists a measurable subset $H \subset \Omega$ such that

$$\overline{\mu}_D = \kappa 1_H.$$

[Mazari -N.- Privat '21]

The constraints are saturated / $\overline{\mu}_D$ is an extremal point of $\mathscr{B}$ (despite non-convexity of the functional).
Earlier results for smooth optimizers [Nagahara-Yanagida '18] and large D [Mazari-N.-Privat '20].

The case of large diffusivities: no fragmentation

Is it possible to characterize further the maximizers? Influence of fragmentation?

Theorem

Assume $d = 1$ *and* $\Omega = (0,1)$. *There exists* $\hat{D} > 0$ *s. t. for* $D \geq \hat{D}$, *the only maximizers are*

$$\kappa 1_{(0,\ell)} \quad \text{and} \quad \kappa 1_{(1-\ell,1)}$$

where $\ell = \frac{\mu_0}{\kappa}$.

[Mazari-N.-Privat '20]

Biological interpretation: If the species diffuses fast enough, it is better for it to gather all its habitat at the same place in order to maximize its total population at equilibrium.

Fragmentation may occur for small diffusivities

Proposition

Let $\mu_{cr} = \kappa 1_{(0,\ell)}$. *There exists* $D > 0$ *(small) such that*

$$G_D\big(\mu_{cr}(2\,\cdot)\big) > G_D(\mu_{cr}).$$

Remark: The maximizer is not the same as that of the principal eigenvalue in this case.

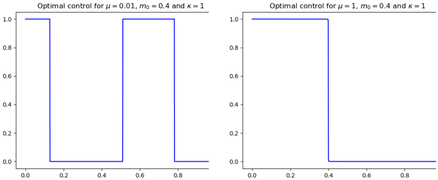
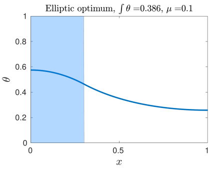


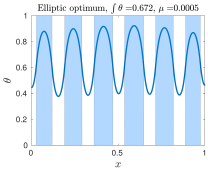
Figure: left $D = 0.01$, Right $D = 1$.

Biological interpretation: If the species diffuses slowly, it is better for it to spare its habitat since the carrying capacity will bound the population in large habitats.

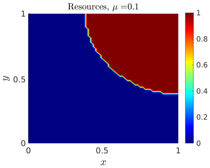
Numerics



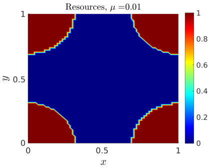
(a)



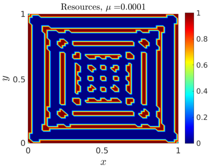
(b)



(c)



(d)



(e)

[Mazari - Ruiz-Balet 2020]

Quantification of the fragmentation rate

$\overline{\mu}_D = \kappa 1_H$, $\|\overline{\mu}_D\|_{BV(\Omega)} := Per(H)$.

Theorem

Consider a family of maximizers $\{\overline{\mu}_D\}_{D>0}$. There holds

$$\|\overline{\mu}_D\|_{BV(\Omega)} \xrightarrow[D\to 0^+]{} +\infty. \quad (1)$$

[Mazari - Ruiz-Balet 2020]

Theorem

Let $d \geq 1$ and let $\Omega = (0;1)^d$. There exist $C_0 > 0$ and $D_0 > 0$ such that, for any $D \in (0, D_0)$, if $\overline{\mu}_D$ is a maximizer of G_D, then

$$\|\overline{\mu}_D\|_{BV(\Omega)} \geq \frac{C_0}{\sqrt{D}}. \quad (2)$$

[Mazari - N. - Privat 2022]

The discrete case has been fully described in [Lou - Nagahara - Yanagida '21], see also [N. '23, Mazari N. in prep].

A meta-analysis on fragmentation

Annual Review of Ecology, Evolution, and Systematics

Ecological Responses to Habitat Fragmentation Per Se

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Keywords

landscape pattern, landscape structure, landscape configuration, landscape complementation, landscape connectivity, landscape heterogeneity, patch area, patch isolation, edge effect, SLOSS

"I found 118 studies reporting 381 significant responses to habitat fragmentation independent of habitat amount. Of these responses, 76% were positive. Most significant fragmentation effects were positive, irrespective of how the authors controlled for habitat amount, the measure of fragmentation, the taxonomic group, the type of response variable, or the degree of specialization or conservation status of the species or species group. No support was found for predictions that most significant responses to fragmentation should be negative in the tropics, for species with larger movement ranges, or when habitat amount is low; most significant fragmentation effects were positive in all of these cases." [Fahrig '17]

Conclusions

First lecture: $\lambda_1(\mu^\star) \geq \lambda_1(\mu)$ fragmentation of the habitat is unfavourable to persistence.

1. When the species persists, could we find notions quantifying this persistence?
2. Could we quantify the influence of the "fragmentation of the habitat" on these quantities?

- $c(\mu^\star) \geq c(\mu)$: the less fragmented habitat yields the largest **invasion speed**.
- **Multi-d**, patch model $\mu = \mu_+ \mathbf{1}_H$: difficult to state a general characterization of the set H^* maximizing λ_1.
- The **total population at equilibrium** is not always maximized by the less fragmented habitat, it depends on the diffusivity D.

Time-periodic reaction-diffusion equations

$$\partial_t u - D\Delta u = u\left(\mu(t,x) - u\right) \text{ in } \Omega, \quad \partial_n u = 0 \text{ on } \partial\Omega$$

$$\mu(t+T,x) = \mu(t,x) \quad \forall (t,x) \in \mathbb{R}\times\Omega$$

$$\begin{cases} -\partial_t\phi + D\Delta_x\phi + \mu(t,x)\phi = \lambda_1(\mu)\phi & \text{in } \Omega, \\ \partial_n\phi = 0 \quad \text{on } \partial\Omega, \\ \phi(\cdot+T,\cdot) = \phi, \\ \phi > 0. \end{cases}$$

Theorem

- *If $\lambda_1(\mu) > 0$, $\exists!$ positive $T-periodic$ solution, attractive for non-negative, bounded initial data $u_0 \not\equiv 0$.* **(persistence)**
- *If $\lambda_1(\mu) \leq 0$, $u \equiv 0$ is globally attractive.* **(extinction)**

[Hess '91, Hutson-Michaikow-Polacik '02, N. '10]

Phenotypically-structured populations in time-periodic environments

$$\partial_t u - D\Delta u = u\left(\mu(t,x) - \rho_u(t)\right) \text{ in } \Omega, \quad \partial_n u = 0 \text{ on } \partial\Omega \tag{3}$$

where $\rho_u(t) := \int_\Omega u(t,x)dx$.

$$\begin{cases} -\partial_t\phi + D\Delta_x\phi + \mu(t,x)\phi = \lambda_1(\mu)\phi, & \phi > 0 \quad \text{in } \Omega, \\ \partial_n\phi = 0 \quad \text{on } \partial\Omega, \\ \phi(\cdot + T, \cdot) = \phi. \end{cases}$$

Theorem

- *If $\lambda_1(\mu) > 0$, $\exists!$ positive $T-periodic$ solution N, attractive for non-negative, bounded initial data $u_0 \not\equiv 0$. Moreover, $\frac{1}{T}\int_t^{t+T} N \to \lambda_1(\mu)$ as $t \to +\infty$.* **(persistence)**
- *If $\lambda_1(\mu) \leq 0$, $\lim_{t\to+\infty} u(t,x) = 0$.* **(extinction)**

[Iglesias-Mirrahimi '21, Carrere-N. '20]

Influence of the coefficients on the periodic principal eigenvalue: Earlier works

$$\begin{cases} -\partial_t \phi + D\Delta_x \phi + \mu(t,x)\phi = \lambda_1(\mu)\phi, \quad \phi > 0 \quad \text{in } \Omega, \\ \partial_n \phi = 0 \quad \text{on } \partial\Omega, \\ \phi(\cdot + T, \cdot) = \phi. \end{cases}$$

- If $\mu(t,x) = b - c|x - \bar{x}(t)|^2$, explicit computation in [Lorenzi-Chisholm-Desvillettes-Hughes '15], variance around $\bar{x}(t)$ is $\sqrt{b/c}$.
- $\lambda_1(\mu) \geq \lambda_1(\langle\mu\rangle)$ where $\langle\mu\rangle(x) := \fint_0^T \mu(t,x)dt$ [Hutson–Shen–Vickers '01]
  $\Rightarrow$ averaging in time decreases the average total population at eq.
- If we consider $\mu(\omega t, x)$, then λ_1 decreases in ω [Liu-Lou-Peng-Zhou '19]
  $\Rightarrow$ oscillations in t decrease the average total population at eq.

Influence of fragmentation on the periodic principal eigenvalue

$$\begin{cases} -\partial_t \phi + D\Delta_x \phi + \big(b(t) - c(t)|x|^2\big)\phi = \lambda_1(b,c)\phi, \quad \phi > 0 \quad \text{in } \Omega, \\ \partial_n \phi = 0 \quad \text{on } \partial\Omega, \\ \phi(\cdot + T, \cdot) = \phi. \end{cases}$$

Example: chemotherapy optimization [Carrère '17]

Theorem

One has $\lambda_1(b,c) = \lambda_1(\langle b\rangle, c)$, *where* $\langle b\rangle := \fint_0^T b(t)dt$, *and*

$$\lambda_1(b,c^*) \geq \lambda_1(b,c)$$

where c^* *is the decreasing rearrangement in time.*

[Bogosel-Mazari-N. preprint]

Interpretation in the phenotypically-structured population model:
"Fragmentation" of the treatment yields a smaller population at equilibrium.

True for general $\mu = \mu(t,x)$ if $D \to 0$ and $D \to +\infty$.

III. Mean-field models for fishery management

Fishermen population: deterministic model

Consider a population of **fishermen** that can control their velocities at time t and position x. The position X_t^α of a fisherman with strategy α starting at x is given by

$$\begin{cases} \frac{dX_t^\alpha}{dt} = \alpha(t, X_t^\alpha), \\ X_0^\alpha = x, \end{cases}$$

He/she wants to maximize its **gain**:

$$J(\alpha) = \int_0^T \big(u(s, X_s) - \alpha_s^2/2\big)\, ds.$$

where

$u(s,x) =$ density of fishes in x at time s,

$-\alpha_s^2/2 =$ cost of movement.

⚠We assume for the moment that the fishermen have a negligible influence on the fish population.

Value function and Hamilton-Jacobi equation

For all $t \in (0,T)$, we define the **gain starting from time** t and position x by

$$J(t,x,\alpha) := \int_t^T \big(u(s,X_s) - \alpha_s^2/2\big)\,ds.$$

and we define the **value function** as

$$v(t,x) := \sup_\alpha J(t,x,\alpha).$$

Define the (convex) **Hamiltonian**

$$H(t,x,p) := \sup_{a\in\mathbb{R}^N}\Big(u(t,x) - a^2/2 - p\cdot a\Big) = u(t,x) + p^2/2.$$

Then v is a (viscosity) solution of the Hamilton-Jacobi equation

$$\partial_t v + H(t,x,\nabla v) = \partial_t v + |\nabla v|^2/2 + u(t,x) = 0 \qquad v(T,x) = 0,$$

any optimal strategy satisfies $\alpha_t^* = \nabla v(t, X_t^{\alpha_t^*})$ (feedback) and the maximal gain is $\sup_\alpha J(\alpha) = J(\alpha^*) = v(0,x)$.

classical differential game theory [Pontryagin '56]

Fishermen population: stochastic model

The **fishermen** movement is now perturbed by a standard Brownian motion:

- α fishermen strategy
- B_t Brownian motion accounts for uncertainty
- $\nu \geq 0$ volatility coefficient
- X_t^α position of fisherman is now determined by the Stochastic Differential Equation

$$\begin{cases} dX_t^\alpha = \alpha(t, X_t^\alpha)\, dt + \sqrt{2\nu}\, dB_t, \\ X_0^\alpha = x, \end{cases}$$

He/she wants to maximize its expected gain:

$$J(\alpha) = \mathbb{E}\left[\int_0^T \left(u(s, X_s) - \alpha_s^2/2\right) ds\right].$$

where

- $u(s,x) =$ density of fishes in x at time s,
- $-\alpha_s^2/2 =$ cost of movement.

Value function and Hamilton-Jacobi equation

For all $t \in (0,T)$, we define the expected gain starting from time t and position x by

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T \left(u(s,X_s) - \alpha_s^2/2\right)) \, ds\right].$$

and we define the **value function** as $v(t,x) := \sup_\alpha J(t,x,\alpha)$ and the **Hamiltonian** $H(t,x,p) = u(t,x) + p^2/2$.

Then v is a (viscosity) solution of the Hamilton-Jacobi equation

$$\partial_t v + \nu \Delta v + |\nabla v|^2/2 + u(t,x) = 0 \qquad v(T,x) = 0.$$

any optimal strategy satisfies $\alpha_t^* = \nabla v(t, X_t^{\alpha_t^*})$ (feedback) and the maximal expected gain is $\sup_\alpha J(\alpha) = J(\alpha^*) = v(0,x)$.

classical stochastic differential game theory, for ex. [Evans-Souganidis '84]

Fish population: reaction-diffusion equation

We now assume that the fishery has a signficant impact on the fish population

We consider a **single fish population**[1], with density u, dispersal rate D, growth rate $f(x,u)$, that is captured at rate $-m(t,x)$, where m is the distribution of the fishermen at (t,x):

$$\partial_t u = D\Delta u + f(x,u) - m(t,x)u \quad \text{in } (0,T)\times\Omega,$$

with $u(0,x)=u_0(x)$ and Neumann boundary conditions $\partial_n u = 0$ on $\partial\Omega$ (= our marine ecosystem is isolated).

It is known that the distribution $m(t,\cdot)$ of the variable X_t^α follows the Fokker-Planck equation:

$$\partial_t m - \nu\Delta m + \nabla\cdot(\alpha m) = 0 \text{ in } (0,T)\times\Omega, \quad m(0,\cdot) = m_0, \quad \partial_n m = 0 \text{ on } (0,T)\times\partial\Omega$$

where m_0 is the distribution of fishermen at time $t=0$.

---

[1] For two or three populations models with constant fishing strategy, see [El Harch - Outaaoui - El Foutayeni '25, Villain-Poggiale-Loeuille preprint, Berestycki - Fagan - Safsten preprint]

Competitive and cooperative gain

The goal of the fishermen is to maximize a given gain with respect to $\alpha = \alpha(t,x)$.

They could either maximize their own individual gain (starting from x):

$$J^T_{comp}(\alpha) := \mathbb{E}\left[\int_0^T \Big(u(t,X_t^\alpha) - \alpha^2(t,X_t^\alpha)/2\Big)dt\right]$$

or the common gain:

$$J^T_{coop}(\alpha) := \int_0^T \int_\Omega \Big(u(t,x) - \alpha^2(t,x)/2\Big)m(t,x)\,dt\,dx.$$

⚠Now u depends on the strategy α through the distribution of fishermen m.

Such pbm are called **mean field game/control models** and are topical in PDEs [Lasry-Lions 2006-07, Caines-Huang-Malhamé 2006].

Questions

The goal of the fishermen is to maximize a given gain with respect to $\alpha = \alpha(t, x)$:

$$J^T_{comp}(\alpha) := \mathbb{E}\left[\int_0^T \left(u(t, X_t^\alpha) - \alpha^2(t, X_t^\alpha)/2\right) dt\right]$$

$$J^T_{coop}(\alpha) := \int_0^T \int_\Omega \left(u(t,x) - \alpha^2(t,x)/2\right) m(t,x)\, dt\, dx.$$

Questions

1. Is it possible to characterize a maximizer α^*? Is it unique ?
2. Could we compare the competitive and the cooperative gains?
3. How do these gains depend on the environment $k = k(x)$?

Earlier works

Such problems have been addressed in [Kobeissi - Mazari - Ruiz-Balet Nonlinearity 2024, Kobeissi - Mazari - Ruiz-Balet preprint 2024] .
They considered $\nu = 0$ (no fluctuations), $\Omega = \mathbb{R}$, a homogeneous **bistable nonlinearity**

$$f(u) = u(1-u)(u-\theta), \quad \theta \in (0, 1/2).$$

and infinite horizon gains with discount $\lambda > 0$:

$$J^{\infty}_{comp}(\alpha) := \mathbb{E}\Big[\int_0^{\infty} e^{-\lambda t}\Big(u(t, X_t) - \alpha^2(t, X_t)/2\Big)dt\Big]$$

$$J^{\infty}_{coop}(\alpha) := \int_0^{\infty}\int_{\Omega} e^{-\lambda t}\Big(u(t,x) - \alpha^2(t,x)/2\Big)m(t,x)\,dt\,dx.$$

Theorem (Kobeissi - Mazari - Ruiz-Balet 2024)

There exists $\lambda > 0$, m_0 and u_0 such that

$$\max_{\alpha} J^{\infty}_{comp}(\alpha) < \max_{\alpha} J^{\infty}_{coop}(\alpha).$$

Moreover, for the strategy α maximizing $J^{\infty}_{comp}(\alpha)$, one has $u(t,x) \to 0$ as $t \to +\infty$ loc. in x.

Earlier works

Theorem (Kobeissi - Mazari - Ruiz-Balet 2024)

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Moreover, for the strategy α maximizing $J^{\infty}_{comp}(\alpha)$, one has $u(t,x) \to 0$ as $t \to +\infty$ loc. in x.

Interpretation: Tragedy of commons.

- If the fishermen each try to maximize their individual gain, they will fish too much and destroy all the fishes, getting an asymptotic gain of 0,
- if they coordinate together to maximize their common gain, the fish population will survive and they will get a higher gain.

Joint work with

Greta Lamonaca (PhD, univ. Orléans)

Idriss Mazari-Fouquer (univ. Paris-Dauphine)

Mean Field Game System in the competitive case

Value function: expected gain starting from time t:

$$v^T(t,x) := \sup_\alpha J^T_{comp}(t,\alpha) = \sup_\alpha \mathbb{E}\left[\int_t^T u(s,X^\alpha_s) - \frac{1}{2}|\alpha_s|^2\, ds\right]$$

By standard MFG's theory, it can be proved that, in $(0,T)\times\Omega$:

$$\begin{cases} -\partial_t v - \nu\Delta v - \frac{1}{2}|\nabla v|^2 = u & \text{Hamilton-Jacobi eq.,} \\ \partial_t m - \nu\Delta m + \mathrm{div}(m\nabla v) = 0, \ \fint_\Omega m = 1 & \text{Fokker-Planck eq.,} \\ \partial_t u - d\Delta u - u(k(x) - m(t,x) - u) = 0, \ u > 0 & \text{Fisher-KPP eq.,} \\ v(T,x) = 0, \ m(0,x) = m_0, \ u(0,x) = u_0(x) \not\equiv 0, \geq 0 \end{cases} \quad \text{(MFG)}$$

with Neumann boundary conditions, and an optimal strategy (if it exists!) satisfies $\alpha^*(t,x) = \nabla v^T(t,x)$.

⚠Forward-backward equation, difficult to investigate, well-posedness is not guaranteed.

[Caines - Huang - Malhamé '06, Lasry - Lions '07, Kobeissi - Mazari-Fouquer - Ruiz-Balet '24]

Expected gain at large time $T \to +\infty$.

Ansatz: When the final time $T \to +\infty$, we expect

$$v^T(t,x) \simeq \bar{\lambda}_{comp}(T-t)+\bar{v}(x),\ u^T(t,x) \to \bar{u}(x),\ m^T(t,x) \to \bar{m}(x) \ \text{ as } T \to +\infty$$

locally in (t,x), where $(\bar{v},\bar{m},\bar{u})$ satisfies the Static Mean Field Game

$$\begin{cases} \bar{\lambda}_{comp} - \nu\Delta\bar{v} - \frac{1}{2}|\nabla\bar{v}|^2 = \bar{u}, & \text{in } \Omega \\ -\nu\Delta\bar{m} + \text{div}(\bar{m}\nabla\bar{v}) = 0, & \text{in } \Omega \\ -d\Delta\bar{u} - \bar{u}(k(x) - \bar{m} - \bar{u}) = 0,\ \bar{u} > 0 & \text{in } \Omega \\ \int_\Omega \bar{v} = 0,\ ⨍_\Omega \bar{m} = 1 \end{cases} \qquad (S_{MFG})$$

where $\bar{\lambda}_{comp} \in \mathbb{R}$ ergodic constant ($\simeq$ principal eigenvalue).

The average expected gain at large time $T \to +\infty$ would then be

$$\frac{\sup_\alpha J^T_{comp}}{T} = \frac{1}{T} v^T(0,x) \simeq \frac{1}{T}\Big(\lambda_{comp}(T-t) + \bar{v}(x)\Big) \simeq \lambda_{comp}.$$

⇒ Stationary equation, easier to investigate.

Simplification of the static system

$$\begin{cases} \bar{\lambda}_{comp} - \nu\Delta\bar{v} - \frac{1}{2}|\nabla\bar{v}|^2 = \bar{u}, & \text{in } \Omega \\ -\nu\Delta\bar{m} + \mathrm{div}(\bar{m}\nabla\bar{v}) = 0, & \text{in } \Omega \\ -d\Delta\bar{u} - \bar{u}(k(x) - \bar{m} - \bar{u}) = 0,\ \bar{u} > 0 & \text{in } \Omega \\ \int_\Omega \bar{v} = 0,\ \fint_\Omega \bar{m} = 1. \end{cases} \qquad (S_{MFG})$$

Let us consider the Hopf-Cole transform $\varphi = e^{\frac{\bar{v}}{2\nu}}$. We could prove that $\varphi^2 = \bar{m}$ and φ is the principal eigenvalue of an elliptic equation:

$$\begin{cases} 2\nu^2\Delta\varphi + \bar{u}(x)\varphi & = \bar{\lambda}_{\mathrm{comp}}\,\varphi, \quad \fint_\Omega \varphi^2 = 1, \quad \varphi > 0 & \text{in } \Omega, \\ -d\Delta\bar{u} - \bar{u}(k(x) - \varphi^2 - \bar{u}) & = 0, \quad \bar{u} > 0 & \text{in } \Omega \end{cases}$$

The average expected gain as $t \to +\infty = \bar{\lambda}_{comp}$ is the principal eigenvalue of some coupled equation. We would like to understand its dependance on the environment $k = k(x)$. But is it well-defined? Does (S_{MFG}) admits a unique solution?

Research Landscape: from Insight to Inquiry

- We could prove by a fixed point argument the existence of solutions.
- Uniqueness. A well-known sufficient condition for uniqueness is the so called **Lasry-Lions monotonicity condition** (aversion to crowd)

$$\forall m_1, m_2, \quad 0 \geq \int_\Omega (u[m_1] - u[m_2])(m_1 - m_2)\,dx. \qquad (LL)$$

Theorem

$LL \Longrightarrow$ Uniqueness. If (LL) holds, then there exists a unique solution $(\bar{v}, \bar{m}, \bar{u})$ to (S_{MFG}).

Sketch of Proof.

$$\begin{aligned}\int_\Omega (u_1 - u_2)(m_1 - m_2) &= \int_\Omega \left[\lambda_1 - \lambda_2 - \nu\Delta(v_1 - v_2) - \frac{1}{2}\left(|\nabla v_1|^2 - |\nabla v_2|^2\right)\right](m_1 - m_2)\\ &= \int_\Omega [-\nu\Delta(m_1 - m_2)]\,(v_1 - v_2) - \frac{1}{2}\iint \left(|\nabla v_1|^2 - |\nabla v_2|^2\right)(m_1 - m_2)\\ &= -\int_\Omega \mathrm{div}\,(m_1\nabla v_1 - m_2\nabla v_2)\,(v_1 - v_2) - \frac{1}{2}\iint \left(|\nabla v_1|^2 - |\nabla v_2|^2\right)(m_1 - m_2)\\ &\dots\\ &= \int_\Omega \frac{m_1 + m_2}{2}|\nabla v_1 - \nabla v_2|^2\end{aligned}$$

Our aim: does (LL) hold? Challenging, no examples for realistic models except for [Kobeissi - Mazari-Fouquer - Ruiz-Balet '24].

Uniqueness of the steady state

Proposition

Let $k \in L^\infty(\Omega)$ *s.t.* $\fint_\Omega k > 1$ *and* $(\bar\lambda, \bar v, \bar m, \bar u)$ *solution of* (S_{MFG}) *such that*

$$d\frac{|\nabla \bar u|^2}{\bar u^3} \leq 4. \qquad (*)$$

Then (LL) *holds and thus* $(\bar\lambda, \bar v, \bar m, \bar u)$ *is the unique solution of* (S_{MFG})

Theorem (Lamonaca-Mazari-N.)

1. *If either* $d = 1$ *and* $\inf_\Omega k$ *is large enough (explicit),*
2. *or any dimension but* k *constant.*

Then (LL) *holds and thus* $(\bar\lambda, \bar v, \bar m, \bar u)$ *is the unique solution of* (S_{MFG})

Main ideas of the proof.

1. If $d = 1$, then $(*)$ holds if $k - \bar m \geq 0$ [Bai-He-Li '15]. Hence, if $\inf_\Omega k$ is large enough and $\bar m$ is not too large w.r.t k, we are done.
2. If k is constant, there exists a constant stationary solution with $\bar m \equiv 1$ and $\bar u \equiv k - 1$. Hence $\nabla \bar u = 0$ and thus $(*)$ is satisfied.

Attractiveness of the steady state

Theorem (Lamonaca - Mazari -N.)

Let $k \in C^1(\bar{\Omega})$ be such that $\fint_\Omega k > 1$, $m_0 \geq 0$ in $L^\infty(\Omega)$, $\fint_\Omega m_0 = 1$. Suppose that $(\bar{\lambda}_{comp}, \bar{v}, \bar{m}, \bar{u})$ is a solution of (S_{MFG}). Let $u_0 \geq 0, \neq 0$ in $L^\infty(\Omega)$ and assume that

$$\forall T > 0 \text{ large, a solution } (u, m, u) \text{ of (MFG) satisfies}$$

$$\int_0^T \int_\Omega (u[m] - \bar{u}[\bar{m}])(m - \bar{m}) - \overbrace{\int_\Omega \left(u_0 - \bar{u} - \bar{u}\ln\left(\frac{u_0}{\bar{u}}\right)\right)}^{\text{correction term} > 0} \leq 0. \tag{C}$$

*Then $(\bar{\lambda}_{comp}, \bar{v}, \bar{m}, \bar{u})$ is **attractive**.*
This is true in particular if $(\bar{\lambda}_{comp}, \bar{v}, \bar{m}, \bar{u})$ satisfies (LL).

Proposition (Lamonaca - Mazari -N.)

In the 1D case, assume $k - \bar{m} > 0$ everywhere. Then $\forall \theta_0 \geq 0, \neq 0$, (C) is satisfied, and so $(\bar{\lambda}_{comp}, \bar{u}, \bar{m}, \bar{u})$ is attractive.

Mean Field Control System in the cooperative case

Assume that the fishermen cooperate in order to maximize a common gain:

$$J^T_{\text{coop}}(0,\alpha) := \int_0^T \int_\Omega \left(u\, m - \frac{1}{2}|\alpha|^2\, m \right) dx\, dt \qquad (J_{MFC})$$

where u is the density of fishes and m is the density of fishermen:

$$\begin{cases} \partial_t u - d\Delta u - u(k(x) - m - u) = 0, \ \ u > 0 & \text{in } (0,T)\times\Omega \\ \partial_t m - \nu\Delta m + \text{div}(m\,\alpha) = 0, \ \ \fint_\Omega m = 1 & \text{in } (0,T)\times\Omega \\ u(0,\cdot) = u_0(\cdot) \not\equiv 0, \quad m(0,\cdot) = m_0(\cdot) & \text{in } \Omega \end{cases}$$

⚠Multiple solutions might exist.
Interpretation: heterogeneous strategies.
⇒ **See Greta's talk.**

Conclusions

The goal of the fishermen is to maximize a given gain w.r.t $\alpha = \alpha(t,x)$:

$$J^T_{comp}(\alpha) := \mathbb{E}\left[\int_0^T \Big(u(t,X^\alpha_t) - \alpha^2(t,X^\alpha_t)/2\Big)dt\right]$$

$$J^T_{coop}(\alpha) := \int_0^T \int_\Omega \Big(u(t,x) - \alpha^2(t,x)/2\Big)m(t,x)\,dt\,dx.$$

1. **Is it possible to characterize a maximizer α^*? Is it unique ?**

   Yes. It could be characterized through a Mean-field game system in the competitive setting. This system converges to some Static MFG when $T \to +\infty$, and

   $$\frac{\sup_\alpha J^T_{comp}}{T} \to \lambda_{comp} \quad \text{as } T \to +\infty.$$

   We provided conditions guaranteeing uniqueness of the solutions of this Static MFG.

2. **Could we compare the competitive and the cooperative gains?**

   ⇒ See Greta's talk.

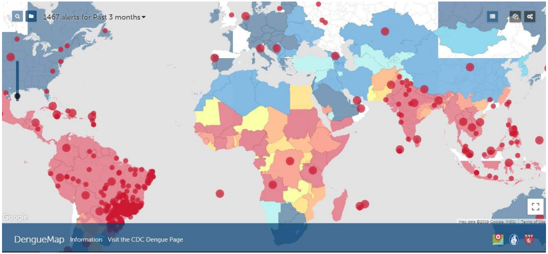
3. **How do these gains depend on the environment $k = k(x)$?**

   The next step is to investigate the dependance of λ_{comp} w.r.t $k = k(x)$... when this quantity is uniquely defined.

IV. Allee effect and invasion blocking

Biological motivation

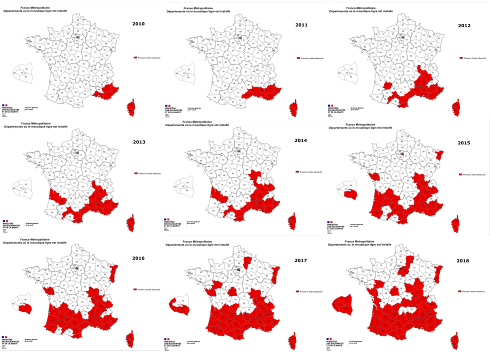
Tiger mosquito killed at CIRM



Mosquitoes are vectors of many diseases (Zika, Dengue, Chikungunya).
They are responsible for 700 000 deaths annually.

Instead of targeting the diseases, a natural idea is to fight against the spreading of the mosquitoes.

Biological motivation



Spreading of tiger mosquitoes between 2010 and 2018

"*Santé publique France pointe une "situation exceptionnelle" en 2022 avec 378 cas de dengue en France métropolitaine (11/07/23).*"

A strategy to avoid these epidemics is to block the invading wave of mosquitoes.

Aim of this research: to model and compare various strategies to block an invasion.

Biological control methods for vector-borne diseases

Several biological control method:

- insecticides (killing strategy)
- Wolbachia bacteria
- sterile (or incompatible) insect techniques

The Wolbachia bacteria

Wolbachia is a symbiotic bacteria reducing infection of mosquitos by pathogens, but causing decrease in fertility and cytoplasmic incompatibility.

♀\♂	Infected	Sane
Infected	I	I
Sane	×	S

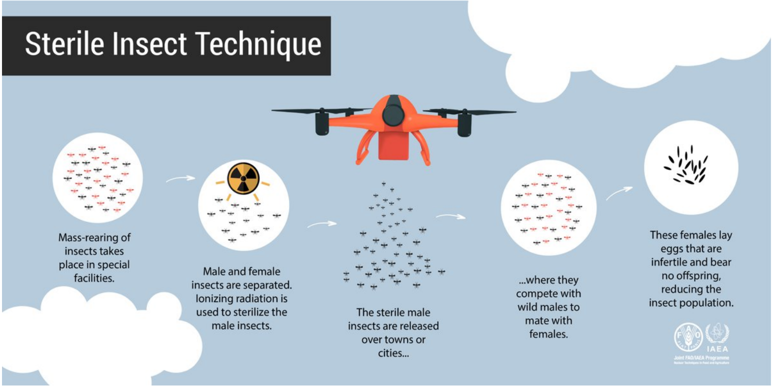
⇒ Bistable reproduction rate

$$f(u) = du(1-u)\frac{u-(1-\sigma)}{(1-u)^2+\sigma u}$$

where $u =$ proportion of infected mosquitoes, $d =$ death rate and $\sigma =$ decrease of fertility. [Barton-Turelli 2011]

Countries where the World Mosquito Program is applied.

The sterile insect technique



Heuristic : As we release sterile males in some place, the birth rate goes to 0 and it remains only a death term in this place

Successful in La Réunion, Tahiti... Contacts with IRD and CIRAD colleagues.

Joint work with

Alexis Léculier (univ. Bordeaux)

Yannick Privat (univ. Strasbourg)

Luis Almeida (Sorbonne univ.)

Optimal control of bistable travelling waves: looking for the best spatial distribution of a killing action to block a pest invasion, SIAM Journal on Control and Optimization 2024

Optimization of the spatial distribution of released of sterile insect to avoid an invasion of pests, in preparation

Bistable reaction-diffusion equations

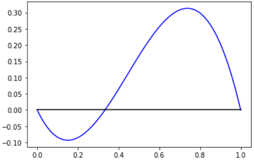
$u(t,x) \equiv$ Density of population at time t and position x

$$\underbrace{\partial_t u - \Delta u}_{\text{Diffusion}} = \underbrace{g(u)}_{\text{Reaction}}$$

Bistable nonlinearity

$ex : g(u) = u(1-u)(u-\theta), \quad \theta \in (0,1/2)$

- 3 steady states : $\{ 0 \, ; \, \theta \, ; \, 1 \}$
- 2 stable : 0 and 1
- 1 unstable : θ



Several optimization or control problems for such models [Almeida, Duprez, Privat, Vauchelet '19, '20], [N., Strugarek, Vauchelet '18], [Almeida, Bellver, Duprez, Privat '20], [Bliman '19]...

General statement of the problem

$u(t,x)$ ≡ Density of mosquitoes
$\mu \equiv \mu(x)$: impact of some control on the pop

$$\partial_t u - \Delta u = g\big(u, \mu(x)\big)$$

We want to construct a "barrier" blocking an invasion coming from $-\infty$, that is, a steady solution $u > 0$ s.t.

$$-u'' = g\big(u, \mu(x)\big) \quad \text{in } \mathbb{R}, \quad u(-\infty) = 1,\ u(+\infty) = 0.$$

Questions

1. How to model the killing / Wolbachia / SIT strategies through the rate μ?
2. Is it possible to find an admissible rate μ such that a barrier exists?
3. Could we "optimize" μ such that a barrier exists?
4. Could we compare the killing / Wolbachia / SIT strategies?

Earlier works on invasion blocking

u: proportion of mosquitoes infected by Wolbachia

N: total population of mosquitoes (assumed constant in time = carrying capacity)

$$\partial_t u - \Delta u - 2\nabla u \cdot \frac{\nabla N}{N} = g(u)$$

[Barton-Turelli '11, Hughes-Britton '13]

What is the effect of an increase of the carrying capacity?

Earlier works on invasion blocking

We assume that $N(x) = e^{Cx/2}$ in $[-L, L]$ and is constant otherwise. We search for steady state solutions of the equation with this N:

$$\begin{cases} -u'' - Cu' &= g(u) \qquad \text{in } (-L, L), \\ -u'' &= g(u) \qquad \text{in } (-\infty, -L) \cup (L, \infty), \\ u(-\infty) = 1, & u(+\infty) = 0 \end{cases} \tag{4}$$

Theorem (N.-Strugarek-Vauchelet '17)

For all $L > 0$, there exists $C_*(L)$ such that for all $C > C_*(L)$, equation (4) *admits two steady positive solutions (the biggest is stable and the smallest is unstable).*
Moreover, we could compute the limits of $C_*(L)$ as $L \to 0^+$ and $L \to +\infty$.

See also [Lewis-Keener '00, Berestycki-Rodriguez-Ryzhik '13, Almeida - Léculier - Vauchelet '23]

How to model the killing / SIT strategies through the rate μ?

We write

$$g(u) = b(u) - d(u) \qquad \text{with} \qquad \begin{cases} b(u) := \text{ birth} \\ d(u) := \text{ death} \end{cases}$$

- **Killing strategy**

$$g(u,\mu(x)) = b(u) - (d(u) + \mu(x)u)$$

- **Sterile males strategy** (toy model)

$$g(u,\mu(x)) = \frac{u}{u+\mu(x)} b(u) - d(u)$$

For a more realistic, but less trackable, model, see [Almeida-Léculier-Vauchelet '23]

The killing strategy: earlier work

Take

$$\mu(x) = \begin{cases} C & \text{on} \quad (0,L), \\ 0 & \text{on} \quad \mathbb{R}\setminus(0,L) \end{cases}$$

Then for any C large enough, there exists L_* such that for all $L \geq L_*$, equation

$$\begin{cases} -u'' = g(u) - \mu(x)u & \text{in } (0,L), \\ -u'' = g(u) & \text{in } (-\infty,0)\cup(L,\infty), \\ u(-\infty) = 1, \quad u(+\infty) = 0, & \end{cases}$$

admits a solution.

[Almeida-Léculier-Vauchelet '23]

Interpretation: providing we act sufficiently strongly on a sufficiently large area, we could block an invasion.

Remark: Not true in the Fisher-KPP framework. We could always restart a population even from a very small density.

The killing strategy

Goal : Assume we know we could block the invasion. How to minimize our control action in order to block an invasion?

Mathematical formulation: Minimize

$$\mathcal{N}(\mu) = \int_0^L \mu(x)dx \quad \text{with } 0 \leq \mu(x) \leq C, \ \operatorname{supp} \mu \subset [0, L]$$

s.t. $\exists u$ solution of

$$\begin{cases} -u'' = g(u) - \mu(x)u & \text{in } (0, L), \\ -u'' = g(u) & \text{in } (-\infty, 0) \cup (L, \infty), \\ u(-\infty) = 1, \quad u(+\infty) = 0. & \end{cases}$$

Optimal result for the killing strategy

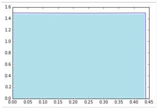
Minimize $\mathcal{N}(\mu) = \int_{\mathbb{R}} \mu(x)\,dx$ with $(\mathscr{P}) \begin{cases} -u'' = g(u) - \mu(x)u, \\ u(-\infty) = 1, \qquad u(+\infty) = 0, \\ 0 \le \mu \le C, \qquad \operatorname{supp}\, \mu \subset [0, L]. \end{cases}$

$$g(u) = u(1-u)(u-\theta), \quad \theta \in (0, 1/2)$$

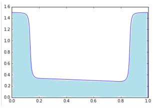
Theorem [Almeida-Léculier-N.-Privat preprint]

If C is large enough, ∃ 2 critical sizes L_*, L^* s.t.

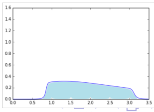
1. if $L < L_*$, no solution to $(\mathscr{P})$,
2. if $L = L_*$, the optimal solution is $\mu = C1_{[0,L]}$,
3. if $L_* < L < L^*$ then the optimal solution satisfies $\mu = C$ in $[0, a[\cup]b, L]$ and $0 < \mu < C$ (singular arc) in $]a, b[$,
4. if $L \ge L^*$ then, up to translation, the optimal solution is a singular arc on $(0, L^*)$, that is $0 < \mu < C$, which is fully determined by a known ODE.



$L = L_*$



$L_* < L < L^*$



$L > L^*$

An earlier result without any constraint on the support

Minimize $\mathscr{N}(\mu)=\int_{\mathbb{R}}\mu(x)\,dx$ with

$$(\mathscr{P})\ \begin{cases} -u''=g(u)-\mu(x)u, & \\ u(-\infty)=1, & u(+\infty)=0, \\ 0\le\mu\ \xcancel{\le C}, & \xcancel{\operatorname{supp}\mu\subset[0,L]}. \end{cases}$$

Theorem

Up to translation, the optimal solution is a singular arc on $(0,L^*)$, that is, $\mu>0$ on $[0,L^*)$, which is fully determined by a known ODE.

[Bressan-Chiri-Salehi, 2021]

Method of proof: Stokes formula in the phase plane.

Reduction to a bounded domain problem

$$\begin{cases} -u'' = g(u) - \mu(x)u & \text{in } (0,L), \\ -u'' = g(u) & \text{in } (-\infty,0)\cup(L,\infty), \\ u(-\infty)=1, \quad u(+\infty)=0. \end{cases}$$

$\Rightarrow (u')^2/2 + G(u) \equiv G(0) = 0$ on (L,∞) and $(u')^2/2 + G(u) \equiv G(1)$ on $(-\infty,0)$ with $G(s) = \int_0^s g$. We thus reduce to the problem on bounded domain $(0,L)$:

$$(P)\begin{cases} -u'' = g(u) - \mu(x)u \quad \text{in } (0,L), \\ \\ u'(0) = -\sqrt{G(1)-G(u(0))}, \\ \\ u'(L) = -\sqrt{-G(u(L))}. \end{cases}$$

New problem : Minimize

$$\mathcal{N}(\mu) = \int_0^L \mu(x)\,dx \quad \text{with } 0 \le \mu(x) \le C$$

s.t. $\exists u$ solution of (P).

The Pontryagin maximum principle

$$\text{Minimize } \mathcal{N}(\mu) = \int_0^L \mu(x)dx \quad \text{with } 0 \le \mu(x) \le C, \ \operatorname{supp} \mu \subset [0,L]$$

$$\text{s.t. } \exists u \text{ solution of } \begin{cases} -u'' = g(u) - \mu(x)u \text{ in } (0,L), \\ u'(0) = -\sqrt{G(1) - G(u(0))}, \\ u'(L) = -\sqrt{-G(u(L))}. \end{cases}$$

Let p the solution of the adjoint problem

$$\begin{cases} -p'' = g'(u)p - \mu(x)p \text{ in } (0,L), \\ p'(0)u'(0) + g(u(0))p(0) = p'(L)u'(L) + g(u(L))p(L) = 0. \end{cases}$$

Then $\exists p^0 \in \{-1,0\}$ s.t.

if $\mu = 0$ then $pu \le -p^0$, if $\mu = C$ then $pu \ge -p^0$, if $0 < \mu < C$, then $pu = -p^0$.

Moreover:

for a.e. $x \in [0,L]$, $p'u' + p(g(u) - \mu u) - p^0\mu =: h$ is constant.

Corollaries of the PMP

Then $\exists p^0 \in \{-1, 0\}$ s.t.

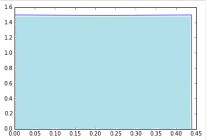
if $\mu = 0$ then $pu \leq -p^0$, if $\mu = C$ then $pu \geq -p^0$, if $0 < \mu < C$, then $pu = -p$

Moreover:

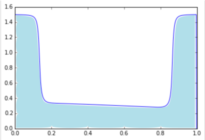
for a.e. $x \in [0, L]$, $p'u' + p(g(u) - \mu u) - p^0\mu =: h$ is constant.

Working (hard) with these characterizations, we could prove:

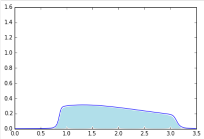
1. $p^0 = 0 \Rightarrow \mu = C1_{[0,L]}$, that is, $L = L_*$
2. $p^0 = -1, h > 0 \Rightarrow \mu = C$ in $[0, a[\cup]b, L]$ and μ is a singular arc in $]a, b[$, that is, $L_* < L < L^*$,
3. $p^0 = -1, h = 0 \Rightarrow \mu$ is a singular arc on $(0, L^*)$, $L \geq L^*$.



$L = L_*$



$L_* < L < L^*$



$L > L^*$

Optimal result for the Sterile Insect Technique

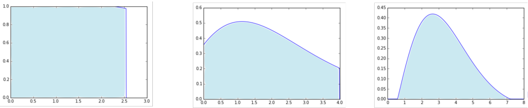
Minimize $\mathcal{N}(\mu) = \int_{\mathbb{R}} \mu(x)\,dx$ with $(\mathscr{P}) \begin{cases} -u'' = \dfrac{u}{u+\mu(x)} b(u) - d(u), \\ u(-\infty) = 1, \qquad u(+\infty) = 0, \\ 0 \le \mu \le C, \qquad \operatorname{supp}\,\mu \subset [0, L]. \end{cases}$

Theorem [Almeida, Léculier, N., Privat]

If C is large enough and under some hyp on b, $\exists$ 2 critical sizes L_*, L^* s. t.

1. if $L < L_*$, there does not exists a solution to $(\mathscr{P})$,
2. if $L = L_*$ then the optimal solution is $\mu = C 1_{[0,L]}$,
3. if $L_* < L < L^*$ then the optimal solution satisfies $\mu > 0$,
4. if $L \ge L^*$ then, up to translation, the optimal solution is a singular arc on $(0, L^*)$, which is fully determined by a known ODE.

Moreover, $\mu(x) = \min\left\{C, \left(\sqrt{-p^0 b(u(x))u(x)p(x)} - u(x)\right)_+\right\}$ is continuous



The Pontryagin maximum principle

$$\text{Minimize } \mathcal{N}(\mu) = \int_0^L \mu(x)\,dx \quad \text{with } 0 \leq \mu(x) \leq C,\ \operatorname{supp}\mu \subset [0,L]$$

$$\text{s.t. } \exists u \text{ solution of } \begin{cases} -u'' = \frac{u}{u+\mu(x)} b(u) - d(u) \text{ in } (0,L), \\ u'(0) = -\sqrt{G(1) - G(u(0))}, \\ u'(L) = -\sqrt{-G(u(L))}. \end{cases}$$

Let p the solution of the adjoint problem

$$\begin{cases} -p'' = p\big(\frac{u}{u+\mu(x)} b'(u) - d'(u) + \frac{\mu}{(u+\mu(x))^2} b(u)\big)p \text{ in } (0,L), \\ p'(0)u'(0) + g(u(0))p(0) = p'(L)u'(L) + g(u(L))p(L) = 0. \end{cases}$$

Then $\exists p^0 \in \{-1,0\}$ s.t. μ minimizes the Hamiltonian

$$H = p'u' + p\Big(\frac{u}{u+\mu} b(u) - d(u)\Big)p - p^0\mu$$

$$\Rightarrow \mu(x) = \min\Big\{C, \Big(\sqrt{-p^0 b(u(x))u(x)p(x)} - u(x)\Big)_+\Big\}$$

Conclusions

1. How to model the killing / Wolbachia / SIT strategies through the rate μ?
   $\Rightarrow$ We introduced 2 toy models where we act either on the mortality or the fecundity.
2. Is it possible to find an admissible rate μ such that a barrier exists?
   $\Rightarrow$ Yes, by taking μ large enough on a sufficient large model [Almeida-Léculier-Vauchelet '23].
3. Could we "optimize" μ such that a barrier exists?
   $\Rightarrow$ Yes. The optimizers could be characterized through a Pontryagin Maximum Principle. They depend on the toy model under consideration. But they are not easily implementable.
4. Could we compare the killing / Wolbachia / SIT strategies?
   $\Rightarrow$ Not yet.

Another perspective

Address the time-heterogeneous problem:

$$\frac{du}{dt} = f(t,u) + \mu(t) g(t,u)$$

where f and g are $1-$periodic with respect to t and

$$\mu \in \mathscr{A} = \{\mu \text{ s.t. } 0 \le \mu \le C, \text{ and associated } u \text{ cv to } 1 \text{ as } t \to +\infty\}.$$

We want to minimize $\int_0^\infty \mu(t)\,dt$ over $\mu \in \mathscr{A}$.

- Fully determined bang-bang control when f and g do not depend on t [Almeida-Privat-Strugarek-Vauchelet '19]
- PhD thesis of D. Nahmani (ANRS Maladies infectieuses émergentes), co-supervised with N. Vauchelet (univ. Paris Nord).

Merci pour votre attention!