Strongly oscillating singularities for the interior transmission eigenvalue problem

Anne-Sophie Bonnet-Ben Dhia[1] **and Lucas Chesnel**[2]

[1]Laboratoire POEMS, UMR 7231 CNRS/ENSTA/INRIA, ENSTA ParisTech, 828, boulevard des Maréchaux, 91762 Palaiseau, France

[2]Department of Mathematics and Systems Analysis, Aalto University, P.O. Box 11100 FI-00076 Aalto, Finland

E-mail: `Anne-Sophie.Bonnet-Bendhia@ensta-paristech.fr`, `lucas.chesnel@aalto.fi`

Abstract. In this paper, we investigate a two-dimensional interior transmission eigenvalue problem for an inclusion made of a composite material. We consider configurations where the difference between the parameters of the composite material and the ones of the background change sign on the boundary of the inclusion. In a first step, under some assumptions on the parameters, we extend the variational approach of the T-coercivity to prove that the transmission eigenvalues form at most a discrete set. In the process, we also provide localization results. Then, we study what happens when these assumptions are not satisfied. The main idea is that, due to very strong singularities that can occur at the boundary, the problem may lose Fredholmness in the natural H^1 framework. Using Kondratiev theory, we propose a new functional framework where the Fredholm property is restored.

1. Introduction

The interior transmission eigenvalue problem (ITEP) has now a long history. In electromagnetism, both the scalar [13, 31, 14, 6, 15] and Maxwell [18, 8, 7, 9] problems have been widely studied. But lot of questions remain. In this paper, we investigate an ITEP in a situation which, up to our knowledge, has not been studied before. Denoting by μ the magnetic permeability of the inclusion D and by μ_0 the one of the reference medium, we consider a two-dimensional scalar problem where the sign of $\mu - \mu_0$ changes on ∂D. This 2D configuration corresponds to electromagnetic scattering in transverse electric polarization. Notice that μ can be a matrix, if the inclusion is filled with an anisotropic material ($\mu = \mu_0 \, A^{-1}$ in the sequel). The difficulty lies in the fact that solutions of the ITEP can exhibit a very singular behaviour at the points of ∂D where $\mu - \mu_0$ changes its sign.

To carry our study, we will rely on the analogy between the ITEP and the sign-changing transmission problem (SCTP) which models the interface between a metamaterial (with electromagnetic constants ε and μ taking negative real values) and a classical material (with positive ε and μ).

We have already pointed out this analogy in previous papers [3, 10]. In particular, we have shown that the $\mathtt{T}$-coercivity technique (that we used extensively for studying the SCTP [4, 1]) allows to establish Fredholm property for the variational formulation of the ITEP, in the classical framework (the fields belong to $\mathrm{H}^1(D)$), as soon as the sign of $\mu - \mu_0$ is constant in a neighbourhood of ∂D. Then the discreteness of interior transmission eigenvalues can be deduced under additional hypotheses on the dielectric permittivity ε. When μ, μ_0 are smooth, these additional hypotheses have been relaxed in [21]. However, in this work, the authors also need to impose that $\mu - \mu_0$ has a constant sign in a neighbourhood of ∂D.

In the opposite case where $\mu - \mu_0$ has not a constant sign in a neighbourhood of ∂D, one expects, still by analogy with the SCTP, that the $\mathrm{H}^1(D)$ functional framework may become inappropriate. For the SCTP, this happens when the interface between the negative material and the positive one has corners. Then there exist so-called black-hole waves which propagate towards the corners, with associated fields which do not belong to $\mathrm{H}^1(D)$. The corresponding theory is detailed in [2] for a model problem: an appropriate functional framework is derived and justified, using Mellin transform and Kondratiev weighted Sobolev spaces.

Our objective in the present paper is twofold.

(i) First we aim at relaxing the condition on $\mu - \mu_0$ (or equivalently on $A - Id$ in the anisotropic case) imposed in [3] to apply the $\mathtt{T}$-coercivity technique. Doing so, we will derive new conditions on μ, allowing a change of sign of $\mu - \mu_0$ on ∂D, such that the ITEP is of Fredholm type in the $\mathrm{H}^1(D)$ framework.

(ii) On the other hand, we want to point out that this Fredholm property of the ITEP in the $\mathrm{H}^1(D)$ framework can fail. Adapting the ideas of [2], we will propose in this case new extended functional frameworks to restore Fredholmness. For each extension, the associated spectrum contains two parts: the set of transmission eigenvalues (with eigenvectors in $\mathrm{H}^1(D)$) and a set of spurious eigenvalues (with a strongly oscillating behaviour in $\mathrm{L}^2(D) \setminus \mathrm{H}^1(D)$). This spurious spectrum depends on the choice of the new functional framework. We will see how to select the good extensions so that no spurious spectrum appears on the real and purely imaginary axes. Let us emphasize that it is necessary to take into account the strongly oscillating behaviour in the functional framework only to recover Fredholmness. However non spurious eigenvalues correspond to regular (non oscillating) eigenvectors and do not depend on the choice of the extension.

The paper is organized as follows. The interior transmission eigenvalue problem is defined in Section 2, and the idea of the $\mathtt{T}$-coercivity method is briefly recalled in a simple configuration. In Section 3, we investigate the extension of this approach to cases where the sign of $A - Id$ changes on ∂D. Building appropriate operators $\mathtt{T}$, we derive a sufficient condition on A which ensures that the ITEP is of Fredholm type. To relax this condition, we turn to a different approach. First, in Section 4, we study in details the singularities of a model problem, around a point of ∂D where $A - Id$ is sign-changing. We focus particularly on the possible existence of "strongly oscillating singularities", which are local solutions of the interior transmission equation which behave like $r^{i\eta}$ with $\eta \in \mathbb{R}$ (where r denotes the distance to the singular point). Such solutions are not in H^1, and this is why the variational $\mathtt{T}$-coercivity method fails in this case. This leads to generalize the definition of the ITEP by changing the functional framework. In Section 5, we introduce well-suited weighted Sobolev spaces, where a strongly oscillating behavior is allowed and where the Fredholm property of the model problem is established. Finally, Section 6 is devoted to the study of discreteness of the transmission eigenvalues. We consider both configurations where the variational approach applies and configurations with strongly oscillating singularities. We discuss the relevance of the problems set in the new functional frameworks.

2. Setting of the problem

2.1. Basic definitions

Consider $D \subset \mathbb{R}^2$ a bounded simply connected domain with Lipschitz boundary ∂D. The unit outward normal vector to ∂D will be denoted ν. We study a time-harmonic electromagnetic scattering problem for an inclusion whose coefficients are given by $A(\boldsymbol{x})$ and $n(\boldsymbol{x})$. To simplify the presentation, we assume that the background is homogeneous. Here, $A \in \mathrm{L}^\infty(D, \mathbb{C}^{2\times 2})$ is a matrix valued function such that $A(\boldsymbol{x})$ is hermitian positive definite for almost all $\boldsymbol{x} \in D$. Moreover, $n \in \mathrm{L}^\infty(D)$ is a strictly positive real valued function. We suppose that $A^{-1} \in \mathrm{L}^\infty(D, \mathbb{C}^{2\times 2})$ and $n^{-1} \in \mathrm{L}^\infty(D)$.
If $\mathscr{O}$ is an open subset of $\mathbb{R}^2$, we denote indistinctly $(\cdot,\cdot)_{\mathscr{O}}$ the inner products of $\mathrm{L}^2(\mathscr{O}) := \mathrm{L}^2(\mathscr{O}, \mathbb{C})$ and $\mathbf{L}^2(\mathscr{O}) := \mathrm{L}^2(\mathscr{O}, \mathbb{C}^2)$, and $\|\cdot\|_{\mathscr{O}}$ the associated norms. We

also denote $\mathrm{H}^1(D)$ instead of $\mathrm{H}^1(D,\mathbb{C})$. Moreover, we define

$$\sup_{\mathscr{O}} A := \sup_{\boldsymbol{x}\in\mathscr{O}} \sup_{\xi\in\mathbb{C}^2,|\xi|=1} (\xi\cdot A(\boldsymbol{x})\overline{\xi}) \quad \text{and} \quad \inf_{\mathscr{O}} A := \inf_{\boldsymbol{x}\in\mathscr{O}} \inf_{\xi\in\mathbb{C}^2,|\xi|=1} (\xi\cdot A(\boldsymbol{x})\overline{\xi}).$$

We will say that the open set $\mathscr{V} \subset \mathbb{R}^2$ is a neighbourhood of ∂D if there holds $\partial D \subset (\mathscr{V} \cap \overline{D})$.

Definition 2.1 *The elements $k \in \mathbb{C}$ such that there exists a pair $(u,w) \neq (0,0)$ solving the problem*

$$\left|\begin{array}{lcll} \text{Find } (u,w) \in \mathrm{H}^1(D)\times\mathrm{H}^1(D) \text{ such that:} & & & \\ \mathrm{div}\,(A\nabla u) + k^2 n u & = & 0 & \text{in } D \\ \Delta w + k^2 w & = & 0 & \text{in } D \\ u - w & = & 0 & \text{on } \partial D \\ \nu\cdot A\nabla u - \nu\cdot\nabla w & = & 0 & \text{on } \partial D \end{array}\right. \tag{1}$$

are called transmission eigenvalues.

Here, w and u denote respectively the incident field which does not scatter and the total field inside the inclusion. One classically proves that (u,w) satisfies (1) if and only if (u,w) satisfies the problem

$$\left|\begin{array}{l} \text{Find } (u,w)\in \mathrm{X} \text{ such that, for all } (u',w')\in\mathrm{X}, \\ a_k((u,w),(u',w')) := (A\nabla u,\nabla u')_D - (\nabla w,\nabla w')_D - k^2\left((nu,u')_D - (w,w')_D\right) = 0, \end{array}\right. \tag{2}$$

with $\mathrm{X} = \{(u,w)\in\mathrm{H}^1(D)\times\mathrm{H}^1(D)\,|\,u-w\in\mathrm{H}^1_0(D)\}$. With the Riesz representation theorem, we define the operator $\mathscr{A}_k : \mathrm{X}\to\mathrm{X}$ such that, for all $((u,w),(u',w'))\in\mathrm{X}\times\mathrm{X}$,

$$(\mathscr{A}_k(u,w),(u',w'))_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)} = a_k((u,w),(u',w')). \tag{3}$$

The spectral problem associated with (1) differs from classical ones because a_k is not coercive on X neither "coercive+compact".

2.2. The T*-coercivity method*

We briefly recall the method of the $\mathtt{T}$-coercivity in the simple case: $\sup_D A < 1$ and $\sup_D n < 1$. The idea is to consider an equivalent formulation of (2) where a_k is replaced by a_k^{T} defined by

$$a_k^{\mathrm{T}}((u,w),(u',w')) := a_k((u,w),\mathtt{T}(u',w')), \quad \forall((u,w),(u',w'))\in\mathrm{X}\times\mathrm{X}, \tag{4}$$

$\mathtt{T}$ being an *ad hoc* isomorphism of X. Indeed, $(u,w)\in\mathrm{X}$ satisfies $a_k((u,w),(u',w')) = 0$ for all $(u',w')\in\mathrm{X}$ if, and only if, it satisfies $a_k^{\mathrm{T}}((u,w),(u',w')) = 0$ for all $(u',w')\in\mathrm{X}$. In the present case, let us take $\mathtt{T}$ such that $\mathtt{T}(u,w) := (u-2w,-w)$. One can check that $\mathtt{T}^2 = Id$. Therefore, $\mathtt{T}$ is an isomorphism which is equal to its inverse. Using Young's inequality, we can write, for $k = i\kappa$ with $\kappa \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$, for $\alpha, \beta > 0$ and for all $(u,w) \in \mathrm{X}$,

$$
\begin{array}{rcl}
\left|a_k^{\mathrm{T}}((u,w),(u,w))\right| & = & |(A\nabla u, \nabla u)_D + (\nabla w, \nabla w)_D - 2(A\nabla u, \nabla w)_D \\
 & & +\kappa^2\left((nu,u)_D + (w,w)_D - 2(nu,w)_D\right)| \\
 & \geq & (A\nabla u, \nabla u)_D + (\nabla w, \nabla w)_D + \kappa^2\left((nu,u)_D + (w,w)_D\right) \\
 & & -2\left|(A\nabla u, \nabla w)_D\right| - 2\kappa^2\left|(nu,w)_D\right| \\
 & \geq & ((1-\alpha)A\nabla u, \nabla u)_D + ((1-\alpha^{-1}\mathrm{sup}_D A)\nabla w, \nabla w)_D \\
 & & +\kappa^2\left(((1-\beta)nu,u)_D + ((1-\beta^{-1}\mathrm{sup}_D n)w,w)_D\right).
\end{array}
\qquad (5)
$$

Taking α and β such that $\sup_D A < \alpha < 1$ and $\sup_D n < \beta < 1$, this estimate proves that a_k^{T} is coercive over X. Using Lax-Milgram theorem and since $\mathtt{T}$ is an isomorphism of X, one deduces that $\mathscr{A}_k$ is an isomorphism of X for $k = i\kappa$ with $\kappa \in \mathbb{R}^*$. Besides, for a general $k \in \mathbb{C}$ and for $\kappa \in \mathbb{R}^*$, the operator $\mathscr{A}_k - \mathscr{A}_{i\kappa}$ is compact since the embedding of X in $\mathrm{L}^2(D) \times \mathrm{L}^2(D)$ is compact. As a consequence of the analytic Fredholm theorem, this proves that the set of transmission eigenvalues is at most discrete with infinity as the only accumulation point when $\sup_D A < 1$ and $\sup_D n < 1$. More generally, in [3], we prove the following results.

Theorem 2.2 *1) Assume that there exists $\mathscr{V}$, a neighbourhood of ∂D, such that* $\sup_{D\cap\mathscr{V}} A < 1$ *or* $1 < \inf_{D\cap\mathscr{V}} A$*. Then for all $k \in \mathbb{C}$, the operator $\mathscr{A}_k : \mathrm{X} \to \mathrm{X}$ defined in (3) is of Fredholm type.*
2) Assume that there exists $\mathscr{V}$, a neighbourhood of ∂D, such that $\sup_{D\cap\mathscr{V}} A < 1$ *and* $\sup_{D\cap\mathscr{V}} n < 1$*, or* $\inf_{D\cap\mathscr{V}} A > 1$ *and* $\inf_{D\cap\mathscr{V}} n > 1$*. Then the set of transmission eigenvalues is at most discrete with infinity as the only accumulation point. Moreover, we can find two positive constants ρ and δ such that if $k \in \mathbb{C}$ verifies $|k| > \rho$ and $|\Re e\, k| < \delta\, |\Im m\, k|$, then k is not a transmission eigenvalue.*

Remark 2.3 *With a stronger assumption on A, we can weaken the condition on n (see [3, Theorem 3.4]).*

The goal in this paper is to understand what happens when $A - Id$ changes sign on ∂D. To study such a configuration, we will first work on a simplified interior transmission problem.

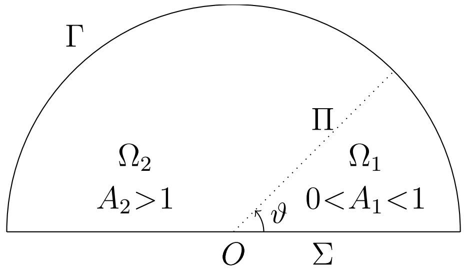

Figure 1. Geometry of the domain Ω.

3. A model problem

3.1. Setting of the problem

Let us denote (r, θ) the polar coordinates centered at the origin O. We define (see Figure 1) the sets:

$$\begin{aligned}
\Omega_1 &:= \{(r\cos\theta, r\sin\theta)\,|\,0<r<1,\ 0<\theta<\vartheta\};\\
\Omega_2 &:= \{(r\cos\theta, r\sin\theta)\,|\,0<r<1,\ \ \vartheta<\theta<\pi\};\\
\Omega &:= \{(r\cos\theta, r\sin\theta)\,|\,0<r<1,\ \ 0<\theta<\pi\};\\
\Gamma &:= \{(r\cos\theta, r\sin\theta)\,|\,r=1,\ \ 0<\theta<\pi\};\\
\Pi &:= \{(r\cos\theta, r\sin\theta)\,|\,0<r<1,\ \ \theta=\vartheta\};\\
\Sigma &:= (-1;1)\times\{0\}.
\end{aligned}$$

In particular, the boundary of Ω verifies $\partial\Omega = \overline{\Gamma}\cup\overline{\Sigma}$. We introduce the function $A : \Omega \to \mathbb{R}$ such that $A = A_1$ in Ω_1 and $A = A_2$ in Ω_2. Here $0 < A_1 < 1$ and $A_2 > 1$ are two constants. Our objective in this section is to understand the properties of the following simplified interior transmission problem:

$$\left|\begin{array}{lcll}
\text{Find } (u,w)\in \mathrm{H}^1(\Omega)\times\mathrm{H}^1(\Omega) \text{ such that:} & & & \\
\mathrm{div}\,(A\nabla u) & = & f & \text{in } \Omega\\
\Delta w & = & g & \text{in } \Omega\\
u-w & = & 0 & \text{on } \Sigma\\
\nu\cdot A\nabla u-\nu\cdot\nabla w & = & 0 & \text{on } \Sigma\\
u & = & 0 & \text{on } \Gamma\\
w & = & 0 & \text{on } \Gamma.
\end{array}\right. \qquad (6)$$

In (6), ν denotes the unit outward normal vector to $\partial\Omega$. Moreover, f and g are two source terms which belong to some functional spaces which will be specified later. Compared to (1), in this interior transmission problem, the transmission conditions are written only on the part Σ of the boundary. This will lead to simplification in the analysis and this is why we say that it is a *simplified* interior transmission problem.

Let us define the linear space $\mathrm{X}_\Sigma = \{(u,w) \in \mathrm{H}^1_\Gamma(\Omega)\times\mathrm{H}^1_\Gamma(\Omega)\,|\,u-w = 0 \text{ on } \Sigma\}$ where $\mathrm{H}^1_\Gamma(\Omega) = \{\varphi\in\mathrm{H}^1(\Omega)\,|\,\varphi = 0 \text{ on } \Gamma\}$. In the sequel, we will also use the space $\mathrm{H}^1_\Gamma(\Omega_2) = \{\varphi|_{\Omega_2}\,|\,\varphi\in\mathrm{H}^1_\Gamma(\Omega)\}$. To (6), we associate the sesquilinear form b on X_Σ such that $b((u,w),(u',w')) := (A\nabla u, \nabla u')_\Omega - (\nabla w, \nabla w')_\Omega$ and we consider the variational problem

$$\left|\begin{array}{l}\text{Find } (u,w) \in \mathrm{X}_\Sigma \text{ such that, for all } (u',w') \in \mathrm{X}_\Sigma,\\ b((u,w),(u',w')) = F((u',w')),\end{array}\right. \tag{7}$$

where $F \in \mathrm{X}_\Sigma^*$, the topological dual space of X_Σ made of the continuous antilinear forms on X_Σ. With the Riesz representation theorem, we define the operator $\mathscr{B} : \mathrm{X}_\Sigma \to \mathrm{X}_\Sigma$ such that, for all $((u,w),(u',w')) \in \mathrm{X}_\Sigma \times \mathrm{X}_\Sigma$,

$$(\mathscr{B}(u,w),(u',w'))_{\mathrm{H}^1(\Omega)\times\mathrm{H}^1(\Omega)} = b((u,w),(u',w')). \tag{8}$$

Again, the study of $\mathscr{B}$ is not simple because b is not coercive on X_Σ neither "coercive+compact".

3.2. A variational approach

In this section, we study the operator $\mathscr{B}$ using the variational approach of the $\mathtt{T}$-coercivity. However, since we have neither $A > 1$ nor $A < 1$ on Σ, the computation presented in §2.2 fails. We must work with a different isomorphism $\mathtt{T} : \mathrm{X}_\Sigma \to \mathrm{X}_\Sigma$. If φ is a measurable function on Ω, we denote $\varphi_1 := \varphi|_{\Omega_1}$, $\varphi_2 := \varphi|_{\Omega_2}$. Set

$$\mathtt{T}(u,w) = (u',w'), \quad \text{with} \quad (u',w') = \left|\begin{array}{ll}(u_1 - 2w_1 + 2R_2w_2, -w_1 + 2R_2u_2) & \text{on } \Omega_1\\ (u_2, -w_2 + 2u_2) & \text{on } \Omega_2\end{array}\right. . \tag{9}$$

In this definition, R_2 is the operator such that, on Ω_1, $(R_2\varphi_2)(r,\theta) = \varphi_2(r, \frac{\vartheta-\pi}{\vartheta}\theta + \pi)$ for $\varphi_2 \in \mathrm{H}^1_\Gamma(\Omega_2)$. First, notice that $R_2\varphi_2 = \varphi_2$ on Π for all $\varphi_2 \in \mathrm{H}^1_\Gamma(\Omega_2)$. As a consequence, one can check that $u' \in \mathrm{H}^1_\Gamma(\Omega)$ and $w' \in \mathrm{H}^1_\Gamma(\Omega)$. Then, one can verify that $u' - w' \in \mathrm{H}^1_0(\Omega)$ so that the operator $\mathtt{T}$ defined in (9) is indeed valued in X_Σ. Finally, noticing that $\mathtt{T}^2$ is equal to the identity of X_Σ, we deduce that $\mathtt{T}$ defines an isomorphism. For all $(u,w) \in \mathrm{X}_\Sigma$, we find

$$\begin{array}{rcl} b((u,w),\mathtt{T}(u,w)) & = & (A\nabla u, \nabla u)_\Omega + (\nabla w, \nabla w)_\Omega \\ & & +2(A_1\nabla u_1, \nabla(-w_1 + R_2w_2))_{\Omega_1} - 2(\nabla w_1, \nabla(R_2u_2))_{\Omega_1} - 2(\nabla w_2, \nabla u_2)_{\Omega_2}. \end{array}$$

Let us introduce $\alpha, \beta, \gamma, \eta$ four strictly positive parameters. Using Young's inequality and setting

$$\|R_2\| := \sup_{\varphi_2 \in \mathrm{H}^1_\Gamma(\Omega_2),\ \|\nabla\varphi_2\|_{\Omega_2}=1} \|\nabla(R_2\varphi_2)\|_{\Omega_1},$$

we can write:

$$\begin{array}{c} |2(A_1\nabla u_1, \nabla(-w_1 + R_2w_2))_{\Omega_1}| \\ \le A_1(\alpha+\beta)(\nabla u_1, \nabla u_1)_{\Omega_1} + A_1\alpha^{-1}(\nabla w_1, \nabla w_1)_{\Omega_1} + A_1\beta^{-1}\|R_2\|^2(\nabla w_2, \nabla w_2)_{\Omega_2}, \end{array}$$

$$|2(\nabla w_1, \nabla(R_2u_2))_{\Omega_1}| \le \gamma(\nabla w_1, \nabla w_1)_{\Omega_1} + \gamma^{-1}\|R_2\|^2(\nabla u_2, \nabla u_2)_{\Omega_2},$$

$$|2(\nabla w_2, \nabla u_2)_{\Omega_2}| \le \eta(\nabla w_2, \nabla w_2)_{\Omega_2} + \eta^{-1}(\nabla u_2, \nabla u_2)_{\Omega_2}.$$

We deduce

$$\begin{array}{rl} & b((u,w),\mathtt{T}(u,w)) \\ \geq & A_1(1-\alpha-\beta)(\nabla u_1,\nabla u_1)_{\Omega_1}+(A_2-\gamma^{-1}\|R_2\|^2-\eta^{-1})(\nabla u_2,\nabla u_2)_{\Omega_2} \\ & +(1-\alpha^{-1}A_1-\gamma)(\nabla w_1,\nabla w_1)_{\Omega_1}+(1-A_1\beta^{-1}\|R_2\|^2-\eta)(\nabla w_2,\nabla w_2)_{\Omega_2}. \end{array} \tag{10}$$

In [1, 11], we obtain $\|R_2\|^2=\Upsilon_\vartheta$ where

$$\Upsilon_\vartheta := \max\Big(\frac{\pi-\vartheta}{\vartheta},\frac{\vartheta}{\pi-\vartheta}\Big). \tag{11}$$

Notice that there holds $\Upsilon_\vartheta\geq 1$ with $\Upsilon_\vartheta=1$ if and only if $\vartheta=\pi/2$. From (10), we are going to prove the following result.

Lemma 3.1 *Assume that A_1, A_2 verify*

$$A_1<\frac{1}{1+\Upsilon_\vartheta}\qquad\text{and}\qquad A_2>\frac{1+\Upsilon_\vartheta}{1-A_1(1+\Upsilon_\vartheta)}, \tag{12}$$

where Υ_ϑ is defined in (11). Then the operator $\mathscr{B}:\mathrm{X}_\Sigma\to\mathrm{X}_\Sigma$ defined in (8) is an isomorphism.

Remark 3.2 *Thus, when $A_1<\min(\vartheta/\pi,(\pi-\vartheta)/\pi)$, the operator $\mathscr{B}:\mathrm{X}_\Sigma\to\mathrm{X}_\Sigma$ defined in (8) is an isomorphism for A_2 large enough.*

Remark 3.3 *Actually the proof we give allows to obtain the following result. Assume that $A\in\mathrm{L}^\infty(\Omega,\mathbb{C}^{2\times 2})$ is a matrix valued function such that $A(\boldsymbol{x})$ is hermitian positive definite for almost all $\boldsymbol{x}\in\Omega$, with $A^{-1}\in\mathrm{L}^\infty(\Omega,\mathbb{C}^{2\times 2})$. Assume that*

$$\sup_{\Omega_1}A<\frac{1}{1+\Upsilon_\vartheta}\qquad\text{and}\qquad \inf_{\Omega_2}A>\frac{1+\Upsilon_\vartheta}{1-(\sup_{\Omega_1}A)(1+\Upsilon_\vartheta)}, \tag{13}$$

then the operator $\mathscr{B}:\mathrm{X}_\Sigma\to\mathrm{X}_\Sigma$ defined in (8) is an isomorphism. Here, we see an advantage of this variational tool of the $\mathtt{T}$-coercivity compared to the Fourier/Mellin approach of §5. It allows to work with quite general parameters.

Proof From (10), to complete the proof, the goal is to find four strictly positive parameters α,β,γ,η such that we have both

$$\begin{array}{ll} 1-\alpha-\beta>0\ ; & A_2-\gamma^{-1}\Upsilon_\vartheta-\eta^{-1}>0\ ; \\ 1-\alpha^{-1}A_1-\gamma>0\ ; & 1-A_1\beta^{-1}\Upsilon_\vartheta-\eta>0\ . \end{array}$$

Let us choose α and β such that $\alpha^{-1}=s(1+\Upsilon_\vartheta)$ and $\beta^{-1}=s(1+\Upsilon_\vartheta)/\Upsilon_\vartheta$ for some $s\in(1;1/(A_1(1+\Upsilon_\vartheta)))$. We find $1-\alpha-\beta=1-1/s>0$. Next, take $\gamma=\eta=t(1-sA_1(1+\Upsilon_\vartheta))$ for some $t<1$. We obtain

$$1-\alpha^{-1}A_1-\gamma=1-A_1\beta^{-1}\Upsilon_\vartheta-\eta=(1-t)(1-sA_1(1+\Upsilon_\vartheta))>0$$

$$\text{and}\qquad A_2-\gamma^{-1}\Upsilon_\vartheta-\eta^{-1}=A_2-\frac{1+\Upsilon_\vartheta}{t(1-sA_1(1+\Upsilon_\vartheta))}. \tag{14}$$

If A_2 satisfies the second relation of (12), taking $s>1$ and $t<1$ close enough to one, we obtain that (14) is strictly positive. By virtue of (10), we deduce that

$((u,w),(u',w')) \mapsto b((u,w),\mathtt{T}(u',w'))$ is coercive on X_Σ. Since $\mathtt{T}$ defines an isomorphism, Lax-Milgram theorem allows us to conclude that the operator $\mathscr{B} : \mathrm{X}_\Sigma \to \mathrm{X}_\Sigma$ defined in (8) is an isomorphism when A_1 and A_2 satisfy the condition (12). ■

Now, the question we propose to investigate is the following. Are the conditions of Lemma 3.1 necessary? In other words, what happens for the operator $\mathscr{B}$ when A does not meet the assumptions (12)? For a precise answer, we refer the reader to Remark 6.2.

4. Study of the singularities

The analysis we will develop in the next section relies on a precise study of the behaviour of functions satisfying problem (7) in a neighbourhood of O by means of Fourier/Mellin transform. In order to facilitate the presentation of the technique, we will work not directly on the simplified interior transmission problem (6) but on an equivalent problem. This equivalent problem is obtained from (6) by *unfolding the domain* Ω. In this section, we first explain this *unfolding* procedure. Then, we compute the singularities in the unfolded domain.

4.1. The unfolding procedure

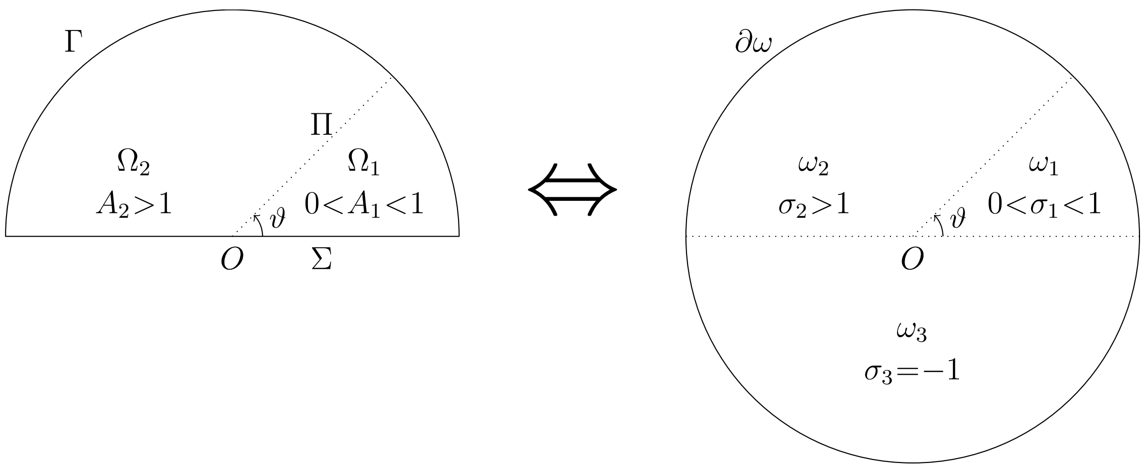

Figure 2. Initial domain Ω and unfolded domain ω.

Let us reintroduce (r,θ) the polar coordinates centered at the origin O. We define (see Figure 2, on right) the sets:

$$\begin{aligned}
\omega_1 &:= \{(r\cos\theta, r\sin\theta)\,|\,0<r<1,\ 0<\theta<\vartheta\};\\
\omega_2 &:= \{(r\cos\theta, r\sin\theta)\,|\,0<r<1,\ \ \vartheta<\theta<\pi\};\\
\omega_3 &:= \{(r\cos\theta, r\sin\theta)\,|\,0<r<1,\ \ \pi<\theta<2\pi\};\\
\omega\ &:= \{(r\cos\theta, r\sin\theta)\,|\,0<r<1,\ \ 0\le\theta<2\pi\}.
\end{aligned}$$

Let σ denote the function such that $\sigma(\theta) = \sigma_1$ for $\theta \in (0;\vartheta)$, $\sigma(\theta) = \sigma_2$ for $\theta \in (\vartheta;\pi)$ and $\sigma(\theta) = \sigma_3 = -1$ for $\theta \in (\pi;2\pi)$. Here $0 < \sigma_1 < 1$ and $\sigma_2 > 1$ are two constants. In the sequel, to simplify, we will make no distinction between σ and the function defined on ω, equal to σ_j on ω_j, $j = 1\dots 3$. Now, we consider the problem:

$$\left|\begin{array}{l}\text{Find } v \in \mathrm{H}^1_0(\omega) \text{ such that:}\\ \mathrm{div}\,(\sigma\nabla v) \;=\; h \quad \text{in } \omega,\end{array}\right. \tag{15}$$

where $\mathrm{H}^1_0(\omega) := \{\varphi \in \mathrm{H}^1(\omega)\,|\,\varphi = 0 \text{ on } \partial\omega\}$. Problem (15) can be seen as a transmission problem between a positive composite material filling the region $\omega_1 \cup \omega_2$ and a negative material in ω_3. Let us assume that the source terms in (6) and (15) are such that $f \in \mathrm{L}^2(\Omega)$, $g \in \mathrm{L}^2(\Omega)$ and $h \in \mathrm{L}^2(\omega)$. Moreover, let us assume that h satisfies $h(x,y) = f(x,y)$ a.e. in $\omega_1 \cup \omega_2$ and $h(x,y) = g(x,-y)$ a.e. in ω_3.

$\star$ Consider (u,w) a solution, if it exists, of the simplified interior transmission problem (6). Then define the function v such that $v = u$ on $\omega_1 \cup \omega_2$ and $v(x,y) = w(x,-y)$ a.e. on ω_3. Is it easy to check that v verifies the transmission problem with a sign-changing coefficient (15) for $\sigma_1 = A_1$ and $\sigma_2 = A_2$.

$\star$ Conversely, if v satisfies the transmission problem with a sign-changing coefficient (15), define the functions u and w such that $u = v$ on Ω, $w(x,y) = v(x,-y)$ a.e. on Ω. Then, one can check that the pair (u,w) is a solution of the simplified interior transmission problem (6) for $A_1 = \sigma_1$ and $A_2 = \sigma_2$.

Although the equivalence between the simplified interior transmission problem (6) and the transmission problem with a sign-changing coefficient (15) is very simple, we decided to introduce the scalar problem (15) to avoid to work with the system of partial differential equations (6). We associate to problem (15) the sesquilinear form m on $\mathrm{H}^1_0(\Omega)$ such that $m(v,v') := (\sigma\nabla v, \nabla v')_\omega$ and we consider the variational problem

$$\left|\begin{array}{l}\text{Find } v \in \mathrm{H}^1_0(\omega) \text{ such that, for all } v' \in \mathrm{H}^1_0(\omega),\\ m(v,v') = H(v'),\end{array}\right. \tag{16}$$

where $H \in \mathrm{H}^{-1}(\omega) := \mathrm{H}^1_0(\omega)^*$. With the Riesz representation theorem, we define the operator $\mathscr{M} : \mathrm{H}^1_0(\omega) \to \mathrm{H}^1_0(\omega)$ such that, for all $(v,v') \in \mathrm{H}^1_0(\omega) \times \mathrm{H}^1_0(\omega)$,

$$(\mathscr{M}v, v')_{\mathrm{H}^1(\omega)} = m(v,v'). \tag{17}$$

The main difficulty in the investigation of the properties of $\mathscr{M}$ comes from the triple point O: at this point, the solutions of problem (16) can be very singular.

4.2. Description of the singularities

When one is interested in studying the regularity of the solutions of problem (16), one is led to compute the *singularities*, *i.e.* one is led to look for the functions $s(r,\theta) = \kappa(r)\phi(\theta)$ with separate variables (in polar coordinates) which satisfy

$$\mathrm{div}(\sigma\nabla s) = r^{-2}\Big(\sigma\phi(r\partial_r)^2\kappa + \kappa(\partial_\theta\sigma\partial_\theta)\phi\Big) = 0.$$

The above equation has been obtained noticing that σ depends only on θ. Using the separation of variables, we deduce that κ and ϕ must satisfy

$$(r\partial_r)^2\kappa = \lambda^2\,\kappa \;\text{ on } (0;+\infty) \quad \text{and} \quad (\partial_\theta\sigma\partial_\theta)\phi = -\lambda^2\,\sigma\phi \;\text{ on } \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}),$$

for some constant $\lambda \in \mathbb{C}$. The problem verified by κ, for which we do not impose boundary conditions, can be easily solved. For $\lambda = 0$, we find $\kappa(r) = A\,\ln r + B$ whereas for $\lambda \neq 0$, we obtain $\kappa(r) = A\,r^{\lambda} + B\,r^{-\lambda}$, A, B being two constants. The problem satisfied by ϕ contains boundary conditions (here, periodic boundary conditions) and no source term. This implies that, for most $\lambda \in \mathbb{C}$, zero is the only solution. Therefore, if we want to compute non trivial singularities, we need to solve the spectral problem:

$$\left|\begin{array}{l} \text{Find } (\lambda,\phi) \in \mathbb{C}\times \mathrm{H}^1(\mathbb{T})\setminus\{0\}, \text{ such that :} \\ \dfrac{\partial}{\partial\theta}\Big(\sigma\dfrac{\partial}{\partial\theta}\phi\Big) \;=\; -\lambda^2\,\sigma\phi \qquad \text{on } \mathbb{T}. \end{array}\right. \tag{18}$$

The study of this problem leads us to consider the symbol $\mathscr{L}$ such that for all $\lambda \in \mathbb{C}$,

$$\begin{array}{rccc} \mathscr{L}(\lambda): & \mathrm{H}^1(\mathbb{T}) & \longrightarrow & \mathrm{H}^1(\mathbb{T})^* \\ & \phi & \longmapsto & \partial_\theta(\sigma\partial_\theta\phi) + \lambda^2\,\sigma\phi. \end{array} \tag{19}$$

We say that $\lambda \in \mathbb{C}$ is an eigenvalue of the symbol $\mathscr{L}$ when there exists a non trivial $\phi \in \mathrm{H}^1(\mathbb{T})$ such that $\mathscr{L}(\lambda)\phi = 0$. The dimension of $\ker\mathscr{L}(\lambda)$ is called the geometric multiplicity of the eigenvalue λ. We denote Λ the set of the eigenvalues of $\mathscr{L}$. This set is also called the set of singular exponents associated to O. As for the interior transmission eigenvalue problem, the study of (18) is not standard because the sign-changing parameter σ appears both in the principal and in the compact part of the equation. However, now it is only a 1D problem. Proceeding as in [12, Lemma 4.9], we can prove the

Proposition 4.1 *Assume that $\sigma_1 \neq 1$ and $\sigma_2 \neq 1$. Then, the set Λ is discrete. Moreover, there exist two positive constants ρ and δ such that if $\lambda \in \mathbb{C}$ verifies $|\lambda| > \rho$ and $|\Re e\,\lambda| < \delta\,|\Im m\,\lambda|$, then λ is not a singular exponent.*

As we will see later, we can decompose the solutions of the source term problem (16) as the sum of a finite number of singularities and a regular term. The computations of the beginning of this paragraph show that for $\lambda \neq 0$, each singularity is proportional to

$$(r,\theta) \mapsto r^{\lambda}\phi(\theta), \tag{20}$$

where (λ,ϕ) corresponds to an eigenpair of problem (18). Therefore, the regularity of the solutions of problem (16) only depends on the set Λ. That is why our work now will consist in describing precisely Λ. In particular, we will be interested by the set $\Lambda \cap \mathbb{R}i \setminus \{0\}$, the reason being that, for $\lambda \in \mathbb{R}i \setminus \{0\}$, the singularity (20) presents a curious oscillating behaviour (see Figure 3). When such singularities exist, we will prove in Proposition 6.16 (actually this was already noticed by Kondratiev in [19]) that Fredholm property in H^1 is lost for problem (16). At this point, we should emphasize that these oscillating singularities do not appear for classical elliptic operators. Their existence here is a consequence of the change of sign of the parameter σ.

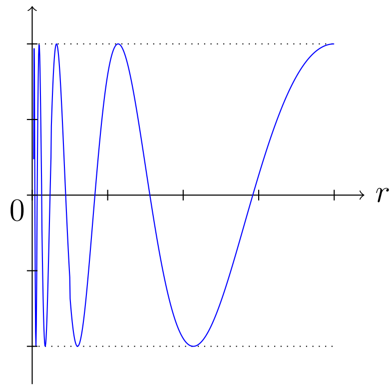

Figure 3. Behaviour of the real part of the radial component of the singularity $r^{\lambda}\phi(\theta)$, for $\lambda \in \mathbb{R}i \setminus \{0\}$, in a neighbourhood of O. To understand these oscillations, observe that $\Re e\, r^{i\eta} = \cos(\eta \ln r)$ for $\eta \in \mathbb{R}^*$.

4.3. Explicit computation of the singularities in the case $\vartheta = \pi/2$

In order to be able to compute explicitly the set Λ, we will restrict our study to the case of an angle

$$\vartheta = \pi/2.$$

In the sequel, we denote $I_1 := (0;\vartheta)$, $I_2 := (\vartheta;\pi)$, $I_3 := (\pi;2\pi)$ and $\phi_j := \phi|_{I_j}$, $j = 1\ldots3$.

$\star$ **The set Λ always contains the value** 0. Indeed, there holds $\mathscr{L}(0)\phi = \partial_\theta(\sigma\partial_\theta\phi) = 0$ when $\phi = cst$.

$\star$ **Cases where** 0 **is an eigenvalue of geometric multiplicity equal to** 2. Let us look for a $\phi \in \mathrm{H}^1(\mathbb{T})$ such that $\mathscr{L}(0)\phi = \partial_\theta(\sigma\partial_\theta\phi) = 0$ with $\phi_1 = A_1\theta$, $\phi_2 = A_2\theta + B_2$ and $\phi_3 = A_3\theta + B_3$. First, writing the continuity of the flux, we obtain $\phi_1 = A_1\theta$, $\phi_2 = A_1(\sigma_1/\sigma_2)\,\theta + B_2$ and $\phi_3 = A_1(\sigma_1/\sigma_3)\,\theta + B_3$. The continuity of ϕ provides $\phi_1 = A_1\theta$, $\phi_2 = A_1(\sigma_1/\sigma_2)\,(\theta-\vartheta) + A_1\vartheta$, $\phi_3 = A_1(\sigma_1/\sigma_3)\,(\theta-\pi) + A_1(\sigma_1/\sigma_2)\,(\pi-\vartheta) + A_1\vartheta$ and $0 = A_1(\sigma_1/\sigma_3)\,\pi + A_1(\sigma_1/\sigma_2)\,(\pi-\vartheta) + A_1\vartheta$. So, we have $\mathscr{L}(0)\phi = 0$ and $\phi \neq 0$ when

$$\frac{\vartheta}{\sigma_1} + \frac{\pi-\vartheta}{\sigma_2} + \frac{\pi}{\sigma_3} = 0.$$

For our configuration where $\sigma_3 = -1$ and $\vartheta = \pi/2$, this relation writes

$$\sigma_1 + \sigma_2 = 2\sigma_1\sigma_2 \quad \Leftrightarrow \quad \sigma_2(2\sigma_1 - 1) = \sigma_1. \tag{21}$$

In order equation (21) to be solvable in $(0;\infty)\times(0;\infty)$, the coefficients σ_1, σ_2 must satisfy $\sigma_1 > 1/2$, $\sigma_2 > 1/2$. Moreover, if $\sigma_1 > 1/2$ is given, then there exists a unique

$$\sigma_2 = \frac{\sigma_1}{2\sigma_1 - 1}, \tag{22}$$

such that we can find a non constant function ϕ satisfying $\mathscr{L}(0)\phi = 0$. From (22), we observe in particular that if $1/2 < \sigma_1 < 1$ then $\sigma_2 > 1$, and if $\sigma_1 > 1$, then $1/2 < \sigma_2 < 1$. Therefore, 0 is an eigenvalue of geometric multiplicity equal to 2 only in cases where the sign of $\sigma - 1$ changes. This is coherent with Theorem 2.2 which states that the operator $\mathscr{A}_k : \mathrm{X} \to \mathrm{X}$ associated with the interior transmission problem is Fredholm if $\sup_{D\cap\mathscr{V}} A < 1$ or if $1 < \inf_{D\cap\mathscr{V}} A$, where $\mathscr{V}$ is a neighbourhood of ∂D. Indeed, configurations where 0 is an eigenvalue of geometric multiplicity equal to 2 correspond to limit configurations where Fredholm property is lost in H^1. For $\sigma_1 = \sigma_2 = 1$, we will see that Λ is equal to the entire complex plane (Remark 4.2).

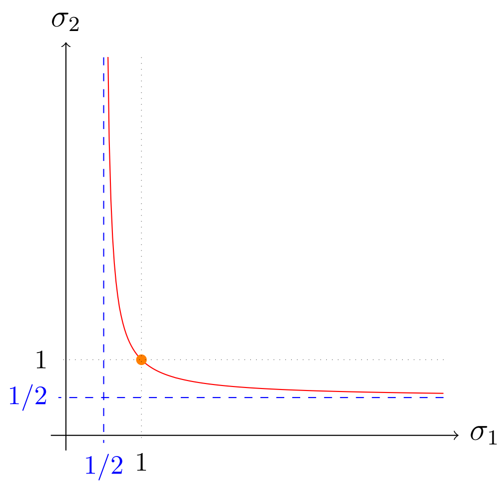

Figure 4. Curve (σ_1, σ_2) when σ_1, σ_2 satisfy the relation (22).

$\star$ **Computation of the non trivial singular exponents.** If ϕ satisfies $\mathscr{L}(\lambda)\phi = \partial_\theta(\sigma\partial_\theta\phi) + \lambda^2\sigma\phi = 0$ for $\lambda \neq 0$, then, for $j = 1\dots 3$, we have $\phi(\theta) = A_j \exp(i\lambda\theta) + B_j \exp(-i\lambda\theta)$ on I_j. Writing the matching conditions, we find that $\lambda \in \mathbb{C}^*$ satisfies $\lambda \in \Lambda$ if and only if the following matrix is not invertible

$$\mathfrak{M}(\lambda) := \begin{bmatrix} e^{i\lambda\vartheta} & e^{-i\lambda\vartheta} & -e^{i\lambda\vartheta} & -e^{-i\lambda\vartheta} & 0 & 0 \\ \sigma_2 e^{i\lambda\vartheta} & -\sigma_2 e^{-i\lambda\vartheta} & -\sigma_1 e^{i\lambda\vartheta} & \sigma_1 e^{-i\lambda\vartheta} & 0 & 0 \\ 0 & 0 & e^{i\lambda\pi} & e^{-i\lambda\pi} & -e^{i\lambda\pi} & -e^{-i\lambda\pi} \\ 0 & 0 & \sigma_1 e^{i\lambda\pi} & -\sigma_1 e^{-i\lambda\pi} & e^{i\lambda\pi} & -e^{-i\lambda\pi} \\ 1 & 1 & 0 & 0 & -e^{2i\lambda\pi} & -e^{-2i\lambda\pi} \\ \sigma_2 & -\sigma_2 & 0 & 0 & e^{2i\lambda\pi} & -e^{-2i\lambda\pi} \end{bmatrix}.$$

For an angle ϑ equal to $\pi/2$, the determinant of this matrix can be explicitly computed. We obtain

$$\begin{aligned} \det\mathfrak{M}(\lambda) \;=\;& 2(\sigma_2-1)(\sigma_1-1)(\sigma_1+\sigma_2)\cos(2\lambda\pi) \\ &+4(\sigma_1-\sigma_2)^2\cos(\lambda\pi) \\ &-2\sigma_2-2\sigma_1-2\sigma_1^2-2\sigma_2^2+12\sigma_2\sigma_1-2\sigma_2\sigma_1^2-2\sigma_2^2\sigma_1. \end{aligned}$$

Remark 4.2 *For $\sigma_1 = \sigma_2 = 1$, we notice that $\det\mathfrak{M}(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. Therefore, in this case, we deduce that $\Lambda = \mathbb{C}$.*

Remark 4.3 *We see that if $\lambda \in \Lambda$, then $-\lambda \in \Lambda$ and $\overline{\lambda} \in \Lambda$. The first point comes from the fact that problem (18) is quadratic with respect to λ. The second statement can be obtained observing that $\overline{\det \mathfrak{M}(\lambda)} = \det \mathfrak{M}(\overline{\lambda})$. Moreover, if $\lambda \in \Lambda$, then $(\lambda + 2) \in \Lambda$. As a consequence, it is sufficient to study $\Lambda \cap \{\lambda \in \mathbb{C} \,|\, 0 \leq \Re e\, \lambda \leq 1,\, 0 \leq \Im m\, \lambda\}$.*

Rewriting the equation $\det \mathfrak{M}(\lambda) = 0$ under the form $a(\cos(\lambda\pi))^2 + b\cos(\lambda\pi) + c = 0$, one obtains that λ should satisfy

$$\cos(\lambda\pi) \in \left\{1, \frac{(\sigma_1+\sigma_2)(1+\sigma_1\sigma_2) - 4\sigma_1\sigma_2}{(1-\sigma_1)(\sigma_2-1)(\sigma_1+\sigma_2)}\right\}.$$

The previous computations lead to the conclusion that the set of eigenvalues of $\mathscr{L}$ is given by the expression

$$\Lambda \;=\; (2\mathbb{Z}) \,\cup\, \{i\eta + 2\mathbb{Z}\} \cup \{-i\eta + 2\mathbb{Z}\} \,\cup\, \{i\overline{\eta} + 2\mathbb{Z}\} \cup \{-i\overline{\eta} + 2\mathbb{Z}\}, \quad (23)$$

where $\eta \in \{z \in \mathbb{C} \,|\, -\pi \leq \Im m\, z \leq 0\}$ denotes the number such that

$$\eta := -\frac{i}{\pi}\,\arccos\left(\frac{(\sigma_1+\sigma_2)(1+\sigma_1\sigma_2) - 4\sigma_1\sigma_2}{(1-\sigma_1)(\sigma_2-1)(\sigma_1+\sigma_2)}\right). \quad (24)$$

To know when there exist purely imaginary singular exponents, it just remains to know when there holds

$$\frac{(\sigma_1+\sigma_2)(1+\sigma_1\sigma_2) - 4\sigma_1\sigma_2}{(1-\sigma_1)(\sigma_2-1)(\sigma_1+\sigma_2)} > 1. \quad (25)$$

Indeed, $\arccos(z) \in \mathbb{R}i \setminus \{0\}$ if and only if z belongs to $(1;+\infty)$. For $(\sigma_1, \sigma_2) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+$, one can check that property (25) is true if and only if (σ_1, σ_2) belongs to the set

$$\mathscr{R} \;:=\; \mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3 \cup \mathscr{R}_4, \quad (26)$$

where

$$\begin{aligned}
\mathscr{R}_1 &= \{(\sigma_1,\sigma_2) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \,|\, \sigma_1 > 1 \quad \text{and} \quad \sigma_2 < 2 - \sigma_1\} \\
\mathscr{R}_2 &= \{(\sigma_1,\sigma_2) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \,|\, \sigma_1 > 1 \quad \text{and} \quad \frac{\sigma_1}{2\sigma_1 - 1} < \sigma_2 < 1\} \\
\mathscr{R}_3 &= \{(\sigma_1,\sigma_2) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \,|\, \sigma_2 > 1 \quad \text{and} \quad \sigma_1 < 2 - \sigma_2\} \\
\mathscr{R}_4 &= \{(\sigma_1,\sigma_2) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \,|\, \sigma_2 > 1 \quad \text{and} \quad \frac{\sigma_2}{2\sigma_2 - 1} < \sigma_1 < 1\}.
\end{aligned}$$

$\star$ **Conclusion.** • For $(\sigma_1, \sigma_2) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \setminus \overline{\mathscr{R}}$, we can prove that $\Lambda \cap \mathbb{R}i = \{0\}$.
• For $(\sigma_1, \sigma_2) \in \mathscr{R}$, we can prove that η defined in (24) is real strictly positive. Therefore, in this case, there holds $\Lambda \cap \mathbb{R}i = \{0, \pm i\eta\}$. The associated singularities take the form

$$r^{\pm i\eta}\phi(\theta) \qquad \text{with} \qquad \phi(\theta) = A_j \exp(-\eta\,\theta) + B_j \exp(\eta\,\theta) \text{ on } I_j, \quad (27)$$

the vector $(A_1, B_1, A_2, B_2, A_3, B_3)^t$ being an eigenvector of the matrix $\mathfrak{M}(i\eta)$. We choose an eigenvector so that $\|\phi\|_{\mathbb{T}} = 1$ (we remind that $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$). Observe that the angular behaviour is the same for the two singularities. In the sequel, the value of the integral $\int_0^{2\pi} \sigma(\theta)\phi(\theta)^2\, d\theta$ will play an important role. More precisely, to avoid technicalities in the analysis we develop, we will need this quantity to be different from zero. We will assume that this property is true. Figure 9 leads us to think that this is not a restricting assumption.

We have computed the singularities for problem (16). Let us present now the framework which will allow us to use them.

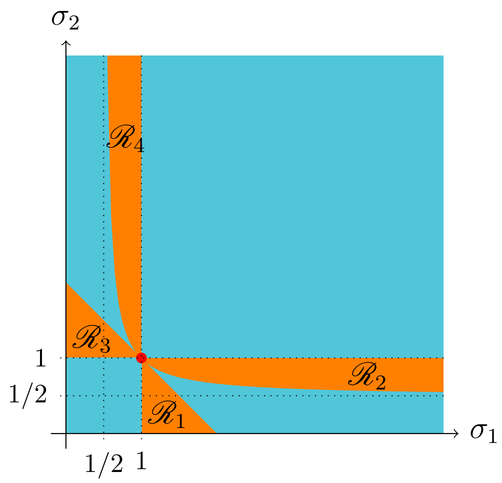

Figure 5. Representation of the set $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3 \cup \mathscr{R}_4$ in orange. For $(\sigma_1, \sigma_2) \in \mathscr{R}$, strongly oscillating singularities appear.

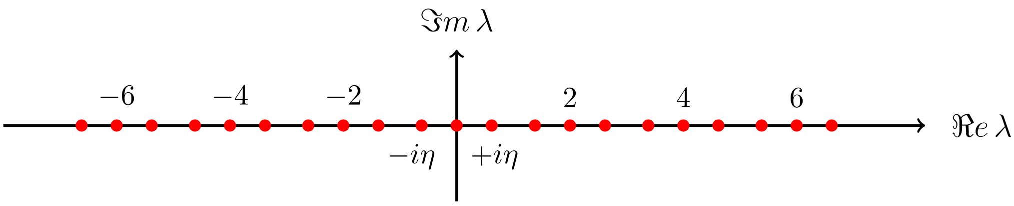

Figure 6. Set Λ for $(\sigma_1, \sigma_2) = (1/4, 2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \setminus \overline{\mathscr{R}}$.

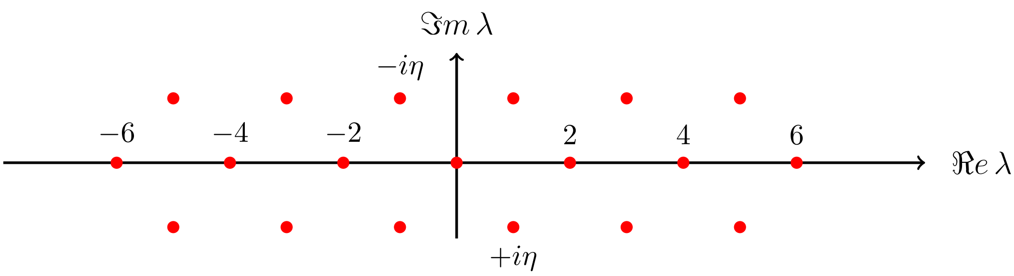

Figure 7. Set Λ for $(\sigma_1, \sigma_2) = (4, 2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \setminus \overline{\mathscr{R}}$.

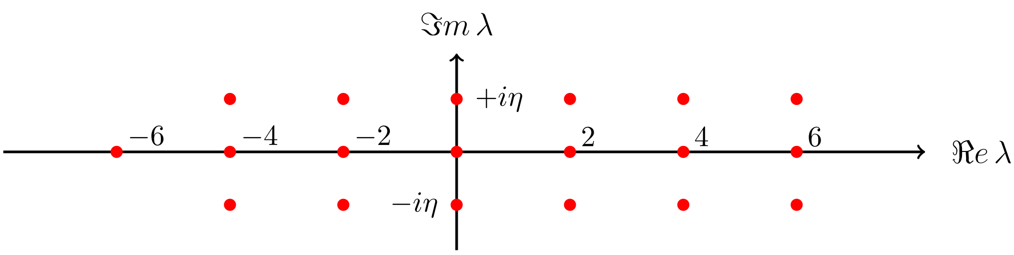

Figure 8. Set Λ for $(\sigma_1, \sigma_2) = (3/4, 2) \in \mathscr{R}$. Notice the two non trivial singular exponents on $\mathbb{R}i$.

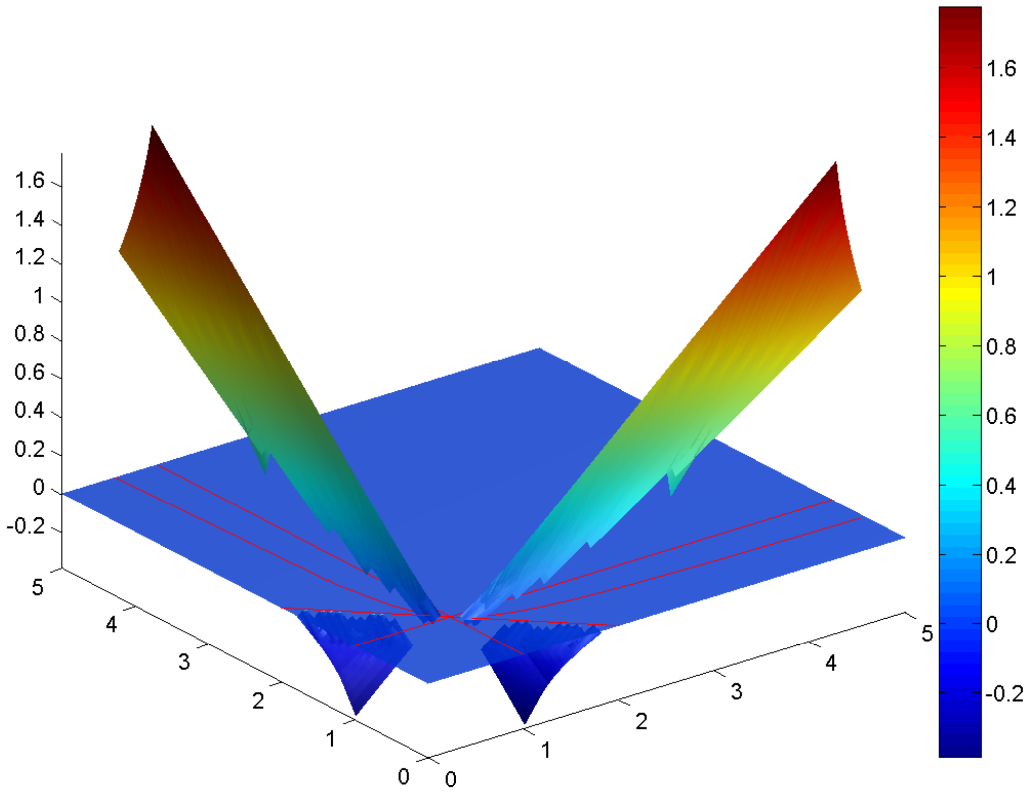

Figure 9. The surface in color represents the value of $\int_0^{2\pi} \sigma(\theta)\phi(\theta)^2\, d\theta$ with respect to (σ_1, σ_2), for $(\sigma_1, \sigma_2) \in \mathscr{R} \cap \big((0;5) \times (0;5)\big)$. Here, $\theta \mapsto \phi(\theta)$ is the angular part of the singularity $(r, \theta) \mapsto r^{i\eta}\phi(\theta)$ (see (27)). The plane in $z = 0$ allows us to see that this integral (at least its approximation) vanishes only on $\partial\mathscr{R}$ but not in $\mathscr{R}$.

5. Fredholm property for the model problem in the unfolded geometry

Problems of singularities usually raise in the study of partial differential equations in non smooth domains as well as in the study of partial differential equations with non smooth coefficients. To handle such problems, Kondratiev developed in the pioneering paper [19] an efficient theory. For more recent references, the reader might consult the monographs [20, 26, 23]. This theory is based on the use of the Mellin transform, which is nothing else than the Fourier transform with respect to $\ln r$, where r denotes the distance to the singular point. The Mellin transform appears very useful in this field because it defines isomorphisms between some *ad hoc* spaces and some weighted Sobolev spaces, the latter being particularly well-suited to measure precisely the behaviour of the functions at O.

5.1. Analysis in weighted Sobolev spaces

We consider the variational problem

$$\left|\begin{array}{l}\text{Find } v \text{ such that:}\\ (\sigma\nabla v, \nabla v')_\omega = H(v'), \qquad \forall v' \in \mathscr{C}_0^\infty(\omega).\end{array}\right. \tag{28}$$

We search for the solution of (28) in the Kondratiev space $\mathrm{V}^1_\beta(\omega)$. This space is defined as the closure of $\mathscr{C}_0^\infty(\omega)$ for the norm

$$\|\varphi\,;\mathrm{V}^1_\beta(\omega)\| = \left(\|r^\beta\nabla\varphi\|^2_\omega + \|r^{\beta-1}\varphi\|^2_\omega\right)^{1/2},$$

where r is the distance to the origin O and $\beta \in \mathbb{R}$ is the weight. Notice that the trace of the elements of $\mathrm{V}^1_\beta(\omega)$ vanishes on $\partial\omega$. Since the linear space $\mathscr{C}_0^\infty(\omega)$ is dense in $\mathrm{V}^1_\beta(\omega)$ for all $\beta \in \mathbb{R}$, the equality (28) is valid for all test functions v' in $\mathrm{V}^1_{-\beta}(\omega)$. Therefore, the source term H in (28) can be chosen in $\mathrm{V}^1_{-\beta}(\omega)^*$, the topological dual space of $\mathrm{V}^1_{-\beta}(\omega)$ made of the continuous antilinear forms on $\mathrm{V}^1_{-\beta}(\omega)$. As a consequence, we can associate with problem (28) the operator

$$\mathscr{M}_\beta : \mathrm{V}^1_\beta(\omega) \to \mathrm{V}^1_{-\beta}(\omega)^* \tag{29}$$

such that $\langle \mathscr{M}_\beta v, v'\rangle_\omega = (\sigma\nabla v, \nabla v')_\omega$ for all $v \in \mathrm{V}^1_\beta(\omega)$, $v' \in \mathrm{V}^1_{-\beta}(\omega)$.

Remark 5.1 *With this definition, one observes that the adjoint of $\mathscr{M}_\beta$ is $\mathscr{M}_{-\beta}$, i.e. $\mathscr{M}_\beta^* = \mathscr{M}_{-\beta}$ for all $\beta \in \mathbb{R}$.*

In order to study the properties of $\mathscr{M}_\beta$, we introduce a transmission problem set on the plane $\mathbb{R}^2$, where we can apply the Mellin transform without concern for boundary. More precisely, let us define the set $\mathring{\mathbb{R}}^2 := \mathbb{R}^2 \setminus \{O\}$ and, for $\beta \in \mathbb{R}$, the operator

$$\mathscr{N}_\beta : \mathrm{V}^1_\beta(\mathring{\mathbb{R}}^2) \to \mathrm{V}^1_{-\beta}(\mathring{\mathbb{R}}^2)^*. \tag{30}$$

such that $\langle \mathscr{N}_\beta v, v'\rangle_{\mathring{\mathbb{R}}^2} = (\hat\sigma\nabla v, \nabla v')_{\mathring{\mathbb{R}}^2}$ for all $v \in \mathrm{V}^1_\beta(\mathring{\mathbb{R}}^2)$, $v' \in \mathrm{V}^1_{-\beta}(\mathring{\mathbb{R}}^2)$. In this definition, $\hat\sigma : \mathring{\mathbb{R}}^2 \to \mathbb{R}$ denotes the extension of σ to $\mathring{\mathbb{R}}^2$ such that

$$\begin{array}{ll}\hat\sigma = \sigma_1 & \text{in } \{(r\cos\theta, r\sin\theta)\,|\,0<r<+\infty,\ \ 0 \le \theta < \vartheta\}\\ \hat\sigma = \sigma_2 & \text{in } \{(r\cos\theta, r\sin\theta)\,|\,0<r<+\infty,\ \ \vartheta \le \theta < \pi\}\\ \hat\sigma = \sigma_3 = -1 & \text{in } \{(r\cos\theta, r\sin\theta)\,|\,0<r<+\infty,\ \ \pi \le \theta < 2\pi\}.\end{array}$$

In the next theorem, we provide a necessary and sufficient condition so that $\mathscr{N}_\beta$ is an isomorphism. For the proof, again, we refer the reader to [19, 26, 20] for the general theory concerning the study of elliptic operators in weighted Sobolev spaces, and to [17, Theorem 3.7], [5], [12, Theorem 4.16], [2, Theorem 4.1] for the extension of this theory to the transmission problem with a sign-changing coefficient. The general idea is to proceed to a Fourier transform with respect to $\ln r$. This leads to study a family of 1D Ordinary Differential Equations, for the angular coordinate θ, which depend on the Fourier parameter. After a precise analysis of the properties of these ODEs, we perform the inverse Fourier transform which provides solutions in the weighted Sobolev spaces we introduced above. In this approach, the sign-changing problem is tackled during the investigation of the properties of the 1D operators.

Theorem 5.2 *Assume that* $\sigma_1 \neq 1$ *and* $\sigma_2 \neq 1$. *Then, the operator* $\mathscr{N}_\beta : \mathrm{V}^1_\beta(\mathring{\mathbb{R}}^2) \to \mathrm{V}^1_{-\beta}(\mathring{\mathbb{R}}^2)^*$ *defined in (30) is an isomorphism if and only if no eigenvalue of the symbol* $\mathscr{L}$ *(defined in (19)) belongs to the line*

$$\ell_\beta := \{\lambda \in \mathbb{C} \,|\, \Re e\,\lambda = \beta\}. \tag{31}$$

In other words, if $\sigma_1 \neq 1$, $\sigma_2 \neq 1$, $\mathscr{N}_\beta$ *is an isomorphism if and only if* $\Lambda \cap \ell_\beta = \emptyset$.

Remark 5.3 *When* $\sigma_1 = 1$ *or/and* $\sigma_2 = 1$, *ellipticity is lost (see [32, 30], [22] and [16, 28]) on the interface where* σ *takes opposite values (we recall that* $\sigma_3 = -1$*). In this case, the functional framework must be modified to recover Fredholmness. This has been achieved naturally when* $\sigma_1 = 1$ *and* $\sigma_2 = 1$ *working with a fourth order formulation for the original interior transmission problem (see [31]). However, when* $\sigma_1 = 1$ *and* $\sigma_2 \neq 1$ *or when* $\sigma_1 \neq 1$ *and* $\sigma_2 = 1$, *the authors do not know any appropriate framework where Fredholmness holds. This seems to be an open question and its treatment is beyond the scope of the present article.*

To characterize the properties of $\mathscr{M}_\beta$, we recall below the definition of a Fredholm operator.

Definition 5.4 *Let* Y *and* W *be two Banach spaces, and let* $L : \mathrm{Y} \to \mathrm{W}$ *be a continuous linear map. The operator* L *is said to be of Fredholm type if and only if the following two conditions are fulfilled*

i) $\dim(\ker L) < \infty$ *and* $\mathrm{range}\, L$ *is closed;*

ii) $\dim(\mathrm{coker}\, L) < \infty$ *where* $\mathrm{coker}\, L := (\mathrm{W}/\mathrm{range}\, L)$.

Besides, the index of a Fredholm operator L *is defined by* $\mathrm{ind}(L) = \dim(\ker L) - \dim(\mathrm{coker}\, L)$.

Using a localization process, Theorem 5.2 to invert locally in a neighbourhood of O and [1, Theorem 5.2] to invert locally away from O, we can build a right regularizer (also called a right parametrix) for $\mathscr{M}_\beta : \mathrm{V}^1_\beta(\omega) \to \mathrm{V}^1_{-\beta}(\omega)^*$ when $\Lambda \cap \ell_\beta \neq \emptyset$. In other words, when $\Lambda \cap \ell_\beta \neq \emptyset$, we can construct an operator $R_\beta : \mathrm{V}^1_{-\beta}(\omega)^* \to \mathrm{V}^1_\beta(\omega)$ such that $\mathscr{M}_\beta R_\beta = Id_\beta + K_\beta$, where $K_\beta : \mathrm{V}^1_{-\beta}(\omega)^* \to \mathrm{V}^1_{-\beta}(\omega)^*$ is compact (here, Id_β is the identity of $\mathrm{V}^1_{-\beta}(\omega)^*$). Since $\Lambda \cap \ell_\beta \neq \emptyset \Leftrightarrow \Lambda \cap \ell_{-\beta} \neq \emptyset$, we can also build a right regularizer for $\mathscr{M}_{-\beta}$ when $\Lambda \cap \ell_\beta \neq \emptyset$. Remembering that $\mathscr{M}_\beta^* = \mathscr{M}_{-\beta}$, we deduce that if $\Lambda \cap \ell_\beta = \emptyset$, we can construct left and right regularizers for $\mathscr{M}_\beta$ (a left regularizer is nothing else than a left inverse modulo a compact operator). This procedure proves the

Theorem 5.5 *Assume that* $\sigma_1 \neq 1$ *and* $\sigma_2 \neq 1$. *Then, the operator* $\mathscr{M}_\beta : \mathrm{V}^1_\beta(\omega) \to \mathrm{V}^1_{-\beta}(\omega)^*$ *defined in (29) is of Fredholm type if and only if* $\Lambda \cap \ell_\beta = \emptyset$, *where* Λ *denotes the set of singular exponents introduced after (19).*
If $\Lambda \cap \ell_\beta \neq \emptyset$, *then the range of* $\mathscr{M}_\beta$ *is not closed.*

The second non trivial result we need from this theory is a result of decomposition. Using a density process and residue formula, the following result can be proved in the same manner as [20, Theorem 5.4.2].

Theorem 5.6 *Assume that $\sigma_1 \neq 1$ and $\sigma_2 \neq 1$. Consider β^1 and β^2 two real numbers such that $\beta^1 < 0 < \beta^2$ and such that $(\Lambda \cap \{\beta^1 \leq \Re e\,\lambda \leq \beta^2\}) \subset \mathbb{R}i$. Let v_{β^2} be an element of $\mathrm{V}^1_{\beta^2}(\omega)$ which satisfies $\mathscr{M}_{\beta^2} v_{\beta^2} \in \mathrm{V}^1_{-\beta^1}(\omega)^*$ (the important point here is that $\mathrm{V}^1_{-\beta^1}(\omega)^*$ is included into $\mathrm{V}^1_{-\beta^2}(\omega)^*$ since $\beta^1 < \beta^2$). Then, there hold the following representations:*

1) if $(\sigma_1, \sigma_2) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \setminus \overline{\mathscr{R}}$, i.e. if $\Lambda \cap \{\beta^1 \leq \Re e\,\lambda \leq \beta^2\} = \{0\}$,

$$v_{\beta^2} = v_{\beta^1} + \zeta(c_0^+ + c_0^- \ln r)\ ; \tag{32}$$

2) if $(\sigma_1, \sigma_2) \in \mathscr{R}$, i.e. if $\Lambda \cap \{\beta^1 \leq \Re e\,\lambda \leq \beta^2\} = \{0, \pm i\eta\}$,

$$v_{\beta^2} = v_{\beta^1} + \zeta(c_0^+ + c_0^- \ln r + c_\eta^+ r^{i\eta}\phi(\theta) + c_\eta^- r^{-i\eta}\phi(\theta))\ . \tag{33}$$

Here, v_{β^1} is an element of $\mathrm{V}^1_{\beta^1}(\omega)$, ϕ is defined in (27), $c_0^\pm$, $c_\eta^\pm$ are some constants and $\zeta \in \mathscr{C}^\infty(\mathbb{R}_+)$ is a cut-off function such that $\zeta(r) = 1$ for $r \leq 1/2$, $\zeta(r) = 0$ for $r > 3/4$.

5.2. Computation of the index

In this section, we precise the result of Theorem 5.5 computing the index of $\mathscr{M}_\beta$ with respect to β.

Theorem 5.7 *Assume that $\sigma_1 \neq 1$ and $\sigma_2 \neq 1$. Then, there exists $\beta^0 > 0$ such that $\Lambda \cap \ell_\beta = \emptyset$ for all $\beta \in (0; \beta^0)$. When $(\sigma_1, \sigma_2) \in \mathscr{R}$, we can take $\beta^0 = 2$.*

1) Assume that $(\sigma_1, \sigma_2) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \setminus \overline{\mathscr{R}}$. For all $\beta \in (0; \beta^0)$, the operator $\mathscr{M}_\beta : \mathrm{V}^1_\beta(\omega) \to \mathrm{V}^1_{-\beta}(\omega)^*$ defined in (29) is of Fredholm type, onto, and of index 1, whereas $\mathscr{M}_{-\beta} : \mathrm{V}^1_{-\beta}(\omega) \to \mathrm{V}^1_\beta(\omega)^*$ is of Fredholm type, injective and of index -1.

2) Assume that $(\sigma_1, \sigma_2) \in \mathscr{R}$. For all $\beta \in (0; 2)$, the operator $\mathscr{M}_\beta : \mathrm{V}^1_\beta(\omega) \to \mathrm{V}^1_{-\beta}(\omega)^*$ defined in (29) is of Fredholm type, onto and of index 2, whereas $\mathscr{M}_{-\beta} : \mathrm{V}^1_{-\beta}(\omega) \to \mathrm{V}^1_\beta(\omega)^*$ is of Fredholm type, injective and of index -2.

Remark 5.8 *Let Y and W be two Banach spaces, and let $L : \mathrm{Y} \to \mathrm{W}$ be a continuous linear map. If L is of Fredholm type and injective, L is called a monomorphism. In this case, there holds the estimate $\|y\|_{\mathrm{Y}} \leq C\,\|Ly\|_{\mathrm{W}}$, for some $C > 0$, and for all $y \in \mathrm{Y}$. If L is of Fredholm type and onto, L is called an epimorphism.*

Proof According to Proposition 4.1, we know that the set of the singular exponents Λ is discrete. Moreover, there exist two positive constants ρ and δ such that if $\lambda \in \mathbb{C}$ verifies $|\lambda| > \rho$ and $|\Re e\,\lambda| < \delta\,|\Im m\,\lambda|$, then λ does not belong to Λ. This allows to prove that we can find $\beta^0 > 0$ small enough such that $\Lambda \cap \ell_\beta = \emptyset$ for all $\beta \in (0; \beta^0)$. In virtue of Theorem 5.5, we deduce that $\mathscr{M}_\beta$ is of Fredholm type for all $\beta \in (0; \beta^0)$. Since $\lambda \in \Lambda$ if and only if $-\lambda \in \Lambda$, this also implies that $\mathscr{M}_{-\beta}$ is of Fredholm type for all $\beta \in (0; \beta^0)$. When $(\sigma_1, \sigma_2) \in \mathscr{R}$, according to formula (23), we can take $\beta^0 = 2$. Let us fix $\beta \in (0; \beta^0)$.

1) Assume that $(\sigma_1, \sigma_2) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \setminus \overline{\mathscr{R}}$. First, we want to prove that $\ker \mathscr{M}_{-\beta} = \{0\}$. Let v be an element of $\ker \mathscr{M}_{-\beta}$. Define $\hat{v}$ such that $\hat{v}(r, \theta) = v(r, \theta)$ on ω and

$\hat{v}(r,\theta) = -v(1/r,\theta)$ on $\mathbb{R}^2 \setminus \omega$. It is easy to check that $\hat{v}$ belongs‡ to $\mathrm{V}^1_{-\beta}(\mathring{\mathbb{R}}^2)$ and satisfies $\mathrm{div}(\hat{\sigma}\nabla\hat{v}) = 0$. This proves that $\hat{v}$ is an element of $\ker \mathscr{N}_{-\beta}$. Since $\mathscr{N}_{-\beta} : \mathrm{V}^1_{-\beta}(\mathring{\mathbb{R}}^2) \to \mathrm{V}^1_{\beta}(\mathring{\mathbb{R}}^2)^*$ is an isomorphism according to Theorem 5.2, we deduce successively that $\hat{v} = 0$, $v = 0$ and $\ker \mathscr{M}_{-\beta} = \{0\}$.

Next, we focus our attention on the set $\ker \mathscr{M}_{\beta}$. Let v belong to $\ker \mathscr{M}_{\beta}$. According to formula (32), v admits the representation $v = c_1 \ln r + \zeta c_2 + \tilde{v}$ for some constants c_1, c_2 and some $\tilde{v} \in \mathrm{V}^1_{-\beta}(\omega)$. Notice that $(r,\theta) \mapsto \ln r$ belongs to the kernel of $\mathscr{M}_{\beta}$. So $\zeta c_2 + \tilde{v}$ must be an element of $\ker \mathscr{M}_{\beta}$. By an energy argument, we prove now that $c_2 = 0$. Let us introduce $\chi \in \mathscr{C}^{\infty}(\mathbb{R}_-)$ a cut-off function such that $\chi(t) = 0$ for $t \le -2$ and $\chi(t) = 1$ for $t \ge -1$. For all $m \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, we define χ_m (see the graph on Figure 10) such that $\chi_m(r) = \chi(\ln r/m)$.

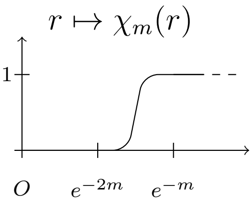

Figure 10. Graph of the function χ_m.

Since, $u := \zeta c_2 + \tilde{v} \in \ker \mathscr{M}_{\beta}$, there holds $(\sigma\nabla u, \nabla(\chi_m \ln r))_{\omega} = 0$. But there also holds $(\sigma\nabla(\chi_m u), \nabla(\ln r))_{\omega} = 0$. Using these two relations, we can write

$$0 = (\ln r\nabla u - u\nabla(\ln r), \sigma\nabla\chi_m)_{\omega}.$$

Since $\tilde{v} \in \mathrm{V}^1_{-\beta}(\omega)$, we can prove that $(\ln r\nabla\tilde{v} - \tilde{v}\nabla(\ln r), \sigma\nabla\chi_m)_{\omega} \to 0$ when $m \to +\infty$. We deduce that $(\ln r\nabla(c_2) - c_2\nabla(\ln r), \sigma\nabla\chi_m)_{\omega} \to 0$ when $m \to +\infty$ (notice that $\nabla\chi_m$ is not null only in a neighbourhood of O and that $\zeta = 1$ in this region for m large enough). But

$$(-c_2\nabla(\ln r), \sigma\nabla\chi_m)_{\omega} = -c_2(\sigma_1\vartheta + \sigma_2(\pi - \vartheta) - \pi). \tag{34}$$

This proves that $c_2 = 0$ and that $v = \tilde{v}$ belongs to $\ker \mathscr{M}_{-\beta}$. Since $\ker \mathscr{M}_{-\beta} = \{0\}$, we deduce that $v = 0$. Thus, there holds $\dim(\ker \mathscr{M}_{\beta}) = 1$ with $\ker \mathscr{M}_{\beta} = \mathrm{span}(\ln r)$.

Finally, we compute the indices. We can write

$$\begin{array}{rcl} \mathrm{ind}(\mathscr{M}_{\beta}) & = & \dim(\ker \mathscr{M}_{\beta}) - \dim(\mathrm{coker}\, \mathscr{M}_{\beta}) \\ & = & \dim(\ker \mathscr{M}_{\beta}) - \dim(\ker \mathscr{M}^*_{\beta}) \\ & = & \dim(\ker \mathscr{M}_{\beta}) - \dim(\ker \mathscr{M}_{-\beta}) \;=\; 1 \end{array} \tag{35}$$

‡ Let us define the property P_{β}: $[v \in \mathrm{V}^1_{-\beta}(\omega) \Rightarrow \hat{v} \in \mathrm{V}^1_{-\beta}(\mathring{\mathbb{R}}^2)]$. It is important to underline that for $\beta > 0$, P_{β} is true but $P_{-\beta}$ is wrong. In other words, for $\beta > 0$, $v \in \mathrm{V}^1_{\beta}(\omega)$ does not imply $\hat{v} \in \mathrm{V}^1_{\beta}(\mathring{\mathbb{R}}^2)$. As a consequence, we can not prove using this approach that $\mathscr{M}_{\beta}$ is injective for $\beta > 0$. And actually, this is reassuring since $\mathscr{M}_{\beta}$ is not injective for $\beta > 0$.

To obtain (35), we have used the following property: if L is a Fredholm operator, then L^* is a Fredholm operator and $\mathrm{coker}\, L$ is isomorphic to $\ker L^*$ (see [24, Theorem 2.13]). In the process, we also obtain

$$\begin{array}{rcl}\mathrm{ind}(\mathscr{M}_{-\beta}) & = & \dim(\ker \mathscr{M}_{-\beta}) - \dim(\mathrm{coker}\, \mathscr{M}_{-\beta}) \\ & = & \dim(\mathrm{coker}\, \mathscr{M}_{\beta}) - \dim(\ker \mathscr{M}_{\beta}) \;=\; -\mathrm{ind}(\mathscr{M}_{\beta}) \;=\; -1.\end{array}$$

2) Now, we assume that $(\sigma_1, \sigma_2) \in \mathscr{R}$. The difference with the case $(\sigma_1, \sigma_2) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+ \backslash \overline{\mathscr{R}}$ is the existence of the oscillating singularities $(r, \theta) \mapsto r^{i\pm\eta}\phi(\theta)$. Proceeding as for point 1), we prove that $\ker \mathscr{M}_{-\beta} = \{0\}$ for $\beta \in (0; 2)$. Now, let us study $\ker \mathscr{M}_{\beta}$. First, we notice that the functions $(r, \theta) \mapsto \ln r$ and

$$s := (r^{i\eta} - r^{-i\eta})\phi(\theta) \tag{36}$$

are two non collinear elements of $\ker \mathscr{M}_{\beta}$. We deduce that $\dim(\ker \mathscr{M}_{\beta}) \geq 2$. Let us consider $v \in \ker \mathscr{M}_{\beta}$. According to formula (33), v admits the representation $v = c_1 \ln r + \zeta(c_2 + c_3 r^{i\eta}\phi(\theta) + c_4 r^{-i\eta}\phi(\theta)) + \tilde{v}$ for some constants c_1, c_2, c_3, c_4 and some $\tilde{v} \in \mathrm{V}^1_{-\beta}(\omega)$. Let us prove that the function $u := v - c_1 \ln r - c_3 s$, which belongs to $\ker \mathscr{M}_{\beta}$, is equal to zero. We have $u = \zeta(c_2 + c_5 r^{-i\eta}\phi(\theta)) + \tilde{u}$ with $c_5 = c_3 + c_4$ and $\tilde{u} = \tilde{v} + c_3(\zeta - 1)s$. Observe that there hold $(\sigma\nabla u, \nabla(\chi_m \overline{s}))_\omega = 0$ and $(\sigma\nabla(\chi_m u), \nabla\overline{s})_\omega = 0$ where χ_m is the cut-off function we introduced in the proof of point 1). This allows us to write

$$0 = (s\nabla u - u\nabla s, \sigma\nabla\chi_m)_\omega.$$

Since $\tilde{u} \in \mathrm{V}^1_{-\beta}(\omega)$, we can prove that $(s\nabla\tilde{u} - \tilde{u}\nabla s, \sigma\nabla\chi_m)_\omega \to 0$ when $m \to +\infty$. We deduce that $(s\nabla(c_2 + c_5 r^{-i\eta}\phi(\theta)) - (c_2 + c_5 r^{-i\eta}\phi(\theta))\nabla s, \sigma\nabla\chi_m)_\omega \to 0$ when $m \to +\infty$ (again, notice that $\nabla\chi_m$ is not null only in a neighbourhood of O where $\zeta = 1$). A simple computation leads to

$$\begin{array}{rl} & (s\nabla(c_5 r^{-i\eta}\phi(\theta)) - (c_2 + c_5 r^{-i\eta}\phi(\theta))\nabla s, \sigma\nabla\chi_m)_\omega \\ = & -c_2 i\eta \displaystyle\int_{e^{-2n}}^{e^{-n}} (r^{i\eta} + r^{-i\eta})\frac{\partial\chi_m}{\partial r}\, dr \int_0^{2\pi} \sigma\phi(\theta)\, d\theta - 2c_5 i\eta \int_0^{2\pi} \sigma\phi(\theta)^2\, d\theta.\end{array} \tag{37}$$

In the distributions sense, there holds $\partial_\theta\sigma\partial_\theta\phi = \lambda^2\sigma\phi$ on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. This implies $\int_0^{2\pi} \sigma\phi(\theta)\, d\theta = 0$. Since $\int_0^{2\pi} \sigma\phi(\theta)^2\, d\theta \neq 0$ (see Figure 9 and the discussion at the end of §4.3), we deduce from (37) that $c_5 = 0$. Working as in point 1), we prove next that $c_2 = 0$. Finally, we obtain that $u = \tilde{u}$ so that u belongs to $\ker \mathscr{M}_{-\beta} = \{0\}$. Hence, $v = c_1 \ln r + c_3 s$ and we deduce that $\dim(\ker \mathscr{M}_{\beta}) = 2$ with $\ker \mathscr{M}_{\beta} = \mathrm{span}(\ln r, s)$.

Now, we compute the indices. We can write

$$\begin{array}{rcl}\mathrm{ind}(\mathscr{M}_{\beta}) & = & \dim(\ker \mathscr{M}_{\beta}) - \dim(\mathrm{coker}\, \mathscr{M}_{\beta}) \\ & = & \dim(\ker \mathscr{M}_{\beta}) - \dim(\ker \mathscr{M}^*_{\beta}) \\ & = & \dim(\ker \mathscr{M}_{\beta}) - \dim(\ker \mathscr{M}_{-\beta}) \;=\; 2 - 0 \;=\; 2.\end{array} \tag{38}$$

Moreover, we have $\mathrm{ind}(\mathscr{M}_{-\beta}) = -\mathrm{ind}(\mathscr{M}_{\beta}) = -2$. ■

5.3. Construction of isomorphisms

In this section, imposing a special behaviour for the solutions at point O, we build isomorphisms. The technique is borrowed from [25], [26, Chapter 5]. If $(\sigma_1, \sigma_2) \in \mathscr{R}$, we define the functions

$$s^+ = \zeta r^{i\eta}\phi(\theta) \qquad \text{and} \qquad s^- = \zeta r^{-i\eta}\phi(\theta), \tag{39}$$

where η, ϕ, ζ are introduced in (24), (27), (33).

Theorem 5.9

1) Assume that $(\sigma_1, \sigma_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \setminus \overline{\mathscr{R}}$. For $\beta \in (0; \beta^0)$, define the unique operator $\mathscr{M}_{-\beta}^{\mathrm{rad}} : \mathrm{span}(\zeta) \oplus \mathrm{V}^1_{-\beta}(\omega) \to \mathrm{V}^1_{\beta}(\omega)^*$ such that $\left\langle \mathscr{M}_{-\beta}^{\mathrm{rad}} v, v' \right\rangle_\omega = (\sigma\nabla v, \nabla v')_\omega$ for all $v \in \mathrm{span}(\zeta) \oplus \mathrm{V}^1_{-\beta}(\omega)$, $v' \in \mathscr{C}_0^\infty(\omega)$. Then, $\mathscr{M}_{-\beta}^{\mathrm{rad}}$ is an isomorphism.

2) Assume that $(\sigma_1, \sigma_2) \in \mathscr{R}$. For $\beta \in (0; 2)$, define the unique operator $\mathscr{M}_{-\beta}^{\mathrm{rad}}$: $\mathrm{span}(\zeta, s^+) \oplus \mathrm{V}^1_{-\beta}(\omega) \to \mathrm{V}^1_{\beta}(\omega)^*$ such that $\left\langle \mathscr{M}_{-\beta}^{\mathrm{rad}} v, v' \right\rangle_\omega = (\sigma\nabla v, \nabla v')_\omega$ for all $v \in \mathrm{span}(\zeta, s^+) \oplus \mathrm{V}^1_{-\beta}(\omega)$, $v' \in \mathscr{C}_0^\infty(\omega)$, where s^+ is the function introduced in (39). Then, $\mathscr{M}_{-\beta}^{\mathrm{rad}}$ is an isomorphism.

Proof 1) If v is an element of $\ker \mathscr{M}_{-\beta}^{\mathrm{rad}}$, then there holds $v = c_1\zeta + \tilde{v}$ for some constant c_1 and some $\tilde{v} \in \mathrm{V}^1_{-\beta}(\omega)$. Proceeding as in the proof of Theorem 5.7, with the energy argument, we prove that $c_1 = 0$. We deduce that $\ker \mathscr{M}_{-\beta}^{\mathrm{rad}} = \ker \mathscr{M}_{-\beta} = \{0\}$. Now, let us consider $H \in \mathrm{V}^1_{\beta}(\omega)^*$. Since the operator $\mathscr{M}_{\beta} : \mathrm{V}^1_{\beta}(\omega) \to \mathrm{V}^1_{-\beta}(\omega)^*$ defined in (29) is onto (see Theorem 5.7), we know that there exists $v \in \mathrm{V}^1_{\beta}(\omega)$ such that $\mathscr{M}_{\beta} u = H$. According to formula (32), v admits the representation $v = c_1 \ln r + \zeta c_2 + \tilde{v}$ for some constants c_1, c_2 and some $\tilde{v} \in \mathrm{V}^1_{-\beta}(\omega)$. Since $\ln r$ belongs to $\ker \mathscr{M}_{\beta}$, the function $u = \zeta c_2 + \tilde{v}$ also satisfies $\mathscr{M}_{\beta} u = H$. But u is an element of $\mathrm{span}(\zeta, s) \oplus \mathrm{V}^1_{-\beta}(\omega)$. Thus, $\mathscr{M}_{-\beta}^{\mathrm{rad}}$ is also onto.

2) Item 2) can be proven following the same lines. Let us just precise the definition of $\mathscr{M}_{-\beta}^{\mathrm{rad}}$ in this case because this is not so direct. The linear form $v' \mapsto (\sigma\nabla s^\pm, \nabla v')_\omega$ is well-defined on $\mathrm{V}^1_{-\beta}(\omega)$. Although $s^\pm \in \mathrm{V}^1_{\beta}(\omega) \setminus \mathrm{V}^1_{-\beta}(\omega)$, we will extend it to $\mathrm{V}^1_{\beta}(\omega)$, and actually to $\mathrm{V}^1_{\gamma}(\omega)$ for all $\gamma \in \mathbb{R}$. Using Green's formula and remembering that $\mathrm{div}(\sigma\nabla(r^{\pm i\eta}\phi)) = 0$ in ω, we can write

$$(\sigma\nabla s^+, \nabla v')_\omega = (\sigma r^{\pm i\eta}\phi\nabla\zeta, \nabla v')_\omega - (\sigma\nabla(r^{\pm i\eta}\phi), v'\nabla\zeta)_\omega, \quad \forall v' \in \mathscr{C}_0^\infty(\omega).$$

Since ζ is equal to one in a neighbourhood of O, the support of $\nabla\zeta$ does not meet O. Therefore, there exists a constant $C > 0$ such that $|(\sigma\nabla s^+, \nabla v')_\omega| \le C\, \|v'\,;\, \mathrm{V}^1_{\beta}(\omega)\|$, for all $v' \in \mathscr{C}_0^\infty(\omega)$. Since by definition, $\mathscr{C}_0^\infty(\omega)$ is dense in $\mathrm{V}^1_{\beta}(\omega)$, we deduce that the linear form $v' \mapsto (\sigma\nabla s^\pm, \nabla v')_\omega$ can be uniquely continuously extended to $\mathrm{V}^1_{\beta}(\omega)$. This justifies that the operator $\mathscr{M}_{-\beta}^{\mathrm{rad}} : \mathrm{span}(\zeta, s^+) \oplus \mathrm{V}^1_{-\beta}(\omega) \to \mathrm{V}^1_{\beta}(\omega)^*$ is well-defined. ■

In Theorem 5.9, the choice of adding the singularity 1 but not the singularity $\ln r$ to the functional framework in which we search for the solution is quite natural. Indeed, the function 1 belongs to $\mathrm{H}^1(\omega)$ whereas the function $\ln r$ does not. However, the choice of adding s^+ instead of s^- in the case 2) is more arbitrary. Assume that $(\sigma_1, \sigma_2) \in \mathscr{R}$. For

$\beta \in (0;2)$ and $\gamma \in \mathbb{C}$, define the unique operator (work as in the item 2) of the proof of Theorem 5.9)

$$\mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma) : \mathrm{span}(\zeta, s^+ + \gamma s^-) \oplus \mathrm{V}^1_{-\beta}(\omega) \to \mathrm{V}^1_{\beta}(\omega)^*$$

such that, for all $v \in \mathrm{span}(\zeta, s^+ + \gamma s^-) \oplus \mathrm{V}^1_{-\beta}(\omega),\ v' \in \mathscr{C}^\infty_0(\omega)$,

$$\left\langle \mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma) v, v' \right\rangle_\omega = (\sigma \nabla v, \nabla v')_\omega.$$

Here, s^+, s^- are the functions introduced in (39). For the operator $\mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma)$, we have the

Proposition 5.10 *i) If $\gamma \in \mathbb{C} \setminus \{-1\}$, then $\mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma)$ is an isomorphism.*
ii) If $\gamma = -1$, then $\mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma)$ is a Fredholm operator of index zero. Moreover, we have $\ker \mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma) = \mathrm{span}(s)$*, where s is the function defined in (36).*

Proof Obviously, there holds $\ker \mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma) \subset \ker \mathscr{M}_\beta = \mathrm{span}(s, \ln r)$. Since, $\ln r \notin \mathrm{span}(\zeta, s^+ + \gamma s^-) \oplus \mathrm{V}^1_{-\beta}(\omega)$, we have $\ker \mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma) \subset \mathrm{span}(s)$. One can check that $s \in \mathrm{span}(\zeta, s^+ + \gamma s^-) \oplus \mathrm{V}^1_{-\beta}(\omega)$ if and only if $\gamma = -1$. Therefore, if $\gamma \in \mathbb{C} \setminus \{-1\}$, then $\mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma)$ is injective, and if $\gamma = -1$, then $\ker \mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma) = \mathrm{span}(s)$.

Now, let us study the question of the ontoness of $\mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma)$. Let us consider $H \in \mathrm{V}^1_\beta(\omega)^*$. Since the operator $\mathscr{M}_\beta : \mathrm{V}^1_\beta(\omega) \to \mathrm{V}^1_{-\beta}(\omega)^*$ defined in (29) is onto in virtue of Theorem 5.7, we know that there exists $v \in \mathrm{V}^1_\beta(\omega)$ such that $\mathscr{M}_\beta v = H$. According to formula (33), v admits the representation $v = c_1 \ln r + \zeta(c_2 + c_3 s^+ + c_4 s^-) + \tilde{v}$ for some constants c_1, c_2, c_3, c_4 and some $\tilde{v} \in \mathrm{V}^1_{-\beta}(\omega)$. Since $\ln r$ and s belong to $\ker \mathscr{M}_\beta$, for all $\alpha \in \mathbb{C}$, the function $u = v - c_1 \ln r - \alpha s$ also satisfies $\mathscr{M}_\beta u = H$. In order H to be in the range of $\mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma)$, we must find α such that $u = v - c_1 \ln r - \alpha s$ belongs to $\mathrm{span}(\zeta, s^+ + \gamma s^-) \oplus \mathrm{V}^1_{-\beta}(\omega)$. This is achievable for all $H \in \mathrm{V}^1_\beta(\omega)^*$ if and only if the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -\gamma \end{pmatrix}$$

is invertible. This condition is equivalent to $\gamma \neq -1$. This proves that $\mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma)$ is onto as soon as $\gamma \neq -1$. This procedure allows also to demonstrate that when $\gamma = -1$, there holds $\dim(\mathrm{coker}\, \mathscr{M}^{\mathrm{rad}}_{-\beta}(\gamma)) = 1$. ■

With these results, we have managed to construct functional frameworks, which take into account the oscillating singularities, where well-posedness for the problem in the unfolded geometry ω holds.

6. Back to the original geometry

Now, we go back to the original interior transmission problem of Section 2. To simplify the notations and to avoid multiple sub-cases, we will focus our attention on quite specific configurations. First, we will assume that the domain D is partitioned into two subdomains D_1, D_2 such that $D_1 \cap D_2 = \emptyset$ and $\overline{D} = \overline{D_1} \cup \overline{D_2}$. The interface $\partial D_1 \cap \partial D_2$ meets ∂D at exactly two points O, O'. At these points, $\partial D_1 \cap \partial D_2$ and the boundary ∂D are locally straight lines. Therefore, at O (resp. O'), the domain D_1 coincides with a sector. We denote ϑ (resp. ϑ') the aperture of this sector (see Figure 11). Define $A_1 := A|_{D_1}$, $A_2 := A|_{D_2}$, where A is introduced in §2.1, and assume that

$$\sup_{D_1 \cap \mathscr{V}_1} A \;<\; 1 \qquad \text{and} \qquad \inf_{D_2 \cap \mathscr{V}_2} A \;>\; 1. \tag{40}$$

Here, for $i = 1, 2$, $\mathscr{V}_i$ denotes a neighbourhood of $\partial D_i \cap \partial D$. At O', we assume that the coefficient A is such that the condition (13) is satisfied locally, *i.e.* we assume that there exists $d' > 0$ such that

$$\begin{aligned} &\sup_{D_1 \cap \mathrm{B}(O',d')} A \;<\; 1/(1+\Upsilon_{\vartheta'}) \\ \text{and} \quad &\inf_{D_2 \cap \mathrm{B}(O',d')} A \;>\; \frac{1+\Upsilon_{\vartheta'}}{1-(\sup_{D_1 \cap \mathrm{B}(O',d')} A)(1+\Upsilon_{\vartheta'})}, \end{aligned} \tag{41}$$

where $\Upsilon_{\vartheta'} = \max((\pi-\vartheta')/\vartheta', \vartheta'/(\pi-\vartheta'))$. In this notation, $\mathrm{B}(O', d')$ refers to the open disk of radius d'.

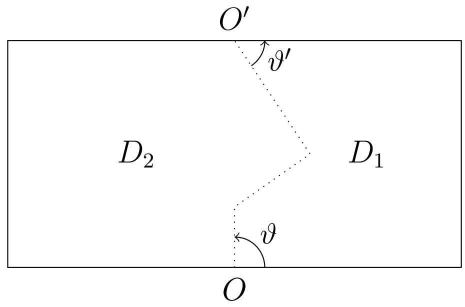

Figure 11. Geometry of the domain D.

Let us introduce a set of assumptions, which will describe different configurations, to precise the values allowed for A and n. In particular, Assumption 1 is constructed so that we can work in a H^1 framework whereas under Assumption 3, two oscillating singularities exist at point O.

Assumption 1 The function A is such that condition (13) is satisfied locally at O. In other words, we assume that there exists $d > 0$ such that

$$\sup_{D_1 \cap \mathrm{B}(O,d)} A < \frac{1}{1+\Upsilon_\vartheta} \quad \text{and} \quad \inf_{D_2 \cap \mathrm{B}(O,d)} A > \frac{1+\Upsilon_\vartheta}{1-(\sup_{D_1 \cap \mathrm{B}(O,d)} A)(1+\Upsilon_\vartheta)}, \tag{42}$$

where $\Upsilon_\vartheta = \max((\pi-\vartheta)/\vartheta, \vartheta/(\pi-\vartheta))$.

Assumption 2 The coefficient n satisfies $\sup_{D_1 \cap \mathscr{V}_1} n < 1$ and $\inf_{D_2 \cap \mathscr{V}_2} n > 1$, where

$\mathscr{V}_i$, $i = 1, 2$, is the neighbourhood of $\partial D_i \cap \partial D$ introduced in (40). Moreover, there holds

$$\begin{aligned}
&\sup_{D_1\cap \mathrm{B}(O,d)} n < \frac{1}{1+\Upsilon_\vartheta}, \quad \inf_{D_2\cap \mathrm{B}(O,d)} n > \frac{1+\Upsilon_\vartheta}{1-(\sup_{D_1\cap \mathrm{B}(O,d)} n)(1+\Upsilon_\vartheta)},\\
&\sup_{D_1\cap \mathrm{B}(O',d')} n < \frac{1}{1+\Upsilon_{\vartheta'}}, \quad \inf_{D_2\cap \mathrm{B}(O',d')} n > \frac{1+\Upsilon_{\vartheta'}}{1-(\sup_{D_1\cap \mathrm{B}(O',d')} n)(1+\Upsilon_{\vartheta'})},
\end{aligned} \tag{43}$$

where Υ_ϑ, d and $\Upsilon_{\vartheta'}, d'$ are respectively defined in (42) and (41).

Assumption 3 There exists $d > 0$ such that $A_1 = A|_{D_1}$ and $A_2 = A|_{D_2}$ are respectively constant in $D_1 \cap \mathrm{B}(O, d)$ and $D_2 \cap \mathrm{B}(O, d)$, with $A_1 = \sigma_1$ and $A_2 = \sigma_2$. Moreover, $\partial D_1 \cap \partial D_2$ is perpendicular to ∂D in this region ($\vartheta = \pi/2$) and (σ_1, σ_2) belongs to $\mathscr{R}$, where $\mathscr{R}$ is defined in (26).

Remark 6.1 *Notice that Assumption 1 and Assumption 3 are mutually exclusive: A cannot satisfy both requirements (see Figure 12).*

Remark 6.2 *With Figure 12, we observe that if $A_1 < 1$ and $A_2 > 1$ are locally constant in a neighbourhood of O and if $\vartheta = \pi/2$, when $(A_1, A_2) \in (0; 1) \times (1; +\infty) \setminus (\overline{\mathscr{F}_1} \cup \overline{\mathscr{R}})$ (notice that this set is not empty), A does not verify Assumption 1 nor Assumption 3. For such A, for which no oscillating singularities exist, we can use a H^1 framework. However, up to now, the authors have failed to handle these configurations with the* T*-coercivity technique.*

Remark 6.3 *To lighten the notation, we do not consider here the case where in Assumption 1, the roles of D_1 and D_2 are exchanged (region $\mathscr{F}_1$ of Figure 12). However, the* T*-coercivity approach we propose allows to consider such configurations.*

6.1. Discreteness of the transmission eigenvalues in the variational framework

Localization process is a classical tool in the theory of elliptic partial differential equations (see [22, Chapter 2, §5], [20, §6.3] or [26, §4.1.2]). Although the operator $\mathscr{A}_k$ associated with the Interior Transmission Problem is not strongly elliptic, we can implement this technique using the T-coercivity approach which allows to restore some ellipticity. Of course, this method can be used only in situations where no oscillating singularities exist.

Theorem 6.4 *Under Assumptions 1 and 2, the operator $\mathscr{A}_k : \mathrm{X} \to \mathrm{X}$ defined in (3) is an isomorphism for $k \in \mathbb{R}i$ such that $|k|$ is large enough.*

Proof First, we introduce a partition of unity adapted to the features of the coefficients A and n. Let ζ_i, $i = 0 \dots 4$ be five elements of $\mathscr{C}^\infty(\overline{D}, [0; 1])$. We assume that ζ_3 (resp. ζ_4) is equal to one in a neighbourhood of O (resp. O') and that the support of ζ_3 is included in $\overline{D \cap \mathrm{B}(O, d)}$ (resp. $\overline{D \cap \mathrm{B}(O', d')}$) where d (resp. d') is introduced in (42) (resp. (41)). The function ζ_1 (resp. ζ_2) is such that its support is included in $\overline{D_1 \cap \mathscr{V}_1} \setminus \{O, O'\}$ (resp. $\overline{D_2 \cap \mathscr{V}_2} \setminus \{O, O'\}$), where $\mathscr{V}_1$ (resp. $\mathscr{V}_2$) denotes the neighbourhood of $\partial D_1 \cap \partial D$ (resp.

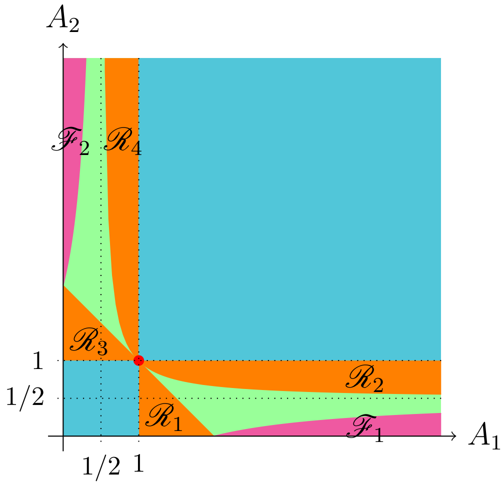

Figure 12. Assume that A_1 and A_2 are locally constant in a neighbourhood of O and that $\vartheta = \pi/2$. In this case, there holds $\Upsilon_\vartheta = 1$ and Assumption 1 boils down to take (A_1, A_2) such that $0 < A_1 < 1/2$, $A_2 > 2/(1 - 2A_1)$ (region $\mathscr{F}_2$). On the other hand, Assumption 2 is equivalent to choose (A_1, A_2) in the region $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3 \cup \mathscr{R}_4$ defined in (26).

$\partial D_2 \cap \partial D$) introduced in (40). The function ζ_0 belongs to $\mathscr{C}_0^\infty(D)$. Finally, these five functions are chosen so that there holds

$$\sum_{i=0}^{4} \zeta_i = 1 \qquad \text{on } \overline{D}.$$

Let us define the operator $\mathtt{T} : \mathrm{X} \to \mathrm{X}$ such that for all $(u, w) \in \mathrm{X}$,

$$\mathtt{T}(u, w) = \zeta_0(u, -w) + \zeta_1(u - 2w, -w) + \zeta_2(u, -w + 2u) + \zeta_3(u_a, w_a) + \zeta_4(u_b, w_b)$$

with

$$\begin{array}{rcl} (u_a, w_a) & = & \left| \begin{array}{ll} (u_1 - 2w_1 + 2R_2 w_2, -w_1 + 2R_2 u_2) & \text{on } D_1 \cap \mathrm{B}(O, d) \\ (u_2, -w_2 + 2u_2) & \text{on } D_2 \cap \mathrm{B}(O, d) \end{array} \right. \\ (u_b, w_b) & = & \left| \begin{array}{ll} (u_1 - 2w_1 + 2R'_2 w_2, -w_1 + 2R'_2 u_2) & \text{on } D_1 \cap \mathrm{B}(O', d) \\ (u_2, -w_2 + 2u_2) & \text{on } D_2 \cap \mathrm{B}(O', d) \end{array} \right. \end{array}.$$

In this definition, the operator R_2 is the one introduced in (9) whereas R'_2 is such that $(R'_2\varphi_2)(r', \theta') = \varphi_2(r', \frac{\vartheta' - \pi}{\vartheta'}\theta' + \pi)$, for $\varphi_2 \in \mathrm{H}^1(D \cap \mathrm{B}(O', d'))$, (r', θ') being the polar coordinates associated with O'.

Let us prove that the form $a^{\mathrm{T}}_{i\kappa}$ defined in (4) is coercive for some $\kappa \in \mathbb{R}$ large enough.

For all $(u,w) \in \mathrm{X}$, one has

$$
\begin{array}{rl}
& a^{\mathrm{T}}_{i\kappa}((u,w),(u,w)) \\
= & (A\nabla u, \nabla u)_D + (\nabla w, \nabla w)_D + \kappa^2(nu,u)_D + \kappa^2(w,w)_D \\
& -\,2(A\nabla u, \nabla(\zeta_1 w))_D - 2\kappa^2(nu,\zeta_1 w)_D \\
& -\,2(\nabla w, \nabla(\zeta_2 u))_D - 2\kappa^2(w,\zeta_2 u)_D \\
& -\,2(A_1\nabla u_1, \nabla(\zeta_3(w_1 - R_2 w_2)))_{D_1} - 2(\nabla w_1, \nabla(\zeta_3 R_2 u_2))_{D_1} - 2(\nabla w_2, \nabla(\zeta_3 u_2))_{D_2} \\
& -\,2\kappa^2(n_1 u_1, \zeta_3(w_1 - R_2 w_2))_{D_1} - 2\kappa^2(w_1, \zeta_3 R_2 u_2)_{D_1} - 2\kappa^2(w_2,\zeta_3 u_2)_{D_2} \\
& -\,2(A_1\nabla u_1, \nabla(\zeta_4(w_1 - R_2' w_2)))_{D_1} - 2(\nabla w_1, \nabla(\zeta_4 R_2' u_2))_{D_1} - 2(\nabla w_2, \nabla(\zeta_4 u_2))_{D_2} \\
& -\,2\kappa^2(n_1 u_1, \zeta_4(w_1 - R_2' w_2))_{D_1} - 2\kappa^2(w_1, \zeta_4 R_2' u_2)_{D_1} - 2\kappa^2(w_2,\zeta_4 u_2)_{D_2}.
\end{array}
\tag{44}
$$

Let us present how to deal with the first "non coercive term" in (44). We first write

$$2(A\nabla u, \nabla(\zeta_1 w))_D = 2(\zeta_1 A\nabla u, \nabla w)_D + 2(A\nabla u, w\nabla\zeta_1)_D.$$

Using Young's inequality, we obtain for all $\alpha > 0$,

$$|2(\zeta_1 A\nabla u, \nabla w)_D| \le \alpha(\zeta_1 A\nabla u, \nabla u)_D + \alpha^{-1}(\zeta_1 A\nabla w, \nabla w)_D.$$

We deduce

$$-|2(A\nabla u, \nabla(\zeta_1 w))_D| \ge -\alpha(\zeta_1 A\nabla u, \nabla u)_D - \alpha^{-1}(\zeta_1 A\nabla w, \nabla w)_D - c\,\|u\|_{\mathrm{H}^1(D)}\|w\|_D,$$

where $c > 0$ is a constant. Similarly, we have

$$-|2(nu,\zeta_1 w)_D| \ge -\alpha(\zeta_1 nu, u)_D - \alpha^{-1}(\zeta_1 nw, w)_D.$$

Since A satisfies property (40) and since n is such that $\sup_{D_1\cap\mathscr{V}_1} n < 1$, one can choose $\alpha > 0$ such that

$$1-\alpha > 0; \quad Id - \alpha^{-1}A > 0 \quad \text{and} \quad 1 - \alpha^{-1}n > 0 \qquad \text{in } D_1 \cap \mathscr{V}_1.$$

This yields

$$
\begin{array}{rl}
& (\zeta_1 A\nabla u, \nabla u)_D + (\zeta_1 \nabla w, \nabla w)_D + \kappa^2(\zeta_1 nu, u)_D + \kappa^2(\zeta_1 w, w)_D \\
& -2(A\nabla u, \nabla(\zeta_1 w))_D - 2\kappa^2(nu, \zeta_1 w)_D \\
\ge & C\Big((\zeta_1 A\nabla u, \nabla u)_D + (\zeta_1 \nabla w, \nabla w)_D + \kappa^2(\zeta_1 nu, u)_D + \kappa^2(\zeta_1 w, w)_D\Big) \\
& -c\,\|u\|_{\mathrm{H}^1(D)}\|w\|_D.
\end{array}
$$

The same idea allows to study the term $-2(\nabla w, \nabla(\zeta_2 u))_D - 2\kappa^2(w,\zeta_2 u)_D$ in (44). To consider the remaining terms, we proceed like in the proof of Lemma 3.1. Collecting all these intermediate estimates, we finally find

$$
\begin{array}{rcl}
a^{\mathrm{T}}_{i\kappa}((u,w),(u,w)) & \ge & C\Big((A\nabla u, \nabla u)_D + (\nabla w, \nabla w)_D + \kappa^2(nu,u)_D + \kappa^2(w,w)_D\Big) \\
& & -c\,(\|u\|_{\mathrm{H}^1(D)}\|w\|_D + \|u\|_D\|w\|_{\mathrm{H}^1(D)}).
\end{array}
\tag{45}
$$

Writing

$$\|u\|_{\mathrm{H}^1(D)}\|w\|_D + \|u\|_D\|w\|_{\mathrm{H}^1(D)} \le \eta\,(\|u\|^2_{\mathrm{H}^1(D)} + \|w\|^2_{\mathrm{H}^1(D)}) + \eta^{-1}(\|u\|^2_D + \|w\|^2_D), \tag{46}$$

for all $\eta > 0$, plugging (46) in (45) and taking η small enough, we obtain

$$\begin{array}{rcl} a^{\mathrm{T}}_{i\kappa}((u,w),(u,w)) & \geq & C\left((A\nabla u,\nabla u)_D + (\nabla w,\nabla w)_D + \kappa^2(nu,u)_D + \kappa^2(w,w)_D\right) \\ & & -c\,(\|u\|^2_D + \|w\|^2_D). \end{array}$$

This proves that if κ is large enough, then $a^{\mathrm{T}}_{i\kappa}$ is coercive. In particular, this implies that $\mathtt{T}^*\mathscr{A}_{i\kappa}$ is an isomorphism of X. Since $\mathscr{A}_{i\kappa}$ is selfadjoint, we deduce that the operator $\mathscr{A}_k$ is an isomorphism for $k \in \mathbb{R}i$ such that $|k|$ is large enough. ■

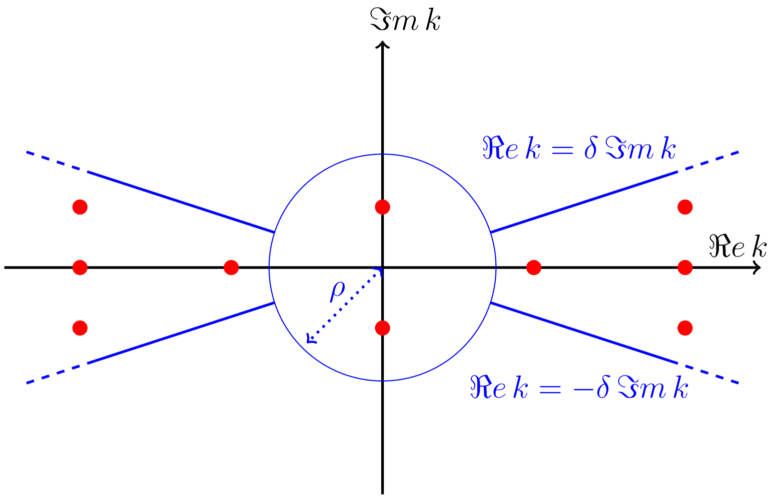

Figure 13. Under Assumptions 1 and 2, all the transmission eigenvalues are located in an infinite bow tie of the complex plane.

As a corollary of this theorem, since in the definition of $\mathscr{A}_k$, the spectral parameter k and the coefficient n appear only in the compact part, we have the

Proposition 6.5 *Under Assumption 1, for all $k \in \mathbb{C}$, the operator $\mathscr{A}_k : \mathrm{X} \to \mathrm{X}$ defined in (3) is a Fredholm operator of index zero.*

Let us conclude by stating and proving the main result of this section.

Theorem 6.6 *Under Assumptions 1 and 2, the set of transmission eigenvalues is at most discrete with infinity as the only accumulation point. Moreover, there exist two positive constants ρ and δ such that if $k \in \mathbb{C}$ verifies $|k| > \rho$ and $|\Re e\, k| < \delta\, |\Im m\, k|$, then k is not a transmission eigenvalue (see Figure 13).*

Proof The first result is a direct consequence of Propositions 6.5, 6.4 and analytic Fredholm theorem. Let us study the question of the localization of the transmission eigenvalues. According to Proposition 6.4, we know that there exists $\tau_0 > 0$ such that if $k_0 \in \mathbb{R}i$ satisfies $|k_0| \geq \tau_0$, then $\mathscr{A}_{k_0}$ is an isomorphism. Let us denote $\mathscr{R}_{k_0}$ the inverse of $\mathscr{A}_{k_0}$. Now, take $k = k_0 e^{i\theta}$ with $\theta \in [-\pi/2; \pi/2]$. It is easy to check that $\|\mathscr{A}_k - \mathscr{A}_{k_0}\| \leq C|1 - e^{2i\theta}|$. Therefore, $\mathscr{R}_{k_0}\mathscr{A}_k = \mathscr{R}_{k_0}(\mathscr{A}_{k_0} + (\mathscr{A}_k - \mathscr{A}_{k_0})) = Id + \mathscr{R}_{k_0}(\mathscr{A}_k - \mathscr{A}_{k_0})$ is invertible for θ small enough. This yields the second result of the theorem. ■

6.2. Fredholmness in the frameworks with strongly oscillating singularities

In this section, we suppose that Assumption 3 holds true. In this case, according to the results we obtained in §4.3, we know that there exist two strongly oscillating singularities at O. Our goal is to understand what is the consequence of the existence of these singularities for the initial transmission eigenvalue problem. Before proceeding, we need to convert to the original geometry the notations introduced in §5.1 to study the transmission problem set in the unfolded domain.

In accordance to §5.1, for $\beta \in \mathbb{R}$, we introduce the space

$$\mathrm{X}_\beta = \left\{(u,w) \in \mathrm{V}^1_\beta(D) \times \mathrm{V}^1_\beta(D) \,|\, u - w = 0 \quad \text{on } \partial D \setminus \{O\}\right\}.$$

In this definition, $\mathrm{V}^1_\beta(D)$ denotes the closure of $\mathscr{C}^\infty(\overline{D} \setminus \{O\})$ for the norm

$$\|\varphi\,;\,\mathrm{V}^1_\beta(D)\| = \left(\|r^\beta \nabla\varphi\|^2_D + \|r^{\beta-1}\varphi\|^2_D\right)^{1/2},$$

where r is the distance to the point O and $\beta \in \mathbb{R}$ is the weight. We denote X^*_β the topological dual space of X_β, made of the continuous antilinear forms on X_β. For all $k \in \mathbb{C}$, we introduce the operator

$$\mathscr{A}_{k,\,\beta} : \mathrm{X}_\beta \to \mathrm{X}^*_{-\beta}. \tag{47}$$

such that, for all $(u,w) \in \mathrm{X}_\beta$, $(u',w') \in \mathrm{X}_{-\beta}$,

$$\langle \mathscr{A}_{k,\,\beta}(u,w),(u',w')\rangle_D = (A\nabla u, \nabla u')_D - (\nabla w, \nabla w')_D - k^2\left((nu,u')_D - (w,w')_D\right).$$

The two oscillating singularities in the unfolded geometry ω were denoted $(r,\theta) \mapsto r^{i\eta}\phi(\theta)$ and $(r,\theta) \mapsto r^{-i\eta}\phi(\theta)$. We define the functions Φ_t, $\Phi_i \in \mathrm{H}^1((0;\pi))$ such that $\Phi_t(\theta) = \phi(\theta)$, $\Phi_i(\theta) = \phi(2\pi - \theta)$ for all $\theta \in (0;\pi)$. Here, we use the subscripts $_t$ and $_i$ because Φ_t, Φ_i correspond respectively to the angular component of the singularities for the total and incident fields (see the sentence right after (1) for the definition of the total and incident fields). We introduce

$$\begin{aligned}
&\mathbf{s}_a := (\mathbf{s}_{a\,t}, \mathbf{s}_{a\,i}) \quad \text{where} \quad (\mathbf{s}_{a\,t}(r,\theta), \mathbf{s}_{a\,i}(r,\theta)) := (1,1),\\
&\mathbf{s}_b := (\mathbf{s}_{b\,t}, \mathbf{s}_{b\,i}) \quad \text{where} \quad (\mathbf{s}_{b\,t}(r,\theta), \mathbf{s}_{b\,i}(r,\theta)) := (\ln r, \ln r),\\
&\mathbf{s}_c := (\mathbf{s}_{c\,t}, \mathbf{s}_{c\,i}) \quad \text{where} \quad (\mathbf{s}_{c\,t}(r,\theta), \mathbf{s}_{c\,i}(r,\theta)) := (\zeta(r)r^{i\eta}\Phi_t(\theta), \zeta(r)r^{i\eta}\Phi_i(\theta)),\\
&\mathbf{s}_d := (\mathbf{s}_{d\,t}, \mathbf{s}_{d\,i}) \quad \text{where} \quad (\mathbf{s}_{d\,t}(r,\theta), \mathbf{s}_{d\,i}(r,\theta)) := (\zeta(r)r^{-i\eta}\Phi_t(\theta), \zeta(r)r^{-i\eta}\Phi_i(\theta)).
\end{aligned} \tag{48}$$

Above, $\zeta \in \mathscr{C}^\infty(\mathbb{R}, [0;1])$ is a cut-off function which is equal to 1 in a neighbourhood of O and whose support is included in $[0;d)$, the parameter d being introduced in Assumption 3. Thanks to this cut-off function, the matching conditions $\mathbf{s}_{c\,t} = \mathbf{s}_{c\,i}$ and $\mathbf{s}_{d\,t} = \mathbf{s}_{d\,i}$ are satisfied on ∂D. Notice that the matching conditions $\mathbf{s}_{a\,t} = \mathbf{s}_{a\,i}$ and $\mathbf{s}_{b\,t} = \mathbf{s}_{b\,i}$ on ∂D are satisfied without need to use ζ.

As in the proof of Theorem 5.5, using a localization process, Theorem 5.5 to invert locally in a neighbourhood of O, Theorem 6.4 to invert locally on $\partial D \setminus \{O\}$, we can build left and right regularizers, *i.e.* left and right inverses modulo a compact operator, for $\mathscr{A}_{0,\,\beta} : \mathrm{X}_\beta \to \mathrm{X}^*_{-\beta}$. This allows to proves the

Theorem 6.7 *The operator* $\mathscr{A}_{0,\beta} : \mathrm{X}_\beta \to \mathrm{X}^*_{-\beta}$ *defined in (47) is of Fredholm type if and only if* $\Lambda \cap \ell_\beta = \emptyset$*, where* Λ*,* ℓ_β *are respectively introduced in (23), (31).*
If $\Lambda \cap \ell_\beta \neq \emptyset$*, then the range of* $\mathscr{A}_{0,\beta} : \mathrm{X}_\beta \to \mathrm{X}^*_{-\beta}$ *is not closed.*

Remark 6.8 *Of course, the same results hold true for* $\mathscr{A}_{k,\beta}$ *for all* $k \in \mathbb{C}$*, since* $\mathscr{A}_{k,\beta} - \mathscr{A}_{0,\beta}$ *is a compact operator.*

Now, we detail the proof of an important result, specific to this configuration where the sign of $A - Id$ changes on ∂D.

Proposition 6.9 *Under Assumption 3, for all* $k \in \mathbb{C}$*, the original operator* $\mathscr{A}_k : \mathrm{X} \to \mathrm{X}$ *defined in (3) in the* H^1 *framework, is not of Fredholm type. Its kernel is of finite dimension but its range is not closed.*

Proof For all $k \in \mathbb{C}$, $\mathscr{A}_k - \mathscr{A}_0$ is a compact operator. Therefore, it is sufficient to prove that $\mathscr{A}_0$ is not a Fredholm operator. First, we notice that there holds $\mathrm{H}^1(D) \subset \mathrm{V}^1_\beta(D)$ for $\beta \geq 1$. This yields $\ker \mathscr{A}_0 \subset \ker \mathscr{A}_{0,\beta}$ for $\beta \geq 1$. According to Proposition 4.1, we know that there exists $\tilde{\beta} \geq 1$ for which $\Lambda \cap \ell_{\tilde{\beta}} = \emptyset$. Consequently, in virtue of Theorem 6.7, $\mathscr{A}_{0,\tilde{\beta}}$ is a Fredholm operator. This allows us to write $\dim(\ker \mathscr{A}_0) \leq \dim(\ker \mathscr{A}_{0,\tilde{\beta}}) < +\infty$.

To prove that the range of $\mathscr{A}_0$ is not closed, we start by recalling a lemma due to J. Peetre [29] (see also Lemma 5.1 in [22, Chapter 2] or Lemma 3.4.1 in [20]).

Lemma 6.10 *Let* Y*,* W*,* Z *be three reflexive Banach spaces, such that* Y *is compactly embedded into* Z*. Let* $L : Y \to W$ *be a continuous linear map. Then the assertions below are equivalent:*

i) $\dim(\ker L) < +\infty$ *and* $\mathrm{range}\, L$ *is closed in* W*;*

ii) there exists $C > 0$ *such that* $\|y\|_Y \leq C\,(\|Ly\|_W + \|y\|_Z)$*,* $\forall y \in Y$*.*

Assume that, for our problem, there exists $C > 0$ such that there holds

$$\|(u,w)\|_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)} \leq C\,(\|\mathscr{A}_0(u,w)\|_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)} + \|(u,w)\|_D), \qquad \forall (u,w) \in \mathrm{X}. \tag{49}$$

For all $m \in \mathbb{N}^*$, define the pair (u_m, w_m) such that

$$(u_m(\boldsymbol{x}), w_m(\boldsymbol{x})) = (\zeta(r) r^{i\eta+1/m}\Phi_t(\theta), \zeta(r) r^{i\eta+1/m}\Phi_i(\theta)),$$

where ζ is the cut-off function introduced after (48). By construction, (u_m, w_m) belongs to X for all $m \in \mathbb{N}^*$. It is clear that there exists a constant $C > 0$ such that

$$\|(u_m, w_m)\|_D \leq C, \qquad \forall m \in \mathbb{N}^*. \tag{50}$$

Moreover, we can write, for a fixed small $\varepsilon > 0$ and some $C > 0$,

$$\|(u_m, w_m)\|^2_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)} \geq \|\nabla u_m\|^2_{\mathrm{B}(O,\varepsilon)} \geq C \int_0^\varepsilon r^{2/m-1}\,dr = C\,\frac{m}{2}\,\varepsilon^{2/m} \underset{m\to+\infty}{\sim} C\,\frac{m}{2}. \tag{51}$$

Our goal is to contradict estimate (49). Therefore, it remains to prove that the sequence $(\|\mathscr{A}_0(u_m, w_m)\|_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)})_{m\in\mathbb{N}^*}$ remains bounded. Since the space

$$\mathrm{X}_\infty := \big\{(u,w) \in \mathrm{X} \,|\, (u,w) \in \mathscr{C}^\infty(\overline{D} \setminus \{O\}) \times \mathscr{C}^\infty(\overline{D} \setminus \{O\})\big\} \tag{52}$$

is dense in X, we can write

$$\|\mathscr{A}_0(u_m, w_m)\|_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)} = \sup_{\substack{(u', w') \in \mathrm{X}_\infty, \\ \|(u', w')\|_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)} = 1}} |(A\nabla u_m, \nabla u')_D - (\nabla w_m, \nabla w')_D|.$$

Let us define $\widehat{u}_m$, $\widehat{w}_m$ such that $(\widehat{u}_m(\boldsymbol{x}), \widehat{w}_m(\boldsymbol{x})) = (r^{i\eta+1/m}\Phi_t(\theta), r^{i\eta+1/m}\Phi_i(\theta))$. With this definition, there holds $(u_m, w_m) = (\zeta\widehat{u}_m, \zeta\widehat{w}_m)$. For all $(u', w') \in \mathrm{X}_\infty$, we can write

$$\begin{array}{rl} & |(A\nabla u_m, \nabla u')_D - (\nabla w_m, \nabla w')_D - ((A\nabla\widehat{u}_m, \nabla(\zeta u'))_D - (\nabla\widehat{w}_m, \nabla(\zeta w'))_D)| \\ = & |(A\widehat{u}_m\nabla\zeta, \nabla u')_D - (\widehat{w}_m\nabla\zeta, \nabla w')_D - ((A\nabla\widehat{u}_m, u'\nabla\zeta)_D - (\nabla\widehat{w}_m, w'\nabla\zeta)_D)| \\ \leq & C\,\|(u', w')\|_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)}, \end{array} \tag{53}$$

where the constant C is independent of $m \in \mathbb{N}^*$. The last line of (53) has been obtained noticing that $\nabla\zeta$ vanishes in a neighbourhood of O. Therefore, to prove that $(\|\mathscr{A}_0(u_m, w_m)\|_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)})_{m\in\mathbb{N}^*}$ remains bounded, it is sufficient to establish that there exists $C > 0$ such that

$$|(A\nabla\widehat{u}_m, \nabla(\zeta u'))_D - (\nabla\widehat{w}_m, \nabla(\zeta w'))_D| \leq C\,\|(u', w')\|_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)},\ \forall (u', w') \in \mathrm{X}_\infty. \tag{54}$$

Integrating twice by parts (remember that the elements of X_∞ vanish at O), we find

$$\begin{array}{rl} & |(A\nabla\widehat{u}_m, \nabla(\zeta u'))_D - (\nabla\widehat{w}_m, \nabla(\zeta w'))_D| \\ = & |(\mathrm{div}(A\nabla\widehat{u}_m), \zeta u')_D - (\Delta\widehat{w}_m, \zeta w')_D| \\ = & (1/m)|(A(2i\eta + 1/m)r^{-2+i\eta+1/m}\Phi_t(\theta), \zeta u')_D \\ & \qquad -((2i\eta + 1/m)r^{-2+i\eta+1/m}\Phi_i(\theta), \zeta w')_D| \\ = & (1/m)|(A(2i\eta + 1/m)r^{-1+i\eta+1/m}\Phi_t(\theta)/(i\eta + 1/m), \partial_r(\zeta u'))_D \\ & \qquad -((2i\eta + 1/m)r^{-1+i\eta+1/m}\Phi_i(\theta)/(i\eta + 1/m), \partial_r(\zeta w'))_D| \\ \leq & C\,\|r^{-1+i\eta+1/m}\|_D(\|u'\|_{\mathrm{H}^1(D)} + \|w'\|_{\mathrm{H}^1(D)})/m. \end{array} \tag{55}$$

Since $\|r^{-1+i\eta+1/m}\|_D \leq C\sqrt{m}$ (see (51)), we deduce (54) from (55). Thus, the sequence $(\|\mathscr{A}_0(u_m, w_m)\|_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)})_{m\in\mathbb{N}^*}$ is bounded. Thanks to (50), (51), this proves that estimate (49) does not hold. Lemma 6.10 allows us to conclude that the range of $\mathscr{A}_0$ is not closed since $\dim(\ker\mathscr{A}_0) < +\infty$ according to the first part of the proof. ■

The result of Proposition 6.9 is interesting because it tells us that, under Assumption 3, we can not hope to apply the analytic Fredholm theorem in a H^1 setting in order to prove discreteness of transmission eigenvalues. We need to change the functional framework to recover Fredholmness. To do this, we take into account the two singularities (plus the constant) prescribing the behaviour of the functions at O, as we did in the unfolded geometry ω (see §5.3). Let us present this procedure.

According to formula (23) and Assumption 3, we know that $\Lambda \cap \ell_\beta = \emptyset$ for all $\beta \in (0; 2)$. For $\beta \in (0; 2)$ and $\gamma \in \mathbb{C}$, define the space

$$\mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma) := \mathrm{span}(\mathbf{s}_a, \mathbf{s}_c + \gamma\mathbf{s}_d) \oplus \mathrm{X}_{-\beta} \tag{56}$$

where the singularities $\mathbf{s}_a$, $\mathbf{s}_c$ and $\mathbf{s}_d$ are set in (48). Let us introduce

$$\mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma) : \mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma) \to \mathrm{X}_\beta{}^*$$

the unique operator such that, for all $(u,w) \in \mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma)$, $(u',w') \in \mathrm{X}_{\infty}$,

$$\left\langle \mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma)(u,w),(u',w')\right\rangle_D = (A\nabla u,\nabla u')_D - (\nabla w,\nabla w')_D - k^2\left((nu,u')_D - (w,w')_D\right). \quad (57)$$

Again, for details concerning the definition of this operator, we refer the reader to item 2) of the proof of Theorem 5.9. Using again a localization process, Proposition 5.10 to invert locally in a neighbourhood of O, Theorem 6.4 to invert locally on $\partial D\setminus\{O\}$ and working as in the proof of [2, theorem 4.4], we obtain the

Proposition 6.11 *Under Assumption 3, the operator* $\mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma)$: $\mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma) \to \mathrm{X}_{\beta}{}^*$ *defined in (57) is Fredholm.*

Now, we want to prove that $\mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma) : \mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma) \to \mathrm{X}_{\beta}{}^*$ is of index zero for all $k \in \mathbb{C}$ and all $\gamma \in \mathbb{C}$. Since $\mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma) - \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)$ is compact, it is sufficient to prove this result for $\mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)$. In paragraph §5 where we worked in the canonical geometry ω, this process was relatively easy to carry on since the elements of $\ker \mathscr{M}_{\beta}$ and $\operatorname{coker} \mathscr{M}_{-\beta}$ (see Proposition 5.10) were explicitely known. In the original geometry, we will need to use some simple algebra to obtain the same results. The approach is borrowed from [26, Chapter 5] (see an example of application in [27]).

First, we define the space

$$\mathbf{W} := \operatorname{span}(\mathbf{s}_a,\mathbf{s}_b,\mathbf{s}_c,\mathbf{s}_d) \oplus \mathrm{X}_{-\beta},$$

where the singularities $\mathbf{s}_a$, $\mathbf{s}_b$, $\mathbf{s}_c$ and $\mathbf{s}_d$ are set in (48). Now, we introduce a practical tool, namely the sesquilinear form q over $\mathbf{W}\times\mathbf{W}$ such that

$$q(\mathbf{v},\mathbf{v}') = \langle \mathscr{A}_{0,\beta}\mathbf{v},\mathbf{v}'\rangle_D - \overline{\langle \mathscr{A}_{0,\beta}\mathbf{v}',\mathbf{v}\rangle_D}, \qquad \forall (\mathbf{v},\mathbf{v}') \in \mathbf{W}\times\mathbf{W}. \quad (58)$$

Let us present the main properties of this form. For all $(\mathbf{v},\mathbf{v}') \in \mathbf{W}\times\mathbf{W}$, we have

$$q(\mathbf{v},\mathbf{v}') = -\overline{q(\mathbf{v}',\mathbf{v})}.$$

In other words, q is a skew-symmetric sesquilinear form. Such a map is called a symplectic form. In case where $\mathbf{v} \in \mathrm{X}_{-\beta}$ or $\mathbf{v}' \in \mathrm{X}_{-\beta}$, there holds

$$q(\mathbf{v},\mathbf{v}') = 0.$$

Indeed, for example if $\mathbf{v} \in \mathrm{X}_{-\beta}$, we can write, remembering that the adjoint of $\mathscr{A}_{0,\beta}$ is $\mathscr{A}_{0,-\beta}$,

$$\begin{aligned} q(\mathbf{v},\mathbf{v}') = \langle \mathscr{A}_{0,\beta}\mathbf{v},\mathbf{v}'\rangle_D - \overline{\langle \mathscr{A}_{0,\beta}\mathbf{v}',\mathbf{v}\rangle_D} &= \langle \mathscr{A}_{0,-\beta}\mathbf{v},\mathbf{v}'\rangle_D - \overline{\langle \mathscr{A}_{0,\beta}\mathbf{v}',\mathbf{v}\rangle_D} \\ &= \langle \mathscr{A}_{0,-\beta}\mathbf{v},\mathbf{v}'\rangle_D - \langle \mathscr{A}_{0,-\beta}\mathbf{v},\mathbf{v}'\rangle_D = 0. \end{aligned}$$

Therefore, introducing the quotient space $\mathscr{W} := \mathbf{W}/\mathrm{X}_{-\beta}$, we can actually see q as a symplectic form defined over $\mathscr{W}\times\mathscr{W}$. By definition of $\mathbf{W}$, the classes of equivalence of $\mathbf{s}_a,\mathbf{s}_b,\mathbf{s}_c,\mathbf{s}_d$ constitute a basis of $\mathscr{W}$ (it is easy to see that these functions are linearly independent). Let us construct a new basis for $\mathscr{W}$ made of functions which satisfy some biorthogonality relations for the symplectic form q. In the sequel, this property will be very useful. Let us define the normalisation parameters

$$\begin{aligned} \alpha_0 &:= 1/(\pi(\sigma_1+\sigma_2-2)), \\ \alpha_\eta &:= (2\eta)^{-1}\Big(\int_0^{\pi/2}\sigma_1\Phi_i^2(\theta)\,d\theta + \int_0^{\pi/2}\sigma_2\Phi_i^2(\theta)\,d\theta - \int_0^{\pi}\Phi_t^2(\theta)\,d\theta\Big)^{-1}, \end{aligned} \quad (59)$$

and the pairs of functions

$$\begin{array}{lll}
\mathfrak{s}_0^+ := (\mathfrak{s}_{0\,t}^+, \mathfrak{s}_{0\,i}^+) & \text{where} & (\mathfrak{s}_{0\,t}^+, \mathfrak{s}_{0\,i}^+) := \sqrt{|\alpha_0|}(\mathrm{s}_{a\,t} + i\mathrm{s}_{b\,t}, \mathrm{s}_{a\,i} + i\mathrm{s}_{b\,i}),\\
\mathfrak{s}_0^- := (\mathfrak{s}_{0\,t}^-, \mathfrak{s}_{0\,i}^-) & \text{where} & (\mathfrak{s}_{0\,t}^-, \mathfrak{s}_{0\,i}^-) := \sqrt{|\alpha_0|}(\mathrm{s}_{a\,t} - i\mathrm{s}_{b\,t}, \mathrm{s}_{a\,i} - i\mathrm{s}_{b\,i}),\\
\mathfrak{s}_\eta^+ := (\mathfrak{s}_{\eta\,t}^+, \mathfrak{s}_{\eta\,i}^+) & \text{where} & (\mathfrak{s}_{\eta\,t}^+, \mathfrak{s}_{\eta\,i}^+) := \sqrt{|\alpha_\eta|}(\mathrm{s}_{c\,t}, \mathrm{s}_{c\,i}),\\
\mathfrak{s}_\eta^- := (\mathfrak{s}_{\eta\,t}^-, \mathfrak{s}_{\eta\,i}^-) & \text{where} & (\mathfrak{s}_{\eta\,t}^-, \mathfrak{s}_{\eta\,i}^-) := \sqrt{|\alpha_\eta|}(\mathrm{s}_{d\,t}, \mathrm{s}_{d\,i}).
\end{array}$$

Under Assumption 3 (see also Figure 9 and the discussion at the end of §4.3), the denominators in (59) do not vanish.

Proposition 6.12 *For $j = 0, \eta$ and $l = 0, \eta$, we have*

$$q(\mathfrak{s}_j^\pm, \mathfrak{s}_l^\pm) = \pm i\,\delta_{j,l}\,\mathrm{sgn}(\alpha_j) \qquad \text{and} \qquad q(\mathfrak{s}_j^\pm, \mathfrak{s}_l^\mp) = 0,$$

where $\delta_{j,l} = 1$ if $j = l$ and $\delta_{j,l} = 0$ if $j \neq l$.

Proof The skew-symmetry of q provides $q(\mathfrak{s}_0^\pm, \mathfrak{s}_0^\mp) = 0$ and $q(\mathfrak{s}_\eta^\pm, \mathfrak{s}_\eta^\mp) = 0$: for example, we can write

$$q(\mathfrak{s}_0^+, \mathfrak{s}_0^-) = -\overline{q(\mathfrak{s}_0^-, \mathfrak{s}_0^+)} = -q(\mathfrak{s}_0^+, \mathfrak{s}_0^-) = 0.$$

Let us compute $q(\mathfrak{s}_0^+, \mathfrak{s}_0^+)$. We reintroduced the cut-off function χ_m of Figure 10. One has

$$\begin{array}{ll}
& \langle \mathscr{A}_{0,\beta}\mathfrak{s}_0^+, \mathfrak{s}_0^+ \rangle_D /|\alpha_0| \\
= & \lim\limits_{m\to+\infty} (A\nabla(\mathrm{s}_{a\,t} + i\mathrm{s}_{b\,t}), \nabla(\chi_m(\mathrm{s}_{a\,t} + i\mathrm{s}_{b\,t})))_D - (\nabla(\mathrm{s}_{a\,i} + i\mathrm{s}_{b\,i}), \nabla(\chi_m(\mathrm{s}_{a\,i} + i\mathrm{s}_{b\,i})))_D
\end{array}$$

and

$$\begin{array}{ll}
& \overline{\langle \mathscr{A}_{0,\beta}\mathfrak{s}_0^+, \mathfrak{s}_0^+ \rangle_D}/|\alpha_0| \\
= & \lim\limits_{m\to+\infty} (A\nabla(\mathrm{s}_{a\,t} - i\mathrm{s}_{b\,t}), \nabla(\chi_m(\mathrm{s}_{a\,t} - i\mathrm{s}_{b\,t})))_D - (\nabla(\mathrm{s}_{a\,i} + i\mathrm{s}_{b\,i}), \nabla(\chi_m(\mathrm{s}_{a\,i} + i\mathrm{s}_{b\,i})))_D.
\end{array}$$

We deduce

$$\begin{array}{rcl}
q(\mathfrak{s}_0^+, \mathfrak{s}_0^+) & = & \lim\limits_{m\to+\infty} -2i|\alpha_0|(A\nabla\mathrm{s}_{a\,t}, \nabla(\chi_m\mathrm{s}_{b\,t}))_D + 2i|\alpha_0|(A\nabla\mathrm{s}_{b\,t}, \nabla(\chi_m\mathrm{s}_{a\,t}))_D \\
& & \qquad +2i|\alpha_0|(\nabla\mathrm{s}_{a\,i}, \nabla(\chi_m\mathrm{s}_{b\,i}))_D - 2i|\alpha_0|(\nabla\mathrm{s}_{b\,i}, \nabla(\chi_m\mathrm{s}_{a\,i}))_D \\
& = & \lim\limits_{m\to+\infty} 2i|\alpha_0|(A\nabla\mathrm{s}_{b\,t}, \mathrm{s}_{a\,t}\nabla\chi_m)_D - 2i|\alpha_0|(\nabla\mathrm{s}_{b\,i}, \mathrm{s}_{a\,i}\nabla\chi_m)_D = i\,\mathrm{sgn}\alpha_0.
\end{array} \tag{60}$$

The last line of (60) is the same as (34). In the computation of $q(\mathfrak{s}_0^\pm, \mathfrak{s}_\eta^\pm)$, $q(\mathfrak{s}_0^\mp, \mathfrak{s}_\eta^\mp)$, when one integrates with respect to the θ coordinate in a neighbourhood of O, there appears the term

$$\int_0^{\pi/2} \sigma_1\Phi_i(\theta)\,d\theta + \int_0^{\pi/2} \sigma_2\Phi_i(\theta)\,d\theta - \int_0^{\pi} \Phi_t(\theta)\,d\theta.$$

According to the discussion after (37), we know that this quantity is equal to zero. This yields $q(\mathfrak{s}_0^\pm, \mathfrak{s}_\eta^\pm) = q(\mathfrak{s}_0^\mp, \mathfrak{s}_\eta^\mp) = 0$. Proceeding as for the computation of $q(\mathfrak{s}_0^+, \mathfrak{s}_0^+)$, we finally find $q(\mathfrak{s}_\eta^\pm, \mathfrak{s}_\eta^\pm) = \pm i\,\mathrm{sgn}(\alpha_\eta)$. ■

Let us use the symplectic form q to describe the quotient space $\ker \mathscr{A}_{0,\beta}/\mathrm{X}_{-\beta}$.

Proposition 6.13 *Under Assumption 3, we have* $\dim(\ker \mathscr{A}_{0,\beta}) - \dim(\ker \mathscr{A}_{0,-\beta}) = 2$ *for all* $\beta \in (0;2)$*. Let* K *be a vector space such that* $\ker \mathscr{A}_{0,\beta} = \mathrm{K} \oplus \ker \mathscr{A}_{0,-\beta}$*. There exists a basis of* K *equal to* $(\mathfrak{w}_0, \mathfrak{w}_\eta)$*, with*

$$\mathfrak{w}_0 := \mathbf{s}_a = (1,1)$$

$$\mathfrak{w}_\eta := (\zeta(r)(r^{i\eta} + \gamma_p r^{-i\eta})\Phi_t(\theta), \zeta(r)(r^{i\eta} + \gamma_p r^{-i\eta})\Phi_i(\theta)) + \tilde{\mathfrak{w}}, \tag{61}$$

where $\gamma_p \in \mathbb{C}$ *and* $\tilde{\mathfrak{w}} \in \mathrm{X}_{-\beta}$*. Moreover, the coefficient* γ_p *in (61) satisfies* $|\gamma_p| = 1$*, i.e.* $\gamma_p \in S_{\text{unity}} := \{e^{i\theta},\, \theta \in [0;2\pi)\}$*.*

Proof Let us assume that the normalisation parameters introduced in (59) verify $\alpha_0 > 0$ and $\alpha_\eta > 0$. When $\alpha_0 < 0$ and/or $\alpha_\eta < 0$, the analysis we present can be easily adapted.

i) Let us prove that $\dim(\ker \mathscr{A}_{0,\beta}) - \dim(\ker \mathscr{A}_{0,-\beta}) \geq 2$. Proceed by contradiction assuming that $\dim(\ker \mathscr{A}_{0,\beta}) - \dim(\ker \mathscr{A}_{0,-\beta}) \leq 1$. Introduce F and G two finite dimensional vector spaces such that

$$\mathrm{X}^*_\beta = \operatorname{range} \mathscr{A}_{0,-\beta} \oplus \mathrm{F}; \tag{62}$$

$$\mathrm{F} = (\mathrm{F} \cap \operatorname{range} \mathscr{A}_{0,\beta}) \oplus \mathrm{G}. \tag{63}$$

According to Lemma 6.14, proved later, we have $\mathrm{X}^*_{-\beta} = \operatorname{range} \mathscr{A}_{0,\beta} + \mathrm{X}^*_\beta$. Consequently, we can write $\mathrm{X}^*_{-\beta} = \operatorname{range} \mathscr{A}_{0,\beta} \oplus \mathrm{G}$. Since $\mathscr{A}_{0,-\beta}$ is the adjoint of $\mathscr{A}_{0,\beta}$, one has $\dim(\mathrm{F}) = \dim(\operatorname{coker} \mathscr{A}_{0,-\beta}) = \dim(\ker \mathscr{A}_{0,\beta})$ and $\dim(\mathrm{G}) = \dim(\operatorname{coker} \mathscr{A}_{0,\beta}) = \dim(\ker \mathscr{A}_{0,-\beta})$. Thus, our hypothesis leads to $\dim(\mathrm{F}) - \dim(\mathrm{G}) \leq 1$ which implies $\dim(\mathrm{F} \cap \operatorname{range} \mathscr{A}_{0,\beta}) \leq 1$ according to (63). Now, recall that $\mathscr{A}_{0,\beta}(\mathfrak{s}_0^+) \in \mathrm{X}^*_\beta$ and $\mathscr{A}_{0,\beta}(\mathfrak{s}_\eta^+) \in \mathrm{X}^*_\beta$. According to the decomposition (62), there exist $\tilde{\boldsymbol{v}}_0$, $\tilde{\boldsymbol{v}}_\eta \in \mathrm{X}_{-\beta}$ and $\boldsymbol{f}_0$, $\boldsymbol{f}_\eta \in \mathrm{F}$ such that $\mathscr{A}_{0,\beta}(\mathfrak{s}_0^+) = \mathscr{A}_{0,-\beta}\,\tilde{\boldsymbol{v}}_0 + \boldsymbol{f}_0$ and $\mathscr{A}_{0,\beta}(\mathfrak{s}_\eta^+) = \mathscr{A}_{0,-\beta}\,\tilde{\boldsymbol{v}}_\eta + \boldsymbol{f}_\eta$. But clearly $\boldsymbol{f}_0$ and $\boldsymbol{f}_\eta$ belong to $\mathrm{F} \cap \operatorname{range} \mathscr{A}_{0,\beta}$. Since, by assumption, the dimension of this vector space is less than one, there exist two coefficients τ_0, τ_η, with $|\tau_0| + |\tau_\eta| \neq 0$ such that $\tau_0 \boldsymbol{f}_0 + \tau_\eta \boldsymbol{f}_\eta = 0$. The function $\boldsymbol{v} := \tau_0(\mathfrak{s}_0^+ - \tilde{\boldsymbol{v}}_0) + \tau_\eta(\mathfrak{s}_\eta^+ - \tilde{\boldsymbol{v}}_\eta)$ belongs to $\ker \mathscr{A}_{0,\beta}$. As a consequence, there holds $q(\boldsymbol{v},\boldsymbol{v}) = 0$. But using Proposition 6.12, one finds $q(\boldsymbol{v},\boldsymbol{v}) = i\,|\tau_0|^2 + i\,|\tau_\eta|^2$. This is absurd since we have $|\tau_0| + |\tau_\eta| \neq 0$.

ii) Now, we establish that $\dim(\ker \mathscr{A}_{0,\beta}) - \dim(\ker \mathscr{A}_{0,-\beta}) \leq 2$. Again, we proceed by contradiction and we assume that $\dim(\ker \mathscr{A}_{0,\beta}) - \dim(\ker \mathscr{A}_{0,-\beta}) \geq 3$. Let us introduce $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ three functions of $\ker \mathscr{A}_{0,\beta}$ which are linearly independent modulo $\mathrm{X}_{-\beta}$. According to formula (33), we know that every element $\boldsymbol{v} \in \ker \mathscr{A}_{0,\beta}$ admits the representation $\boldsymbol{v} = c_0^+\mathfrak{s}_0^+ + c_0^-\mathfrak{s}_0^- + c_\eta^+\mathfrak{s}_\eta^+ + c_\eta^-\mathfrak{s}_\eta^- + \tilde{\boldsymbol{v}}$ for some constants c_0^+, c_0^-, c_η^+, c_η^- and some $\tilde{\boldsymbol{v}} \in \mathrm{X}_{-\beta}$. Using some simple algebra, we can find three coefficients τ_1, τ_2, τ_3, with $|\tau_1| + |\tau_2| + |\tau_3| \neq 0$, such that $\boldsymbol{v} := \tau_1\boldsymbol{v}_1 + \tau_2\boldsymbol{v}_2 + \tau_3\boldsymbol{v}_3$ admits the decomposition $\boldsymbol{v} = c_0^+\mathfrak{s}_0^- + c_\eta^+\mathfrak{s}_\eta^- + \tilde{\boldsymbol{v}}$ for some constants c_0^+, c_η^+ and some $\tilde{\boldsymbol{v}} \in \mathrm{X}_{-\beta}$. Since $\boldsymbol{v} \in \ker \mathscr{A}_{0,\beta}$, we have $q(\boldsymbol{v},\boldsymbol{v}) = 0$. Using Proposition 6.12, one finds $q(\boldsymbol{v},\boldsymbol{v}) = i\,|c_0^+|^2 + i\,|c_\eta^+|^2$. Thus, there holds $c_0^+ = c_\eta^+ = 0$ and $\boldsymbol{v}$ is an element of $\ker \mathscr{A}_{0,-\beta}$. In others words, $\tau_1\boldsymbol{v}_1 + \tau_2\boldsymbol{v}_2 + \tau_3\boldsymbol{v}_3$ is equal to zero modulo $\mathrm{X}_{-\beta}$. This is absurd since $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are linearly independent modulo $\mathrm{X}_{-\beta}$ and since $|\tau_1| + |\tau_2| + |\tau_3| \neq 0$.

iii) Let K be a vector space such that $\ker \mathscr{A}_{0,\beta} = \mathrm{K} \oplus \ker \mathscr{A}_{0,-\beta}$. Clearly, $\mathfrak{w}_0 = \mathbf{s}_a = (1,1)$ is a non trivial element of K. Let us introduce $\mathfrak{w}_\eta$ a second element of K such that $(\mathfrak{w}_0, \mathfrak{w}_\eta)$ constitutes a basis of K (we know that this vector space is of dimension 2). According to formula (33), $\mathfrak{w}_\eta$ admits the representation $\mathfrak{w}_\eta = c_a\,\mathbf{s}_a + c_b\,\mathbf{s}_b + c_c\,\mathbf{s}_c + c_d\,\mathbf{s}_d + \tilde{\mathfrak{w}}$ for some constants c_a, c_b, c_c, c_d and some $\tilde{\mathfrak{w}} \in \mathrm{X}_{-\beta}$. Since the first element of our basis $(\mathfrak{w}_0, \mathfrak{w}_\eta) = (\mathbf{s}_a, \mathfrak{w}_\eta)$ is $\mathbf{s}_a$, we can impose $c_a = 0$. Now, observe that $\mathfrak{w}_0$ and $\mathfrak{w}_\eta$ belong to $\ker \mathscr{A}_{0,\beta}$. Thus, we have $q(\mathfrak{w}_\eta, \mathfrak{w}_0) = 0$. Noticing that, $\mathbf{s}_a = |\alpha_0|^{-1/2}(\mathfrak{s}_0^+ + \mathfrak{s}_0^-)/2$, $\mathbf{s}_b = i|\alpha_0|^{-1/2}(\mathfrak{s}_0^- - \mathfrak{s}_0^+)/2$, $\mathbf{s}_c = |\alpha_\eta|^{-1/2}\mathfrak{s}_\eta^+$ and $\mathbf{s}_d = |\alpha_\eta|^{-1/2}\mathfrak{s}_\eta^-$, one computes

$$0 = q(\mathfrak{w}_\eta, \mathfrak{w}_0) = ic_b|\alpha_0|^{-1}q(\mathfrak{s}_0^- - \mathfrak{s}_0^+, \mathfrak{s}_0^+ + \mathfrak{s}_0^-)/4 = c_b|\alpha_0|^{-1}\mathrm{sgn}(\alpha_0)/4.$$

This yields $c_b = 0$. On the other hand, we find

$$0 = q(\mathfrak{w}_\eta, \mathfrak{w}_\eta) = i\,|\alpha_\eta|^{-1}\mathrm{sgn}(\alpha_\eta)\Big(|c_c|^2 - |c_d|^2\Big). \qquad (64)$$

We deduce there should hold both $|c_c| \neq 0$ and $|c_d| \neq 0$ (otherwise $\mathfrak{w}_\eta \in \ker \mathscr{A}_{0,\beta} \cap \mathrm{K} = \{0\}$ and $\mathfrak{w}_\eta$ cannot be an element of the basis of K). Since $\mathfrak{w}_\eta$ is defined up to a multiplicative constant, we can take $c_c = 1$. It follows from (64) that $|c_d| = 1$. Summing up, we can take $\mathfrak{w}_\eta$ admitting the representation $\mathfrak{w}_\eta = \mathbf{s}_c + \gamma_p\,\mathbf{s}_d + \tilde{\mathfrak{w}}$. In this case, the parameter γ_p must satisfy $|\gamma_p| = 1$. ■

The following lemma is a technical result needed in the proof of Proposition 6.13.

Lemma 6.14 *There holds* $\mathrm{X}^*_{-\beta} = \mathrm{range}\,\mathscr{A}_{0,\beta} + \mathrm{X}^*_\beta$.

Proof Consider a source term $\boldsymbol{f} \in \mathrm{X}^*_{-\beta}$. Our goal is to build $\boldsymbol{v} \in \mathrm{X}_\beta$ such that $\boldsymbol{f} - \mathscr{A}_{0,\beta}\,\boldsymbol{v} \in \mathrm{X}^*_\beta$. To proceed, we will localize and unfold $\boldsymbol{f}$ to obtain a source term, defined on the entire plane, with the same behaviour as $\boldsymbol{f}$ at O. Using the well-posedness of the transmission problem in the plane, we will construct a preimage of this source term. Folding and multiplying by a cut-off function this solution to return to the original domain, this procedure will provide a preimage of $\boldsymbol{f}$ modulo a smooth perturbation.

Let us translate this into equations. Define the map $g \in \mathrm{V}^1_{-\beta}(\mathring{\mathbb{R}}^2)^*$ such that $\langle g, \varphi\rangle_{\mathring{\mathbb{R}}^2} = \langle \boldsymbol{f}, \zeta\tau(\varphi)\rangle_{D\times D}$, $\forall \varphi \in \mathrm{V}^1_{-\beta}(\mathring{\mathbb{R}}^2)$. In this definition, ζ is the cut-off function introduced in (48), $\mathring{\mathbb{R}}^2 = \mathbb{R}^2\backslash\{O\}$ and $\tau(\varphi)$ is such that $\tau(\varphi(x,y)) = (\varphi_p(x,y), \varphi_m(x,-y))$, where $\varphi_p = \varphi|_{\mathbb{R}\times(0;+\infty)}$ and $\varphi_m = \varphi|_{\mathbb{R}\times(-\infty;0)}$. According to Theorem 5.2, we know that there exists a unique $v \in \mathrm{V}^1_\beta(\mathring{\mathbb{R}}^2)$ such that $\mathscr{N}_\beta\, v = g$. Define $\boldsymbol{v} := \zeta\tau(v)$. We have both $\mathscr{A}_{0,\beta}\,\boldsymbol{v} \in \mathrm{range}\,\mathscr{A}_{0,\beta}$ and $\boldsymbol{f} - \mathscr{A}_{0,\beta}\,\boldsymbol{v} \in \mathrm{X}^*_\beta$. ■

Now, we are ready to prove well-posedness in the frameworks with oscillating singularities.

Theorem 6.15 *Under Assumption 3, the operator* $\mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma) : \mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma) \to \mathrm{X}_\beta{}^*$ *defined in (57) is Fredholm of index zero for all* $\gamma \in \mathbb{C}$.

Proof Since the index is constant with respect to compact perturbations, let us prove this result for $\mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)$. First, we assume that $\gamma \neq \gamma_p$, where γ_p is defined by the statement of Proposition 6.13 (in particular, we know that $|\gamma_p| = 1$). In this case, we have $\mathfrak{w}_0 \in \ker \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)$ and $\mathfrak{w}_\eta \notin \ker \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)$. Since $\ker \mathscr{A}_{0,-\beta} \subset \ker \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)$, this yields

$$\dim(\ker \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)) = \dim(\ker \mathscr{A}_{0,-\beta}) + 1. \tag{65}$$

Now, let us establish that $\dim(\operatorname{coker} \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)) = \dim(\operatorname{coker} \mathscr{A}_{0,\beta}) + 1$. We introduce $\tilde{\mathrm{F}}$, $\tilde{\mathrm{G}}$ two finite dimensional vector spaces such that

$$\mathrm{X}^*_\beta = \operatorname{range} \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma) \oplus \tilde{\mathrm{F}}; \tag{66}$$

$$\tilde{\mathrm{F}} = (\tilde{\mathrm{F}} \cap \operatorname{range} \mathscr{A}_{0,\beta}) \oplus \tilde{\mathrm{G}}. \tag{67}$$

As in point *i*) of the proof of Proposition 6.13, one obtains $\mathrm{X}^*_{-\beta} = \operatorname{range} \mathscr{A}_{0,\beta}(\gamma) \oplus \tilde{\mathrm{G}}$, $\dim(\tilde{\mathrm{F}}) = \dim(\operatorname{coker} \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma))$ and $\dim(\tilde{\mathrm{G}}) = \dim(\operatorname{coker} \mathscr{A}_{0,\beta})$. Let us prove that $\dim(\tilde{\mathrm{F}}) = \dim(\tilde{\mathrm{G}}) + 1$. According to (67), this is equivalent to $\dim(\tilde{\mathrm{F}} \cap \operatorname{range} \mathscr{A}_{0,\beta}) = 1$. First, let us define $\boldsymbol{g} = \mathscr{A}_{0,\beta}\mathbf{s}_b = \mathscr{A}_{0,\beta}(\zeta \ln r, \zeta \ln r) \in \mathrm{X}^*_\beta$. Assume that $\boldsymbol{g} \in \operatorname{range} \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)$. In this case, there exists $\boldsymbol{v} \in \mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma)$ such that $\mathscr{A}^{\mathrm{rad}}_{0,-\beta}\boldsymbol{v} = \boldsymbol{g}$. The function $\boldsymbol{v} - \mathbf{s}_b$ belongs to $\ker \mathscr{A}_{0,\beta}$. This is not possible by virtue of Proposition 6.13. Indeed, we have computed explicitly the kernel of $\mathscr{A}_{0,\beta}$ and no term involving $\mathbf{s}_b$ appears. Therefore, we conclude that $\boldsymbol{g} \notin \operatorname{range} \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)$. Thus, $\tilde{\mathrm{F}} \cap \operatorname{range} \mathscr{A}_{0,\beta}$ is not equal to $\{0\}$. This implies $\dim(\tilde{\mathrm{F}} \cap \operatorname{range} \mathscr{A}_{0,\beta}) \geq 1$. In view of the sequel, using (66), we decompose $\boldsymbol{g}$ under the form $\boldsymbol{g} = \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)\boldsymbol{v_g} + \boldsymbol{g}^\sharp$, with $\boldsymbol{v_g} \in \mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma)$ and $\boldsymbol{g}^\sharp \in \tilde{\mathrm{F}}$. Now, consider $\boldsymbol{f}$ an element of $\tilde{\mathrm{F}} \cap \operatorname{range} \mathscr{A}_{0,\beta}$. There exists $\boldsymbol{v} \in \mathrm{X}_\beta$ such that $\boldsymbol{f} = \mathscr{A}_{0,\beta}\,\boldsymbol{v}$. Moreover, $\boldsymbol{v}$ admits the representation $\boldsymbol{v} = c_a\,\mathbf{s}_a + c_b\,\mathbf{s}_b + c_c\,\mathbf{s}_c + c_d\,\mathbf{s}_d + \tilde{\boldsymbol{v}}$ for some constants c_a, c_b, c_c, c_d and some $\tilde{\boldsymbol{v}} \in \mathrm{X}_{-\beta}$. Let us look for $\tau \in \mathbb{C}$ such that $\boldsymbol{v} - c_a\,\mathbf{s}_a - c_b\,\mathbf{s}_b - \tau\,(\mathbf{s}_c + \gamma_p\,\mathbf{s}_d + \tilde{\mathfrak{w}})$ belongs to $\mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma)$ (we remind that $\mathbf{s}_a$ and $\mathbf{s}_c + \gamma_p\,\mathbf{s}_d + \tilde{\mathfrak{w}}$ are functions of $\ker \mathscr{A}_{0,\beta}$). One can easily check that we can solve this problem if and only if the matrix

$$\begin{pmatrix} 1 & 1 \\ \gamma_p & \gamma \end{pmatrix}$$

is invertible. This condition is equivalent to $\gamma \neq \gamma_p$, which is precisely the case we are dealing with. Now, observe that $\boldsymbol{v} - c_a\,\mathbf{s}_a - c_b\,\mathbf{s}_b - \tau\,(\mathbf{s}_c + \gamma_p\,\mathbf{s}_d + \tilde{\mathfrak{w}}) + c_b\,\boldsymbol{v_g}$ is an element of $\mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma)$ such that $\mathscr{A}^{\mathrm{rad}}_{0,-\beta}\,(\boldsymbol{v} - c_a\,\mathbf{s}_a - c_b\,\mathbf{s}_b - \tau\,(\mathbf{s}_c + \gamma_p\,\mathbf{s}_d + \tilde{\mathfrak{w}}) + c_b\,\boldsymbol{v_g}) = \boldsymbol{f} - c_b\,\boldsymbol{g}^\sharp$. One deduces that $\boldsymbol{f} - c_b\,\boldsymbol{g}^\sharp \in \tilde{\mathrm{F}} \cap \operatorname{range} \mathscr{A}^{\mathrm{rad}}_{0,-\beta}$ and, since $\tilde{\mathrm{F}} \cap \operatorname{range} \mathscr{A}^{\mathrm{rad}}_{0,-\beta} = \{0\}$, we have $\boldsymbol{f} - c_b\,\boldsymbol{g}^\sharp = 0$. Thus, there holds $\tilde{\mathrm{F}} \cap \operatorname{range} \mathscr{A}_{0,\beta} = \operatorname{span}(\boldsymbol{g}^\sharp)$ and we have finished to prove that

$$\dim(\operatorname{coker} \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)) = \dim(\operatorname{coker} \mathscr{A}_{0,\beta}) + 1. \tag{68}$$

From (65) and (68), we can write

$$\begin{aligned} \operatorname{ind}(\mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)) &= \dim(\ker \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)) - \dim(\operatorname{coker} \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)) \\ &= (\dim(\ker \mathscr{A}_{0,-\beta}) + 1) - (\dim(\operatorname{coker} \mathscr{A}_{0,\beta}) + 1) \\ &= \dim(\ker \mathscr{A}_{0,-\beta}) - \dim(\ker \mathscr{A}_{0,-\beta}) \;=\; 0. \end{aligned}$$

The case where $\gamma = \gamma_p$ can be handled analogously: we can prove that we have both $\dim(\ker \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)) = \dim(\ker \mathscr{A}_{0,-\beta}) + 2$, $\dim(\operatorname{coker} \mathscr{A}^{\mathrm{rad}}_{0,-\beta}(\gamma)) = \dim(\operatorname{coker} \mathscr{A}_{0,\beta}) + 2$. ■

6.3. Eigenvalues in the frameworks with strongly oscillating singularities: discussion

At this stage, we have found a family of formulations of the interior transmission problem, parametrized by $\gamma \in \mathbb{C}$, which satisfy the Fredholm property. Can we prove discreteness of the set of eigenvalues for these formulations? Then, a natural question is: do these different formulations provide the same eigenvalues, and if not, has our result any meaning in relation to the initial question? A part of the answer is given by Proposition 6.16.

For all $\gamma \in \mathbb{C}$, define

$$\mathrm{spc}(\gamma) := \left\{k \in \mathbb{C} \,|\, \ker \mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma) \neq \{0\}\right\}.$$

Moreover, recall that we denote $S_{\mathrm{unity}} := \{e^{i\theta},\, \theta \in [0; 2\pi)\}$.

Proposition 6.16 *Suppose Assumption 3 holds true and* $\gamma \in \mathbb{C} \setminus S_{\mathrm{unity}}$. *Let* $k \in \mathbb{C}$ *be such that* $k^2 \in \mathbb{R}$. *If* (u, w) *is a non trivial element of* $\ker \mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma)$, *then* (u, w) *belongs to* $\mathrm{H}^1(D) \times \mathrm{H}^1(D)$. *In this case,* (u, w) *is a classical transmission eigenvalue in the sense of Definition 2.1. As a consequence, for* $\gamma \in \mathbb{C} \setminus S_{\mathrm{unity}}$, $\mathrm{spc}(\gamma) \cap (\mathbb{R} \cup \mathbb{R}i)$ *is independent of* γ *and is exactly the set of real or purely imaginary transmission eigenvalues, in the classical sense of Definition 2.1.*

Proof Let $k \in \mathbb{C}$ be such that $k^2 \in \mathbb{R}$. Let $\boldsymbol{v} = (u, w)$ be a non trivial element of $\ker \mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma)$. By definition of $\mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma)$ (see (56)), we have $\boldsymbol{v} = c_a\, \mathbf{s}_a + c_s\, (\mathbf{s}_c + \gamma \mathbf{s}_d) + \tilde{\boldsymbol{v}}$ for some constants c_a, c_s and some $\tilde{\boldsymbol{v}} \in \mathrm{X}_{-\beta}$. Since k^2 is real, there holds $q(\boldsymbol{v}, \boldsymbol{v}) = 0$ (here, q is the symplectic form defined in (58)). By a simple computation, working like in point *iii)* of the proof of Proposition 6.13, and using Proposition 6.12, we find $q(\boldsymbol{v}, \boldsymbol{v}) = i|c_s|^2(1 - |\gamma|^2)$. By assumption, we have $|\gamma| \neq 1$. This implies $c_s = 0$ and $\boldsymbol{v} \in \mathrm{H}^1(D) \times \mathrm{H}^1(D)$. ■

According to the analytic Fredholm theorem, for all $\gamma \in \mathbb{C}$, there holds the following alternative: either $\mathrm{spc}(\gamma)$ is at most discrete with infinity as the only accumulation point, or $\mathrm{spc}(\gamma)$ is equal to the entire complex plane. From the previous proposition, we deduce the

Corollary 6.17 *Suppose Assumption 3 holds true. If there exist* $\gamma_0 \in \mathbb{C}$ *and* $k \in \mathbb{C}$ *such that* $\ker \mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma_0) = \{0\}$, *then, for all* $\gamma \in \mathbb{C} \setminus S_{\mathrm{unity}}$, $\mathrm{spc}(\gamma)$ *is at most discrete with infinity as the only accumulation point. Moreover, in this case the set of transmission eigenvalues, in the classical sense of Definition 2.1, is at most discrete with infinity as the only accumulation point.*

Proof Let $\gamma_0 \in \mathbb{C}$ and $k \in \mathbb{C}$ be such that $\ker \mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma_0) = \{0\}$. In this case, with the analytic Fredholm theorem, we can prove that $\mathrm{spc}(\gamma_0)$ is at most discrete with infinity as the only accumulation point. Now, if γ belongs to $\mathbb{C} \setminus S_{\mathrm{unity}}$,we know, thanks to Proposition 6.16, that $(\mathrm{spc}(\gamma) \cap \mathbb{R}) \subset (\mathrm{spc}(\gamma_0) \cap \mathbb{R})$. This proves that $(\mathrm{spc}(\gamma) \cap \mathbb{R})$ is discrete. Therefore, there exists $k \in \mathbb{R}$ such that $\mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma)$ is injective. From the analytic Fredholm theorem, we deduce that $\mathrm{spc}(\gamma)$ is at most discrete with infinity as the only accumulation point. Using the theorem of decomposition 5.6, we can prove that the classical H^1 transmission eigenvalues, in the classical sense of Definition 2.1, actually belong to $\mathrm{span}(\mathbf{s}_a) \oplus \mathrm{X}_{-\beta}$. Noticing that this space is included in $\mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma)$ for all $\gamma \in \mathbb{C}$, we obtain the second part of the corollary. ■

Let us emphasize that finding k such that $\ker \mathscr{A}^{\mathrm{rad}}_{k,-\beta}(\gamma_0) = \{0\}$ cannot be done using the variational approach of the proof of Theorem 6.1. For now, this remains an open problem. What could be proven in the present case is that a given value of k cannot belong to $\mathrm{spc}(\gamma)$ for all parameters A_1 or A_2 varying in a small interval. The idea is to derive the eigenvalue equation with respect to A_i, $i = 1$ or 2. In other words, we are not able to prove the discreteness of transmission eigenvalues for some given configuration. But we can prove the existence of a small perturbation of the configuration such that the discreteness holds.

Let us now go back to the question stated above: does the set of eigenvalues $\mathrm{spc}(\gamma)$ really depend on γ, and if it does, how can we explain and use our result? By Proposition 6.16, if $\gamma_0 \notin S_{\mathrm{unity}}$, real values and purely imaginary values of $\mathrm{spc}(\gamma)$ are independent of γ and correspond to transmission eigenvalues, in the classical sense of Definition 2.1. But what can we say concerning complex values? We conjecture that part of them depend on γ while the others do not. The latest are transmission eigenvalues, in the classical sense of Definition 2.1, while the first ones are not, because they are associated to strongly singular fields (u, w) which are not in $\mathrm{H}^1(D) \times \mathrm{H}^1(D)$. As a consequence, we also conjecture that Herglotz waves are not dense in the functional spaces $\mathrm{X}^{\mathrm{rad}}_{-\beta}(\gamma)$ which contain strongly singular fields.

A numerical way to identify "true" complex transmission eigenvalues among all values of $\mathrm{spc}(\gamma)$ could be to compute the values of $\mathrm{spc}(\gamma)$ for different γ. True eigenvalues are those which are independent of γ. Finally, let us underline that such numerical computations cannot be achieved with a standard approach (a finite element discretization for instance) which will not be able to capture the possible strongly oscillating behavior of the solutions. Again we can benefit from the previous work on the SCTP (sign changing transmission problem) for which we have proved the efficiency of using Perfeclty Matched Layers around the singular points in the case $\gamma = 0$ (see [10, Chapter 5]). We are currently investigating how to extend this approach to deal with the configurations $\gamma \neq 0$.

Acknowledgments

Part of this work has been realized while the second author was visiting the Department of Mathematical Sciences of the University of Delaware thanks to the NSF Grant DMS-1106972. The support of the Associate Team ISIP between DeFi-INRIA at École

Polytechnique and the University of Delaware is gratefully acknowledged. The second author also wishes to express its gratitude to Fioralba Cakoni and David Colton for their very warm welcome.

References

[1] A.-S. Bonnet-Ben Dhia, L. Chesnel, and P. Ciarlet Jr. T-coercivity for scalar interface problems between dielectrics and metamaterials. *Math. Model. Numer. Anal.*, 46(06):1363–1387, 2012.

[2] A.-S. Bonnet-Ben Dhia, L. Chesnel, and X. Claeys. Radiation condition for a non-smooth interface between a dielectric and a metamaterial. *Math. Models Meth. App. Sci.*, 23(09):1629–1662, 2013.

[3] A.-S. Bonnet-Ben Dhia, L. Chesnel, and H. Haddar. On the use of T-coercivity to study the Interior Transmission Eigenvalue Problem. *C. R. Acad. Sci. Paris, Ser. I*, 340(11-12):647–651, 2011.

[4] A.-S. Bonnet-Ben Dhia, P. Ciarlet Jr., and C.M. Zwölf. Time harmonic wave diffraction problems in materials with sign-shifting coefficients. *J. Comput. Appl. Math.*, 234(9):1912–1919, 2010. Corrigendum *J. Comput. Appl. Math.*, 234(9):2616, 2010.

[5] A.-S. Bonnet-Ben Dhia, M. Dauge, and K. Ramdani. Analyse spectrale et singularités d'un problème de transmission non coercif. *C. R. Acad. Sci. Paris, Ser. I*, 328(8):717–720, 1999.

[6] F. Cakoni, D. Colton, and H. Haddar. The linear sampling method for anisotropic media. *J. Comput. Appl. Math.*, 146(2):285–299, 2002.

[7] F. Cakoni, D. Colton, and H. Haddar. The computation of lower bounds for the norm of the index of refraction in an anisotropic media from far field data. *Jour. Integral Equations and Appl.*, 21(2):203–227, 2009.

[8] F. Cakoni and H. Haddar. A variational approach for the solution of the electromagnetic interior transmission problem for anisotropic media. *Inverse Problems and Imaging*, 1(3):443–456, 2007.

[9] F. Cakoni and A. Kirsch. On the interior transmission eigenvalue problem. *Int. Jour. Comp. Sci. Math.*, 3(1-2):142–167, 2010.

[10] L. Chesnel. Interior transmission eigenvalue problem for Maxwell's equations: the T-coercivity as an alternative approach. *Inv. Probl.*, 28(6):065005, 2012.

[11] L. Chesnel. *Investigation of some transmission problems with sign-changing coefficients. Application to metamaterials.* PhD thesis, École Polytechnique, 2012.

[12] L. Chesnel and P. Ciarlet Jr. Compact imbeddings in electromagnetism with interfaces between classical materials and metamaterials. *SIAM J. Math. Anal.*, 43(5):2150–2169, 2011.

[13] D. Colton and P. Monk. The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium. *Quarterly Jnl. of Mechanics & App. Maths.*, 41(1):97–125, 1988.

[14] D. Colton and L. Païvärinta. Transmission eigenvalues and a problem of Hans Lewy. *J. Comput. Appl. Math.*, 117(2):91–104, 2000.

[15] D. Colton, L. Païvärinta, and J. Sylvester. The interior transmission problem. *Inverse Problems and Imaging*, 1(1):13, 2007.

[16] M. Costabel and E. Stephan. A direct boundary integral method for transmission problems. *J. Math. Anal. and Appl.*, 106(2):367–413, 1985.

[17] M. Dauge and B. Texier. Problèmes de transmission non coercifs dans des polygones. Technical Report 97-27, Université de Rennes 1, IRMAR, Campus de Beaulieu, 35042 Rennes Cedex, France, 1997. *http://hal.archives-ouvertes.fr/docs/00/56/23/29/PDF/BenjaminT_arxiv.pdf.*

[18] H. Haddar. The interior transmission problem for anisotropic Maxwell's equations and its applications to the inverse problem. *Math. Meth. Appl. Sci.*, 27(18):2111–2129, 2004.

[19] V. A. Kondratiev. Boundary-value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.*, 16:227–313, 1967.

[20] V. A. Kozlov, V. G. Maz'ya, and J. Rossmann. *Elliptic Boundary Value Problems in Domains*

with Point Singularities, volume 52 of *Mathematical Surveys and Monographs*. AMS, Providence, 1997.

[21] E. Lakshtanov and B. Vainberg. Ellipticity in the interior transmission problem in anisotropic media. *SIAM J. Math. Anal.*, 44(2):1165–1174, 2012.

[22] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications*. Dunod, 1968.

[23] V. Mazya, S.A Nazarov, and B. Plamenevskiĭ. *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains, Vol. 1, 2*. Birkhäuser, Basel, 2000.

[24] W. McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[25] S.A. Nazarov and B. A. Plamenevskiĭ. Radiation conditions for selfadjoint elliptic problems. *Dokl. Akad. Nauk SSSR*, 311(3):532–536, 1990.

[26] S.A. Nazarov and B. A. Plamenevskiĭ. *Elliptic problems in domains with piecewise smooth boundaries*, volume 13 of *Expositions in Mathematics*. De Gruyter, Berlin, Germany, 1994.

[27] S.A. Nazarov and J. Taskinen. Radiation conditions at the top of a rotational cusp in the theory of water-waves. *Math. Model. Numer. Anal.*, 45(5):947–979, 2011.

[28] P. Ola. Remarks on a transmission problem. *J. Math. Anal. Appl.*, 196:639–658, 1995.

[29] J. Peetre. Another approach to elliptic boundary problems. *Commun. Pure Appl. Math.*, 14(4):711–731, 1961.

[30] Ja. A. Roĭtberg and Z. G. Šeftel′. General boundary-value problems for elliptic equations with discontinuous coefficients. *Dokl. Akad. Nauk SSSR*, 148:1034–1037, 1963.

[31] B.P. Rynne and B.D. Sleeman. The interior transmission problem and inverse scattering from inhomogeneous media. *SIAM J. Math. Anal.*, 22(6):1755–1762, 1991.

[32] M. Schechter. A generalization of the problem of transmission. *Ann. Scuola Norm. Sup. Pisa*, 14(3):207–236, 1960.