Feasibility and stability in foodwebs: a Large Random Matrix approach

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joint work with Pierre Bizeul

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The Lotka-Volterra model

A popular model to describe the dynamics of interacting species in foodwebs is given by a system of Lotka-Volterra equations:

$$\frac{dx_i(t)}{dt} = x_i \left(r_i - \theta x_i + \sum_{j=1}^{N} \frac{A_{ij}}{N^\delta} x_j \right)$$

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- δ is a parameter controlling the interaction $j \to i$ strength.

Interaction	**Value of δ**	**Comment**
strong	$\delta \in (0, 1/2)$	-
moderate	$\delta = 1/2$	RMT regime
weak	$\delta \in (1/2, 1)$	Perturbation theory

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- The equilibrium $\boldsymbol{x}^*$ (if it exists) is given by

$$\boxed{\frac{dx_i(t)}{dt} = 0} \quad \Longleftrightarrow \quad x_i \left(r_i - \theta x_i + \sum_{\ell \in [N]} \frac{A_{i\ell}}{N^\delta} x_\ell \right) = 0 \quad \forall i \in [N]$$

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Feasibility

- The equilibrium is **feasible** if $\boxed{x_i^* > 0}$ for all i.
- If the equilibrium is feasible, then

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Stability

- Given the jacobian $\mathcal{J}(\boldsymbol{x}^*)$, which is explicit for Lotka-Voltera systems

$$\boxed{\mathcal{J}(\boldsymbol{x}^*) = \mathrm{diag}(\boldsymbol{x}^*) \left(-\theta I_N + \frac{A}{N^\delta} \right)}$$

The model is stable if $\boxed{\mathrm{Re}\,(\text{eigenvalues of } \mathcal{J}(\boldsymbol{x}^*)) \; < \; 0}$

No feasible equilibrium for moderate interactions

Consider a LV system with moderate interactions:

$$\frac{dx_i(t)}{dt} = 0 \quad \Longrightarrow \quad \theta \boldsymbol{x}^* = \boldsymbol{r} + \frac{A}{\sqrt{N}} \boldsymbol{x}^* \quad \Longrightarrow \quad \boxed{\boldsymbol{x}^* = \left(\theta I_N - \frac{A}{\sqrt{N}} \right)^{-1} \boldsymbol{r}}$$

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An puzzling result from Mazza et al.

Building upon Geman and Hwang, Dougoud et al. establish that **there is no feasible equilibrium with proba 1**

$$\boxed{\mathbb{P}\{x_i^* < 0 \text{ for some } i \in [n]\} \xrightarrow[N \to \infty]{} 1}$$

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Reference

- "The feasibility of equilibria in large ecosystems: A primary but neglected concept in the complexity-stability debate",
  Dougoud, Vikenbosch, Rohr, Bersier, Mazza, PLoS Comput. Biology, **2018**

Elements of proof

Consider the equation of feasible equilibrium for (simplified) LV ($\theta = 1, \boldsymbol{r} = \mathbf{1}$)

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where

- $\boldsymbol{x}$ is a $N \times 1$ unknown vector,
- $\mathbf{1}$ is a $N \times 1$ vector of ones,
- A is a $N \times N$ matrix with i.i.d. entries $\mathcal{N}(0,1)$,
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- Does this system admit a solution $\boxed{\boldsymbol{x} = \left(I - \frac{A}{\boldsymbol{\alpha}\sqrt{N}}\right)^{-1}\mathbf{1}}$?

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- Does this system admit a solution $\boxed{\boldsymbol{x} = \left(I - \frac{A}{\boldsymbol{\alpha}\sqrt{N}}\right)^{-1}\mathbf{1}}$?
- Is this solution feasible?

Non-Hermitian random matrices I

Matrix model

Let A_N be a $N \times N$ matrix

$$A_N = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix}$$

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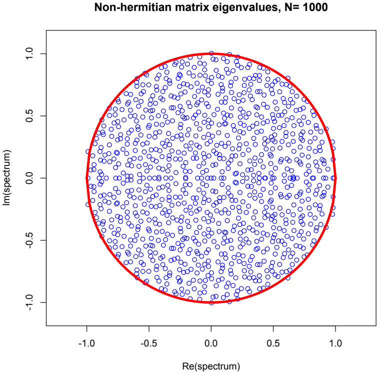

Figure: Distribution of $A_N/\sqrt{N}$'s eigenvalues and the circular law (in red)

Theorem: The Circular Law (Ginibre, Metha, Girko, Götze et al., Tao & Vu, etc.)

The spectrum of $\mathbf{Y}_N$ converges to **the uniform probability on the disc**

Non-Hermitian random matrices II

Spectral radius and spectral norm

- Theorem (Geman)

$$\rho\left(\frac{A}{\sqrt{N}}\right) \xrightarrow[N\to\infty]{a.s.} 1 \ .$$

- Theorem (Bai, Yin)

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Corollary

As a consequence, if $\boldsymbol{\alpha} > 1$ then $\left(I - \frac{A}{\alpha\sqrt{N}}\right)$ is eventually invertible and

$$\boxed{\boldsymbol{x} = \left(I - \frac{A}{\boldsymbol{\alpha}\sqrt{N}}\right)^{-1} \mathbf{1}}$$

is well-defined.

Non-Hermitian random matrices III: Fluctuations of x's components

Theorem (Geman, Hwang)

- Let M fixed, $\boldsymbol{\alpha} > 4$ and $x_k = \left[\left(I - \frac{A}{\boldsymbol{\alpha}\sqrt{N}} \right)^{-1} \mathbf{1} \right]_k$

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$$\begin{pmatrix} x_1 \\ \cdot \\ x_M \end{pmatrix} \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}_M\left(\mathbf{1}_M, \frac{1}{\alpha^2 - 1} I_M\right)$$

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Corollary

- If $\boldsymbol{\alpha} > 4$ fixed, the probability to obtain a positive solution goes to zero:

$$\mathbb{P}\left\{\inf_{k\in[N]} x_k > 0\right\} \le \mathbb{P}\left\{\inf_{k\in[M]} x_k > 0\right\} \sim \Phi^M \xrightarrow[M\to\infty]{} 0\,.$$

where $\Phi = \int_{-\sqrt{\alpha^2-1}}^{\infty} \mathcal{N}(dx)$.

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Conclusion

- Feasible solutions for $\boxed{\boldsymbol{x} = \mathbf{1} + \frac{A}{\boldsymbol{\alpha}\sqrt{N}}\boldsymbol{x}}$ are eventually extremely rare.

Feasibility of the solution

Consider the system

$$\boldsymbol{x} = \mathbf{1} + \frac{A}{\boldsymbol{\alpha}\sqrt{N}}\boldsymbol{x} \quad \text{where} \quad \boldsymbol{\alpha} = \boldsymbol{\alpha}_N \xrightarrow[N\to\infty]{} \infty\,.$$

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- If $\boxed{\frac{\boldsymbol{\alpha}_N}{\boldsymbol{\alpha}_N^*} \le 1-\delta}$ for $N \gg 1$ then $\boxed{\mathbb{P}\left\{\inf_{k\in[N]} x_k > 0\right\} \xrightarrow[N\to\infty]{} 0}$.

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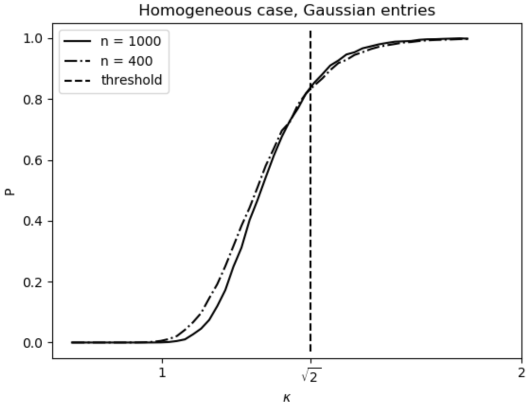
About the logarithmic factor

N	10^2	10^3	10^4	10^5	10^6
$\frac{1}{\alpha_N^*}$	0.33	0.27	0.23	0.21	0.19

- The quantity $\frac{1}{\alpha_N^*} = \frac{1}{\sqrt{2\log N}}$ vanishes extremely slowly as N increases.

Phase transition (gaussian case)



- We plot the frequency of positive solutions over 10000 trials for the system

$$\boldsymbol{x} = \mathbf{1} + \frac{1}{\kappa\sqrt{\log(N)}}\frac{A}{\sqrt{N}}\boldsymbol{x}$$

as a function of the parameter κ.

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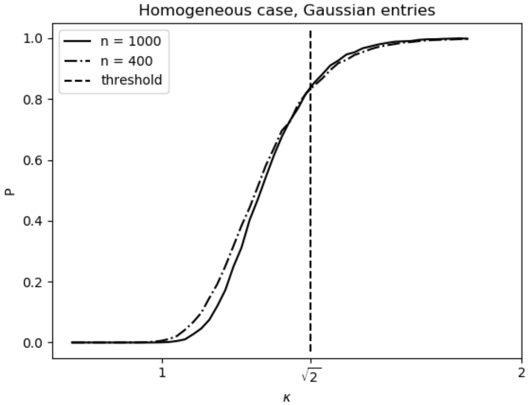

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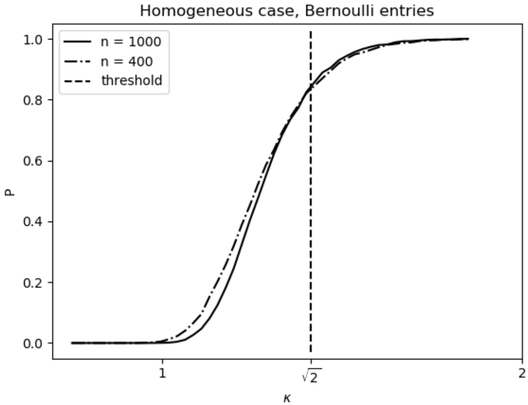
  as a function of the parameter κ.
- A phase transition occurs at the critical value $\kappa = \sqrt{2}$.

Phase transition (non-gaussian case)



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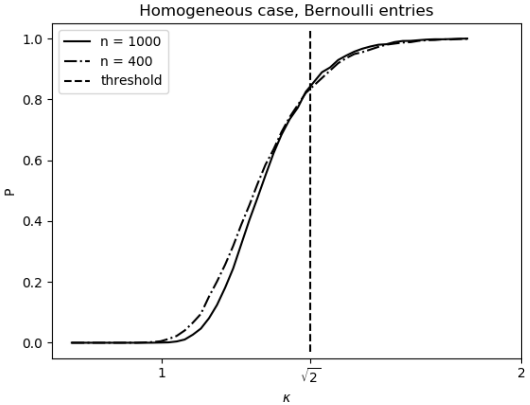
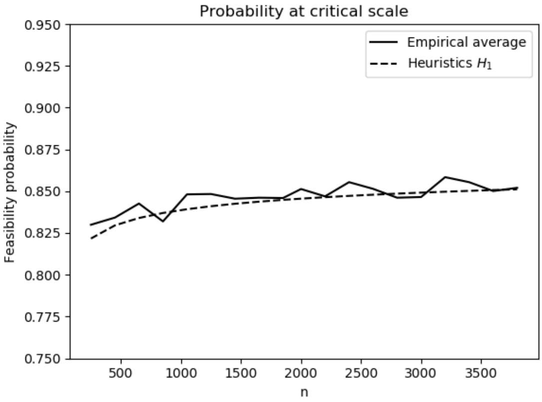


- Same simulations for (centered and normalized) Bernouilli entries.
- The phase transition does not seem to depend on the distribution of the entries.

A heuristics at critical scaling $\boldsymbol{\alpha}_N^* = \sqrt{2\log(N)}$



- At the critical scaling, we have the heuristics

$$\mathbb{P}\left\{\inf_{k\in[N]} x_k > 0\right\} \approx 1 - \sqrt{\frac{e}{4\pi\log(N)}} + \frac{e}{8\pi\log(N)}$$

  based on Gumbel approximation of the minimum of independent $\mathcal{N}(0,1)$.
- Solid line corresponds to the frequency of positive solutions over 10000 simulations at critical scaling - dotted line corresponds to the heuristics formula

Stability

Theorem (Bizeul, N.)

- Recall $\alpha_N^* = \sqrt{2\log(N)}$. Let

$$\boldsymbol{x} = \mathbf{1} + \frac{A}{\alpha\sqrt{N}}\boldsymbol{x} \qquad \text{and} \qquad \boldsymbol{\ell}^+ = \liminf_{N\to\infty} \frac{\alpha_N}{\alpha_N^*}$$

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- In particular, **feasibility implies stability**.

Reminder on Gaussian extreme values

- Let $(Z_k)_{k\in[N]}$ i.i.d. $\mathcal{N}(0,1)$ random variables, Denote by

$$M_N = \max_{k\in[N]} Z_k \quad \text{and} \quad \check{M}_N = \min_{k\in[N]} Z_k \ ,$$

$$\alpha_N^* = \sqrt{2\log(N)} \quad \text{and} \quad \beta_N^* = \alpha_N^* - \frac{1}{2\alpha_N^*}\log(4\pi\log(N))$$

- Then

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- and

$$\mathbb{E}M_N \sim \sqrt{2\log(N)} \quad \text{and} \quad \mathbb{E}\check{M}_N \sim -\sqrt{2\log(N)}$$

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1. **Unfold the resolvent.**

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$$\begin{aligned} x_k &= \left[\left(I - \frac{A}{\alpha\sqrt{N}} \right)^{-1} \mathbf{1} \right]_k \\ &= 1 + \frac{1}{\alpha} \underbrace{\frac{[A\mathbf{1}]_k}{\sqrt{N}}}_{:=Z_k} + \frac{1}{\alpha^2} \underbrace{\left[\left(\frac{A}{\sqrt{N}} \right)^2 \left(I - \frac{A}{\alpha\sqrt{N}} \right)^{-1} \mathbf{1} \right]_k}_{:=R_k} \end{aligned}$$

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Elements of proof

Recall that the feasible solution $\boldsymbol{x} = (x_k)$ writes

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$\Rightarrow$ The main effort is to prove that $A \mapsto R_k(A)$ is K-Lipschitz.

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Let $\boldsymbol{r}$ is $N \times 1$ deterministic. We are interested in the equation

$$\boxed{\boldsymbol{x} = \boldsymbol{r} + \frac{A}{\boldsymbol{\alpha}\sqrt{N}}\boldsymbol{x}} \quad \text{where} \quad \begin{cases} \boldsymbol{r}_{\min}(n) = \min_k r_k \\ \boldsymbol{r}_{\max}(n) = \max_k r_k \\ \sigma_{\boldsymbol{r}}(n) = \sqrt{\frac{1}{N}\sum_k r_k^2} \end{cases}$$

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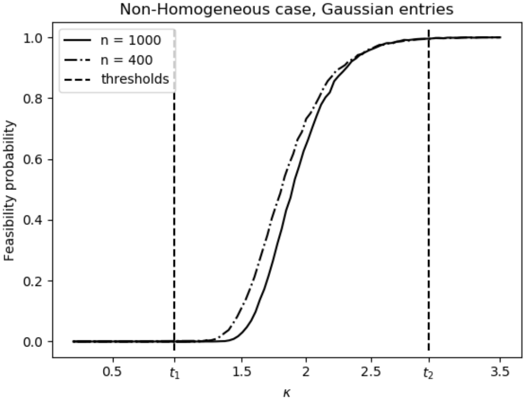
Non-homogeneous case II



- In the non-homogeneous case, there is a transition buffer

$$\frac{\boldsymbol{\alpha}_N}{\boldsymbol{\alpha}_N^*} \in \left[\frac{\sigma_{\boldsymbol{r}}(n)}{\boldsymbol{r}_{\max}(n)}, \frac{\sigma_{\boldsymbol{r}}(n)}{\boldsymbol{r}_{\min}(n)}\right]$$

and not a sharp transition at $\frac{\boldsymbol{\alpha}_N}{\boldsymbol{\alpha}_N^*} \sim 1$.

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References

- Positive solutions for Large Random Linear Systems, Bizeul and Najim, `arXiv:1904.04559`
- On-going project 80 PRIME — CNRS (KARATE) with François Massol and others

  Lot**KA**-Volter**RA** models – when random ma**T**rix theory meets theoretical **E**cology