Crack monitoring using transmission eigenvalues with artificial backgrounds

Lorenzo Audibert[1], **Lucas Chesnel**[2], **Houssem Haddar**[2], **Kevish Napal**[2,*]

[1]Prisme, EDF R&D, Chatou, France
[2]CMAP, École Polytechnique, Palaiseau, France
[*]Email: kevish.napal@polytechnique.edu

Abstract

We propose a new method to localize cracks from far field data based on Transmission Eigenvalues (TEs) associated with a carefully chosen artificial background. It relies on scanning the probed domain with a fictitious inclusion and exploits the fact that TEs change only if the inclusion intersects the cracks. We explain how these TEs can be identified from measured far field data and validate our procedure in the case of extended cracks. The method also allows for the detection and quantification of small cracks aggregates.

Keywords: inverse scattering, crack monitoring, transmission eigenvalues.

1 Introduction

In this work, we are interested in the problem of identifying cracks embedded in some homogeneous background from far field data at multiple frequencies. We rely on the notion of Transmission Eigenvalues. For the considered inverse problem, TEs were usually presented as the frequencies one should avoid to guarantee the success of some inversion methods such as the Linear Sampling Method or the Factorization Method. Several recent works have tried to use TEs to recover quantitative information on the material properties of probed domains. In this approach, the difficulty lies in the fact that the link between TEs and the physical parameters is not straightforward. To bypass this problem, it has been proposed in the literature to work with a modified far field operator constructed from an artificial background and for which the corresponding TEs have a more direct connection to the physical parameters [1]. We shall exploit this idea for the imaging problem related to cracks. Let us recall that because the cracks have empty interior, one cannot define usual TEs if the background is homogeneous. Working with a non homogeneous artificial background, for instance containing an obstacle, we show that we can define new TEs whose values depend on the relative positions between the crack and the artificial obstacle. The numerical procedure is then as follows. First, we subtract to the measured far field F a numerically computed far field F^{num} corresponding to an artificial sound-soft obstacle located in some arbitrary domain Ω and we set $F^{\text{rel}} := F - F^{\text{num}}$. The TEs associated with the relative far field operator F^{rel} are then defined as the frequencies such that there are generalized incident fields that have the same far field both for the true reference medium and for the artificial one. In absence of crack in Ω, these TEs are simply the Dirichlet eigenvalues of Ω, otherwise they are different. These TEs can be evaluated from F^{rel} using the framework of generalized linear sampling method. Varying the position of Ω, and comparing the TEs with the Dirichlet eigenvalues of Ω, one is able to identify the cracks position. In the case of small cracks aggregates, one obtains an indicator function on the cracks density.

2 The forward scattering problem

We start by presenting the scattering problem for a crack embedded in a homogeneous medium. Let $\Gamma \subset \mathbb{R}^3$ be a portion of a non-intersecting surface that encloses a domain with smooth boundary. The scattering of the plane wave $u_i(\theta,x) := e^{ik\theta\cdot x}$ of incident direction $\theta \in \mathbb{S}^2 := \{\theta \in \mathbb{R}^3 \,|\, |\theta| = 1\}$ by Γ leads us to consider the problem

$$\left|\begin{array}{l} \text{Find } u = u_i + u_s \text{ such that} \\ \Delta u + k^2 u \;=\; 0 \quad \text{in } \mathbb{R}^3 \setminus \Gamma \\ \quad\sigma(u) \;=\; 0 \quad \text{on } \Gamma \\ \displaystyle\lim_{|x|\to+\infty} |x|\big(\partial_{|x|} u_s - iku_s\big) = 0. \end{array}\right. \qquad (1)$$

Here $\sigma(u)$ is a generic boundary condition (Neumann, Dirichlet, ...) and $k > 0$ is the wave number. Moreover, the last line of (1) corresponds to the Sommerfeld radiation condition. The scattered field $u_s(\theta,x)$ has the expansion

$$u_s(\theta,x) = e^{ik|x|}|x|^{-1}\left(u_s^\infty(\theta,\widehat{x}) + O(1/|x|)\right) \qquad (2)$$

as $|x| \to +\infty$, uniformly in $\widehat{x} = x/|x|$, where $u_s^\infty(\theta,\widehat{x}) \in \mathbb{C}$ is the far field pattern in the direction $\widehat{x}$. The inverse problem we consider consists in reconstructing Γ from the knowledge of $u_s^\infty(\cdot,\cdot) : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{C}$. We define the far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$

such that

$$(Fg)(\hat{x}) = \int_{\mathbb{S}^2} g(\theta)\, u_s^\infty(\theta,\hat{x})\, ds(\theta). \qquad (3)$$

By linearity of (1), Fg is nothing but the far field pattern of the scattered field associated with the incident field $u_i(g) := \int_{\mathbb{S}^2} g(\theta) e^{ik\theta\cdot x} ds(\theta)$ (Herglotz wave function), with $g \in L^2(\mathbb{S}^2)$.

3 The relative far field operator

Let Ω be an arbitrary bounded domain of $\mathbb{R}^3$. Introduce $F^{\text{num}} : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ the far field operator defined as F in (3) replacing u_s by $\tilde{u}_s$, $\tilde{u}_s$ being the solution of the exterior Dirichlet problem

$$\left|\begin{array}{rcl} \Delta\tilde{u}_s + k^2\tilde{u}_s &=& 0 \quad \text{in } \mathbb{R}^3\setminus\Omega \\ \tilde{u}_s + u_i &=& 0 \quad \text{on } \partial\Omega \\ \lim\limits_{|x|\to+\infty} |x|(\partial_{|x|}\tilde{u}_s - ik\tilde{u}_s) = 0. && \end{array}\right. \qquad (4)$$

Note that F^{num} does not depend on the data and can be computed numerically. Finally, we define the relative far field operator

$$F^{\text{rel}} := F - F^{\text{num}}.$$

One can show that F^{rel} admits a factorization $F^{\text{rel}} = G^{\text{rel}}H^{\text{rel}}$ for certain operators G^{rel}, H^{rel} that we do not explicit here. The TE are then defined as the values of $k > 0$ such that G^{rel} has a non trivial kernel. Using the Rellich lemma, one can prove the following characterization.

Theorem 1 *TEs coincide with the $k > 0$ such that there is a non trivial $w \in H^1(\Omega\setminus\Gamma)$ solving*

$$\left|\begin{array}{rcl} \Delta w + k^2 w &=& 0 \quad \textit{in } \Omega\setminus\Gamma \\ w &=& 0 \quad \textit{on } \partial\Omega \\ \sigma(w) &=& 0 \quad \textit{on } \Gamma\cap\Omega. \end{array}\right. \qquad (5)$$

From this proposition, we observe that when $\Gamma\cap\Omega = \emptyset$, the TEs are nothing but the eigenvalues of the Dirichlet laplacian in Ω. In the next section, we will explain how to compute TEs from F. Hence scanning the probed domain with different Ω, one can identify the cracks using TEs. In the particular case of sound-hard cracks, i.e. when $\sigma(w) = \partial_\nu w$, ν being a unit normal vector to Γ, the spectrum of (5) consists of real positive eigenvalues $0 \le \tau_1^\Gamma \le \tau_2^\Gamma \le \dots$ satisfying the following min-max principle:

$$\tau_j^\Gamma = \min_{W\in\mathscr{W}_j^\Gamma} \max_{w\in W\setminus\{0\}} \frac{\|\nabla w\|^2_{L^2(\Omega\setminus\Gamma)}}{\|w\|^2_{L^2(\Omega\setminus\Gamma)}}, \qquad (6)$$

where $\mathscr{W}_j^\Gamma$ denotes the sets of j-dimensional subspaces of

$$V^\Gamma := \{v \in H^1(\Omega\setminus\Gamma) | v = 0 \text{ on } \partial\Omega\}.$$

Consequently, given two cracks $\Gamma_1 \subset \Gamma_2 \subset \Omega$, since $V^{\Gamma_1} \subset V^{\Gamma_2}$, we obtain that $\tau_j^{\Gamma_1} \ge \tau_j^{\Gamma_2}$ for all $j \in \mathbb{N}$. This monotonicity result, which also holds in the case of sound-soft cracks, allow us to quantify crack densities in Ω.

4 Imaging with TE's computed from far field data

The computation of TEs exploits the behaviour of the solution $g \in L^2(\mathbb{S}^2)$ of the far field equation $F^{\text{rel}}g \approx \Phi_z^\infty$, where Φ_z^∞ is the far field of the fundamental solution Φ_z of the Helmholtz equation (with a Dirac source term at $z \in \mathbb{R}^3$). To state our result, we define the cost function such that, for $\alpha > 0$, $g \in L^2(\mathbb{S}^2)$,

$$J_\alpha(g) = \alpha P(g) + \|F^{\text{rel}}g - \Phi_z^\infty\|_{L^2(\mathbb{S}^2)}, \qquad (7)$$

with the penalty term $P(g) = |\langle F^{\text{num}}g, g\rangle| + |\langle Fg, g\rangle|$. Let g_z^α be a minimizing sequence of J_α.

Theorem 2 *Assume that k is such that $F^{\text{rel}} : L^2(\mathbb{S}^2) \longrightarrow L^2(\mathbb{S}^2)$ has dense range. Then k^2 is an eigenvalue of* (5) *if and only if the set of point z for which $P(g)$ is bounded as $\alpha \to 0$ is nowhere dense in Ω.*

As a consequence, the eigenvalues τ_j^Γ coincide with the peaks in the curve $k \mapsto \int_\Omega P(g_z^\alpha)\, dz$ for small values of α. In Figure 1, we provide a numerical result in 2D. We identify a sound-hard crack by working with a collection of artificial backgrounds with sound-soft disks. For each disk, the distance between TEs and Dirichlet eigenvalues is materialized by the contrast in the red colour. Note that when the radius of the disks tends to zero, the Dirichlet eigenvalues blow up. Therefore, to obtain a thin resolution, it is necessary to work at high frequencies with a k large band.

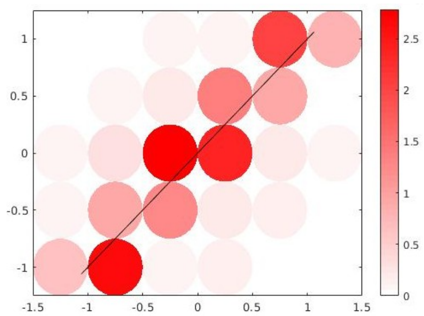

Figure 1: Detection of a sound-hard crack using TEs for artificial backgrounds with sound-soft disks.

References

[1] L. Audibert, L. Chesnel, H. Haddar, Transmission eigenvalues with artificial background for explicit index identification, *C. R. Acad. Sci. Paris, Ser. I*, vol. 356, 6:626-631, 2018.