Inverse Problems in Wave Propagation - IWaP 2015

Non-scattering wavenumbers and far field invisibility for a finite set of incident/scattering directions

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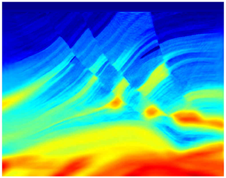
General setting

- We are interested in methods based on the propagation of waves to determine the shape, the physical properties of objects, in an exact or qualitative manner, from given measurements.

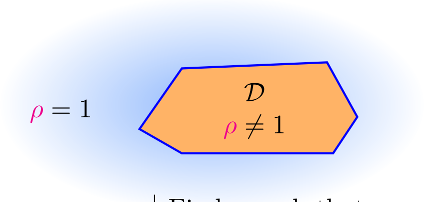
- GENERAL PRINCIPLE OF THE METHODS:
  - i) send waves in the medium;
  - ii) measure the scattered field;
  - iii) deduce information on the structure.



- Many techniques: Xray, ultrasound imaging, seismic tomography, ...
- Many applications: biomedical imaging, non destructive testing of materials, geophysics, ...

Model problem

- Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\mathbb{R}^2$.



$$\left|\begin{array}{rcll} \multicolumn{4}{l}{\text{Find } u \text{ such that}} \\ -\Delta u & = & k^2 \rho\, u & \text{in } \mathbb{R}^2, \\ u & = & u_{\mathrm{i}} + u_{\mathrm{s}} & \text{in } \mathbb{R}^2, \\ \multicolumn{4}{l}{\displaystyle\lim_{r \to +\infty} \sqrt{r} \left(\frac{\partial u_{\mathrm{s}}}{\partial r} - ik u_{\mathrm{s}} \right) = 0.} \end{array}\right. \tag{1}$$

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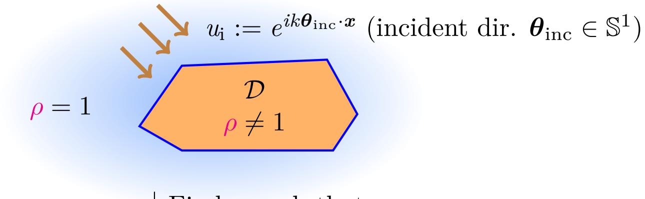

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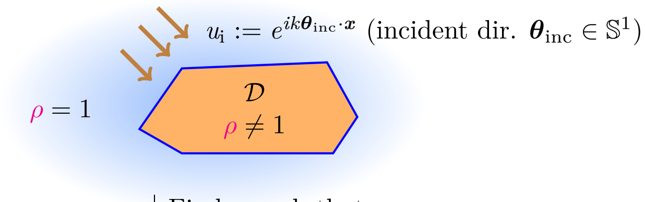


Find u such that

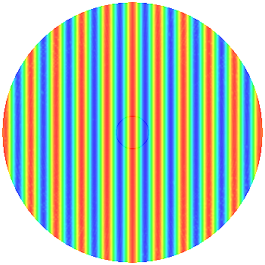
$$\begin{aligned} -\Delta u &= k^2 \rho\, u \quad &\text{in } \mathbb{R}^2, \\ u &= u_{\mathrm{i}} + u_{\mathrm{s}} \quad &\text{in } \mathbb{R}^2, \\ \lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}}\right) &= 0. \end{aligned} \tag{1}$$

DEFINITION:
- u_{i} = incident field (data)
- u = total field (uniquely defined by (1))
- u_{s} = scattered field (uniquely defined by (1)).

Illustration of the scattering of a plane wave

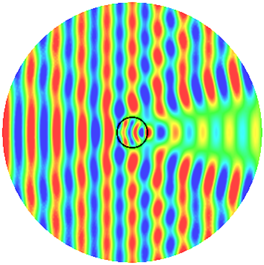
- Below, the movies represent a numerical approximation of the solution of the previous problem.

Incident field



$t \mapsto \Re e \left(e^{-i\omega t} u_{\mathrm{i}}(\boldsymbol{x}) \right)$

Total field



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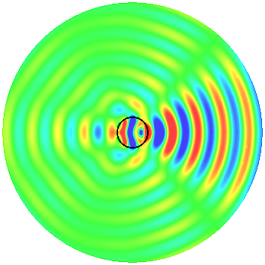
Scattered field



$t \mapsto \Re e \left(e^{-i\omega t} u_{\mathrm{s}}(\boldsymbol{x}) \right)$

- The pulsation ω is defined by $\omega = k/c$ where $c = 1$ is the celerity of the waves in the homogeneous medium.

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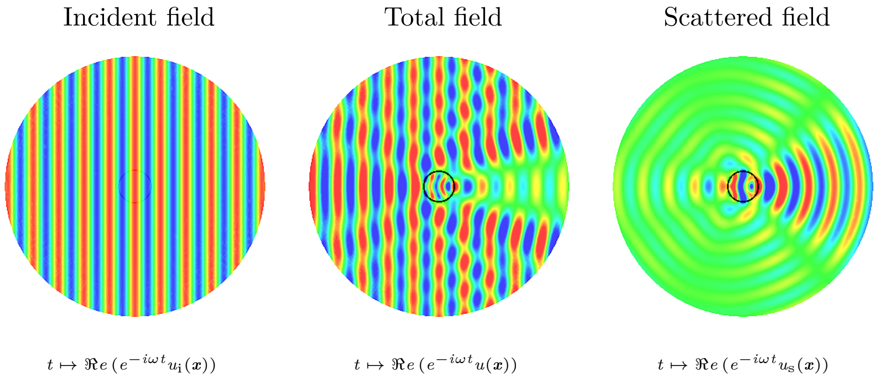

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Can we recover information on the inclusion from far field measurements?

Issue considered in this talk 1/2

We are interested in defects that cannot be detected and in invisibility.

1) Is there an incident wave which does not scatter at infinity?

2) Can it be that all incident waves do not scatter at infinity?

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- These questions have been studied when one can produce incident plane waves and measure the resulted scattered fields in all directions.

Question 1) leads to the analysis of the Interior Transmission Problem. (Cakoni, Colton, Gintides, Haddar, Hu, Kirsch, Kress, Lakshtanov, Lechleiter, Monk, Sylvester, Païvärinta, Rynne, Sleeman, Sun,...)

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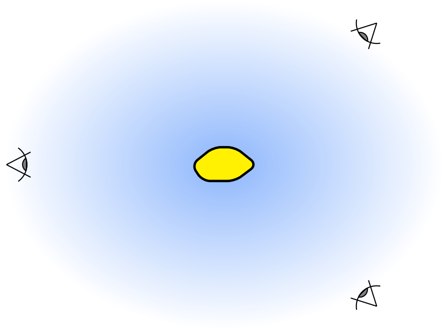
Question 2) is related to the design of cloaking devices.

► Extensions to consider the case of partial aperture (but still with a continuum of data).

Issue considered in this talk 2/2

- In practice, one has always a finite number of emitters and receivers.



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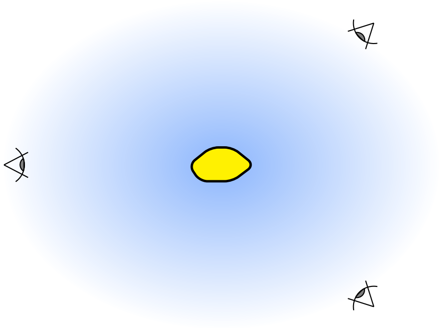

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when one can produce incident plane waves and measure the resulting scattered field only in a finite number of given directions.

Outline of the talk

1. Introduction

2. Non-scattering wavenumbers

   Is there an incident wave which does not scatter at infinity?

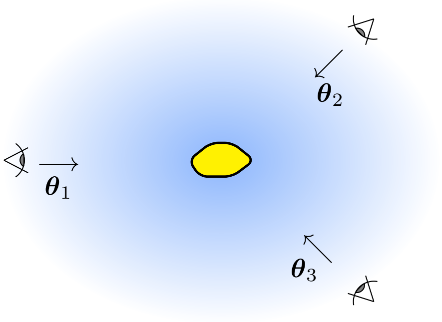
3. Invisible inclusions

   Can it be that all incident waves do not scatter at infinity?

4. Conclusion

Setting

- Let $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ be given directions of the unit circle $\mathbb{S}^1$.



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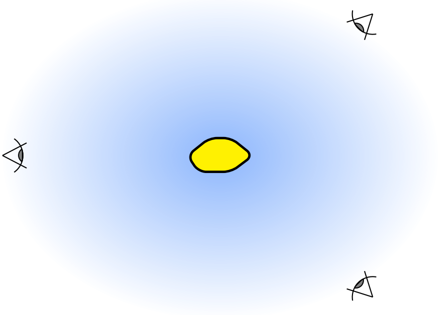

- We assume that emitters and receivers coincide:
  - We send the plane wave $e^{ik\boldsymbol{\theta}_1 \cdot \boldsymbol{x}}$ (direction $\boldsymbol{\theta}_1$) and measure the resulted scattered fields in the directions $-\boldsymbol{\theta}_1, \ldots, -\boldsymbol{\theta}_N$.

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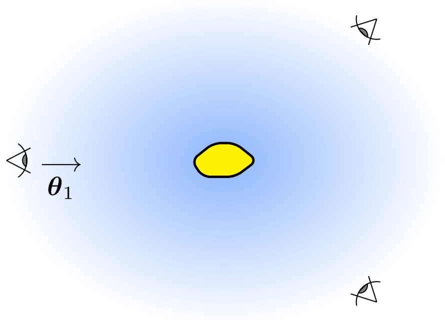

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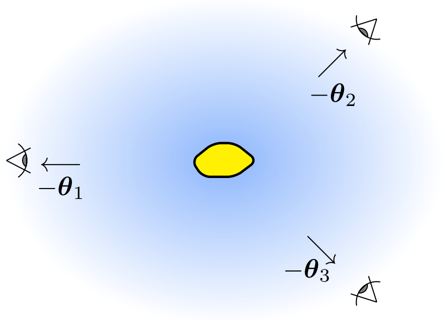

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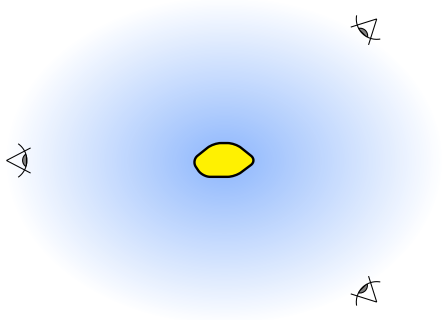

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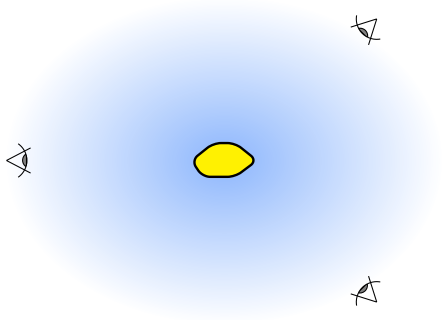

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$N \times N$ multistatic backscattering measurements

Far field pattern

- For a given incident direction $\boldsymbol{\theta}_{\mathrm{inc}}$, the scattered field $u_{\mathrm{s}}(\cdot, \boldsymbol{\theta}_{\mathrm{inc}})$ admits the asymptotic expansion

$$u_{\mathrm{s}}(\boldsymbol{x}, \boldsymbol{\theta}_{\mathrm{inc}}) = \frac{e^{ikr}}{\sqrt{r}} \left(\; u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_{\mathrm{sca}}, \boldsymbol{\theta}_{\mathrm{inc}}) \; + O(1/r) \right)$$

as $r = |\boldsymbol{x}| \to +\infty$, uniformly in $\boldsymbol{\theta}_{\mathrm{sca}} \in \mathbb{S}^1$.

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The far field pattern is the quantity one can measure at infinity (the other terms are too small).

- In practice, the goal of imaging techniques is to find features of the inclusion from the knowledge of $u_{\text{s}}^{\infty}(\cdot, \cdot)$ on a finite subset of $\mathbb{S}^1 \times \mathbb{S}^1$.

Relative scattering matrix

- For $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ given directions of $\mathbb{S}^1$, we introduce the relative scattering matrix $\mathscr{S}(k) \in \mathbb{C}^{N\times N}$ defined via

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DEFINITION. Values of $k > 0$ for which $\mathscr{S}(k)$ has a non trivial kernel are called non-scattering wavenumbers.

- For k non-scat. wavenumber, there is some $(\alpha_1, \dots, \alpha_N) \in \mathbb{C}^N \setminus \{0\}$ s.t.

$$\sum_{n=1}^{N} \alpha_n e^{ik\boldsymbol{\theta}_n \cdot \boldsymbol{x}}$$

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does not scatter at infinity in the directions $-\boldsymbol{\theta}_1, \dots, -\boldsymbol{\theta}_N$.

- Unlike in the continuous setting, the scattered field does not vanish identically at infinity.

Discreteness of non-scattering wavenumbers

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Idea of the approach:

1. We show that $k \mapsto \mathscr{S}(k)$ can be meromorphically extended to $\mathbb{C} \setminus \{0\}$.

2. For $k \in \mathbb{R}i \setminus \{0\}$, using integration by parts, we prove the energy identity

$$\boxed{c\,\overline{\alpha}^\top \mathscr{S}(k)\,\alpha = \int_{\mathbb{R}^2} |\nabla u_\mathrm{s}|^2 + |k|^2 \rho\,|u_\mathrm{s}|^2 + |k|^2 \int_{\mathcal{D}} (1-\rho)|u_\mathrm{i}|^2.}$$

where $u_\mathrm{i} = \sum_{n=1}^{N} \alpha_n e^{ik\boldsymbol{\theta}_n\cdot\boldsymbol{x}}$, $\alpha = (\alpha_1, \ldots, \alpha_N)^\top$ and $c \neq 0$ is a constant.

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PROPOSITION. Suppose that $\rho < 1$. Then the set of non-scattering wavenumbers is discrete and countable.

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TWO REMARKS:

- Unlike in the continuous setting, this problem does not reduce to a problem set on the (compact) support of the inclusion.
- Unlike in the continuous setting, the cases $A = 1$ and $A \neq 1$ (for $\operatorname{div}(A\nabla u) + k^2\rho u = 0$) do not require different functional frameworks.

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Invisible inclusions: setting

- In the previous section, for a given obstacle, we have studied the k such that $\ker \mathscr{S}(k) \neq \{0\}$ ($\mathscr{S}(k)$ is the relative scattering matrix).

- Now, we assume that k and the support of the inclusion $\overline{\mathcal{D}}$ are given.

We explain how to construct non trivial inclusions such that $\mathscr{S}(k) = 0$.

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- These inclusions cannot be detected from far field measurements.

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Formulation of the problem:

Find a real valued function $\rho \not\equiv 1$, with $\rho - 1$ supported in $\overline{\mathcal{D}}$, such that the solution of the problem

$$\left|\begin{array}{l} \text{Find } u = u_{\text{s}} + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u \quad = \quad k^2 \rho\, u \quad \text{ in } \mathbb{R}^2, \\ \displaystyle\lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u_{\text{s}}}{\partial r} - iku_{\text{s}}\right) = 0 \end{array}\right.$$

verifies $u_{\text{s}}^{\infty}(\boldsymbol{\theta}_1) = \dots = u_{\text{s}}^{\infty}(\boldsymbol{\theta}_N) = 0$.

Invisible inclusions: setting

► In the previous section, for a given obstacle, we have studied the k such that $\ker \mathscr{S}(k) \neq \{0\}$ ($\mathscr{S}(k)$ is the relative scattering matrix).

► Now, we assume that k and the support of the inclusion $\overline{\mathcal{D}}$ are given.

We explain how to construct non trivial inclusions such that $\mathscr{S}(k) = 0$.

► To simplify the presentation, assume that there is only one incident direction $\boldsymbol{\theta}_{\mathrm{inc}}$. Let $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ be given scattering directions.

Origin of the method:

• The idea we will use has been introduced in `Nazarov 11` to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.

• It has been adapted in `Bonnet-Ben Dhia & Nazarov 13` to build invisible perturbations of waveguides (see also `Bonnet-Ben Dhia, Nazarov & Taskinen 14` for an application to a water-wave problem).

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector

$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

(N complex measurements $\Rightarrow$ $2N$ real measurements)

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$
- No obstacle leads to null measurements $\Rightarrow$ $F(0) = 0$.

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector

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Our goal: find $\sigma \in \mathrm{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0$ (with $\sigma \not\equiv 0$).

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- We look for small perturbations of the reference medium: $\sigma = \varepsilon\mu$ where $\varepsilon > 0$ is a small parameter and where μ has be to determined.

Sketch of the method

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- Taylor: $F(\varepsilon\mu) = F(0) + \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$

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Assume that there are $\mu_0, \mu_1, \dots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D})$ such that $dF(0)(\mu_0) = 0$, $[dF(0)(\mu_1), \dots, dF(0)(\mu_{2N})] = Id_{2N}$.

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- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the τ_n are real parameters to set:

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where $\vec{\tau} = (\tau_1, \ldots, \tau_{2N})^\top$ and $G^\varepsilon(\vec{\tau}) = -\varepsilon \tilde{F}^\varepsilon(\mu)$.

If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.

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where $\vec{\tau} = (\tau_1, \ldots, \tau_{2N})^\top$ and $G^\varepsilon(\vec{\tau}) = -\varepsilon \tilde{F}^\varepsilon(\mu)$.

If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$. Set $\sigma^{\mathrm{sol}} := \varepsilon\mu^{\mathrm{sol}}$. We have $F(\sigma^{\mathrm{sol}}) = 0$ (existence of an invisible inclusion).

Application: step 1

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re e\, u_\mathrm{s}^\infty(\boldsymbol{\theta}_1), \ldots, \Re e\, u_\mathrm{s}^\infty(\boldsymbol{\theta}_N), \Im m\, u_\mathrm{s}^\infty(\boldsymbol{\theta}_1), \ldots, \Im m\, u_\mathrm{s}^\infty(\boldsymbol{\theta}_N)).$$

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To compute $dF(0)(\mu)$, we take $\boxed{\rho^\varepsilon = 1 + \varepsilon\mu}$ with μ supported in $\overline{\mathcal{D}}$.

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To compute $dF(0)(\mu)$, we take $\boxed{\rho^\varepsilon = 1 + \varepsilon\mu}$ with μ supported in $\overline{\mathcal{D}}$.

- We denote u^ε, u_s^ε the functions satisfying

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u_s^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u^\varepsilon \quad = \quad k^2\rho^\varepsilon\, u^\varepsilon \qquad \text{in } \mathbb{R}^2, \\ \qquad u_s^\varepsilon \text{ outgoing.} \end{array}\right.$$

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- As $r \to +\infty$, we have $u_{\mathrm{s}}^{\varepsilon}(\boldsymbol{x}) = \dfrac{e^{ikr}}{\sqrt{r}}\Big(u_{\mathrm{s}}^{\varepsilon\,\infty}(\boldsymbol{\theta}_{\mathrm{sca}}) + O(1/r)\Big)$

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with $u_{\mathrm{s}}^{\varepsilon\,\infty}(\boldsymbol{\theta}_{\mathrm{sca}}) = c\, k^2 \displaystyle\int_{\mathcal{D}} (\rho^{\varepsilon} - 1)\,(u_{\mathrm{i}} + u_{\mathrm{s}}^{\varepsilon})\, e^{-ik\boldsymbol{\theta}_{\mathrm{sca}}\cdot\boldsymbol{x}}\, d\boldsymbol{x}$ $\qquad \left(c = \dfrac{e^{i\pi/4}}{\sqrt{8\pi k}}\right)$.

Application: step 1

- For our problem, we have $(\sigma = \rho - 1)$

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with $u_s^{\varepsilon\,\infty}(\boldsymbol{\theta}_{\text{sca}}) = c\,k^2\displaystyle\int_{\mathcal{D}}(\rho^\varepsilon - 1)\,(u_i + u_s^\varepsilon)\,e^{-ik\boldsymbol{\theta}_{\text{sca}}\cdot\boldsymbol{x}}\,d\boldsymbol{x}$.

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- For our problem, we have ($\sigma = \rho - 1$)

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- We can prove that $u_s^\varepsilon = O(\varepsilon)$.

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- We can prove that $u_{\mathrm{s}}^{\varepsilon} = O(\varepsilon)$.

Application: step 1

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$$\left|\begin{array}{l} \text{Find } u^{\varepsilon} = u_{\mathrm{s}}^{\varepsilon} + e^{ik\boldsymbol{\theta}_{\mathrm{inc}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u^{\varepsilon} \quad = \quad k^2 \rho^{\varepsilon}\, u^{\varepsilon} \quad \text{ in } \mathbb{R}^2, \\ \qquad u_{\mathrm{s}}^{\varepsilon} \text{ outgoing.} \end{array}\right.$$

- With this choice, we obtain the expansion (Born approx.), for small ε

$$u_{\mathrm{s}}^{\varepsilon\,\infty}(\boldsymbol{\theta}_{\mathrm{sca}}) = 0 + \varepsilon\, c\, k^2 \int_{\mathcal{D}} \mu\, e^{ik(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_{\mathrm{sca}})\cdot\boldsymbol{x}}\, d\boldsymbol{x} + O(\varepsilon^2).$$

Application: step 1

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re e\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Re e\, u_s^\infty(\boldsymbol{\theta}_N), \Im m\, u_s^\infty(\boldsymbol{\theta}_1), \dots, \Im m\, u_s^\infty(\boldsymbol{\theta}_N)).$$

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- It is easy to find functions $\mu_0 \in \ker dF(0)$ (i.e., s.t. $\boxed{u_s^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = O(\varepsilon^2)}$ for $n = 1, \dots, N$). But we want $\boxed{u_s^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = 0}$...

Application: step 2

- In the expression $\rho^\varepsilon = 1 + \varepsilon\mu$, we redecompose μ as

$$\boxed{\mu = \mu_0 + \sum_{m=1}^{N} \tau_{1,m}\, \mu_{1,m} + \sum_{m=1}^{N} \tau_{2,m}\, \mu_{2,m}}$$

where | $\tau_{1,m}$, $\tau_{2,m}$ are real parameters that we will tune

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$$\begin{aligned}
\int_{\mathcal{D}} \mu_0 \cos(k(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_n)\cdot \boldsymbol{x})\, d\boldsymbol{x} &= 0, & \int_{\mathcal{D}} \mu_0 \sin(k(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_n)\cdot \boldsymbol{x})\, d\boldsymbol{x} &= 0 \\
\int_{\mathcal{D}} \mu_{1,m} \cos(k(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_n)\cdot \boldsymbol{x})\, d\boldsymbol{x} &= \delta^{mn}, & \int_{\mathcal{D}} \mu_{1,m} \sin(k(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_n)\cdot \boldsymbol{x})\, d\boldsymbol{x} &= 0 \\
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Assume that there are $\mu_0, \mu_1, \ldots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D})$ such that $dF(0)(\mu_0) = 0$, $[dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = Id_{2N}$.

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We introduce $2N$ real parameters because we want to cancel N complex coefficients.

Application: step 3

- With this decomposition, we obtain

$$u_{\mathrm{s}}^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = \ \varepsilon\, c\, k^2\, (\tau_{1,n} + i\tau_{2,n}) \ + \varepsilon^2 c\, k^2\, (\tilde{F}^{\varepsilon}_{1,n}(\vec{\tau}) + i\tilde{F}^{\varepsilon}_{2,n}(\vec{\tau})),$$

where $\tilde{F}^{\varepsilon}_{1,n}$, $\tilde{F}^{\varepsilon}_{2,n}$ are real-valued functions depending (non linearly) on ε, $\vec{\tau} := (\tau_{1,1}, \ldots, \tau_{1,N}, \tau_{2,1}, \ldots, \tau_{2,N})^{\top}$.

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Use the term at order ε whose dependence with respect to ρ is simple to control and cancel the whole expansion.

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- Now, we can impose $\boxed{u_{\rm s}^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = 0}$ solving the fixed point problem:

$$\boxed{\text{Find } \vec{\tau} \in \mathbb{R}^{2N} \text{ such that } \vec{\tau} = G^{\varepsilon}(\vec{\tau}),} \tag{2}$$

with $G^{\varepsilon}(\vec{\tau}) := -\varepsilon\,(\tilde{F}^{\varepsilon}_{1,1}(\vec{\tau}), \ldots, \tilde{F}^{\varepsilon}_{1,N}(\vec{\tau}), \tilde{F}^{\varepsilon}_{2,1}(\vec{\tau}), \ldots, \tilde{F}^{\varepsilon}_{2,N}(\vec{\tau}))^{\top}$.

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- We can prove that the map $G^{\varepsilon} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ verifies the estimate $|G^{\varepsilon}(\vec{\tau}) - G^{\varepsilon}(\vec{\tau}')| \leq C\, \varepsilon\, |\vec{\tau} - \vec{\tau}'|$. Therefore G^{ε} is a contraction for ε small enough and (2) has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.

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- With this decomposition, we obtain

Proposition: For ε small enough, define $\rho^{\text{sol}} = 1 + \varepsilon\mu^{\text{sol}}$ with

$$\mu^{\text{sol}} = \mu_0 + \sum_{m=1}^{N} \tau_{1,m}^{\text{sol}}\,\mu_{1,m} + \sum_{m=1}^{N} \tau_{2,m}^{\text{sol}}\,\mu_{2,m}.$$

Then the solution of the scattering problem

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u_{\text{s}}^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u \quad = \quad k^2\rho^{\text{sol}}\,u \quad \text{ in } \mathbb{R}^2, \\ \displaystyle\lim_{r\to+\infty}\sqrt{r}\left(\frac{\partial u_{\text{s}}}{\partial r} - iku_{\text{s}}\right) = 0 \end{array}\right.$$

verifies $u_{\text{s}}^\infty(\boldsymbol{\theta}_1) = \cdots = u_{\text{s}}^\infty(\boldsymbol{\theta}_N) = 0$.

- We can prove that the map $G^\varepsilon : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ verifies the estimate $|G^\varepsilon(\tau) - G^\varepsilon(\tau')| \le C\varepsilon|\tau - \tau'|$. Therefore G^ε is a contraction for ε small enough and (2) has a unique solution τ^{sol}.

How to build the shape functions?

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$$\int_{\mathcal{D}} \mu_{1,m} \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_n) \cdot \boldsymbol{x}) \, d\boldsymbol{x} = \delta^{mn}, \quad \int_{\mathcal{D}} \mu_{1,m} \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_n) \cdot \boldsymbol{x}) \, d\boldsymbol{x} = 0$$

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On $\mathcal{D}$, $\{e_n\}_{n=1}^{2N}$ is linearly independent $\Rightarrow$ the matrix $\mathbb{B} \in \mathbb{R}^{2N \times 2N}$ s.t.

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where $\mu_0^{\#} \notin \mathrm{span}\{\mu_{1,1}, \ldots, \mu_{1,N}, \mu_{2,1}, \ldots, \mu_{2,N}\}$.

Remarks

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- The inclusion can be built and does not involve singular materials ($\neq$ cloaking techniques). Moreover, μ^{sol} is just a small perturbation of μ_0:

$$\mu^{\mathrm{sol}} = \mu_0 + O(\varepsilon).$$

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The case $\boldsymbol{\theta}_{\text{inc}} = \boldsymbol{\theta}_{\text{sca}}$

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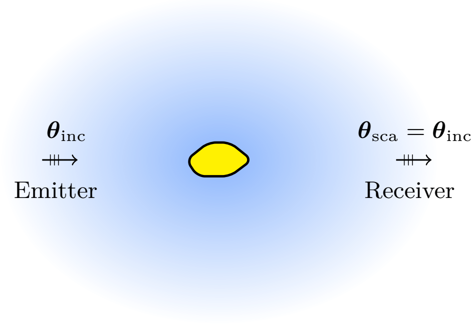

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- No solution if $\mathcal{D}$ has corners and under certain assumptions on ρ.

- Corners always scatter, E. Blåsten, L. Päivärinta, J. Sylvester, 2014
- Corners and edges always scatter, J. Elschner, G. Hu, 2015

- And if $\mathcal{D}$ is smooth? $\Rightarrow$ The problem seems open.

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Data and algorithm

- We can solve the fixed point problem using an iterative procedure: we set $\vec{\tau}^{\,0} = (0, \ldots, 0)^\top$ then define

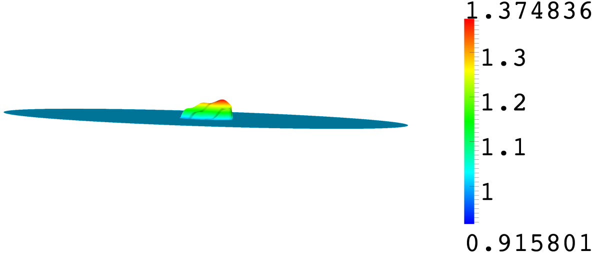
$$\boxed{\vec{\tau}^{\,n+1} = G^{\varepsilon}(\vec{\tau}^{\,n}).}$$

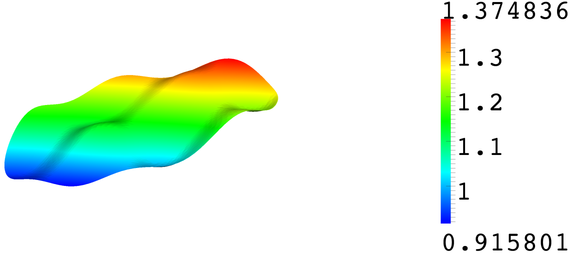
- At each step, we solve a scattering problem. We use a P2 finite element method set on the ball B_8. On $\partial\mathrm{B}_8$, a truncated Dirichlet-to-Neumann map with 13 harmonics serves as a transparent boundary condition.

- For the numerical experiments, we take $\mathcal{D} = \mathrm{B}_1$, $M = 3$ (3 directions of observation) and

$$\left|\begin{array}{ll} \boldsymbol{\theta}_{\mathrm{inc}} = (\cos(\psi_{\mathrm{inc}}), \sin(\psi_{\mathrm{inc}})), & \psi_{\mathrm{inc}} = 0^\circ \\ \boldsymbol{\theta}_1 = (\cos(\psi_1), \sin(\psi_1)), & \psi_1 = 90^\circ \\ \boldsymbol{\theta}_2 = (\cos(\psi_2), \sin(\psi_2)), & \psi_2 = 180^\circ \\ \boldsymbol{\theta}_3 = (\cos(\psi_3), \sin(\psi_3)), & \psi_3 = 225^\circ \end{array}\right.$$

Results: coefficient ρ at the end of the process





Results: scattered field

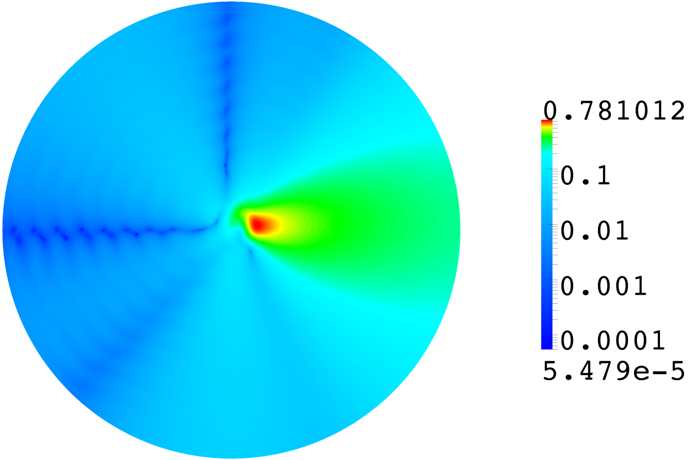


Figure: $|u_{\mathrm{s}}|$ at the end of the fixed point procedure in logarithmic scale. As desired, we see it is very small far from $\mathcal{D}$ in the directions corresponding to the angles $90°$, $180°$ and $225°$. The domain is equal to B_8.

Results: far field pattern

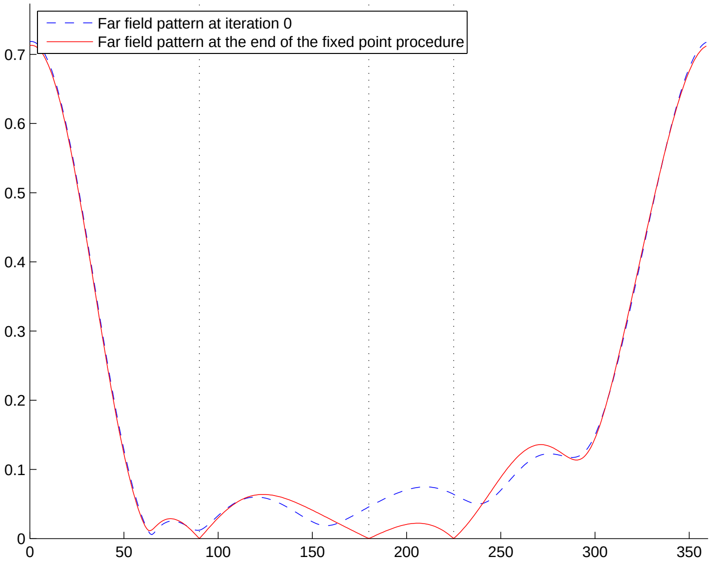


Figure: The dotted lines show the directions where we want u_s^∞ to vanish.

Application to EIT

- In Chesnel, Hyvönen & Staboulis 14, we adapted the approach to build invisible conductivities in Electrical Impedance Tomography.

✉ Goal of EIT: find perturbations of the reference conductivity from boundary measurements of current and potential.

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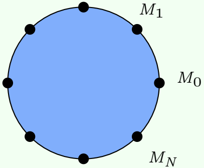

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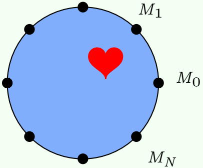

$$\left|\begin{array}{l}\mathrm{div}\,(\sigma\nabla v^n) = 0\\ \nu\cdot\sigma\nabla v^n = \delta_n - \delta_0\end{array}\right.$$

Find $\sigma \not\equiv 1$, with $\mathrm{supp}\,\sigma \subset \mathcal{D}$, s.t.

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for $m, n = 1, \ldots, N$.

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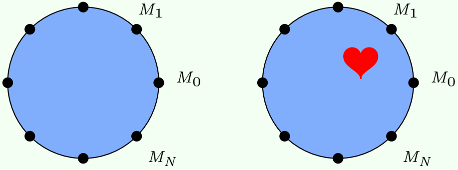

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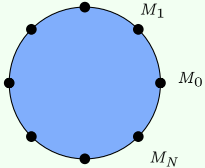

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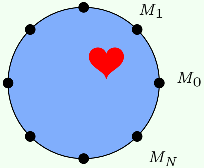

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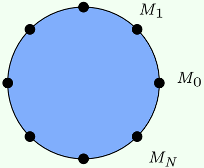

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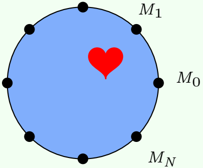

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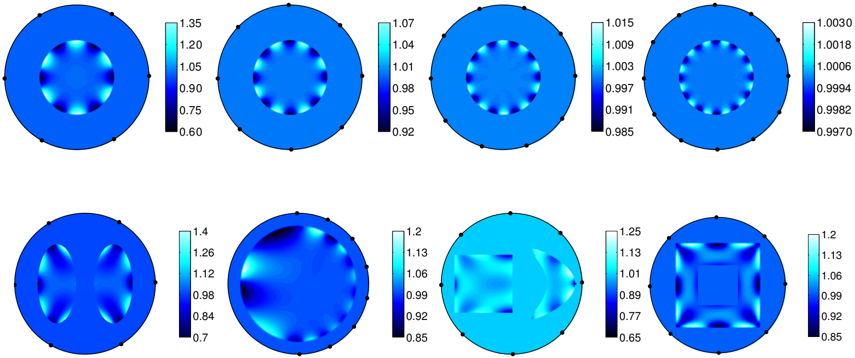
  - Open problem in 3D.

Numerical results

Examples of conductivities which provide the same measurements as the reference conductivity $\sigma = 1$.



- The dots corresponds to the positions of the electrodes.

Conclusion

Discreteness of non-scattering eigenvalues

For a given obstacle, is there an incident field that does not scatter?

- How to proceed to prove discreteness of non-scattering wavenumbers for situations other than multistatic backscattering measurements?
- Can we relax assumptions on ρ?
- Can we prove existence of non-scattering wavenumbers in this setting?
- Do non-scattering wavenumbers (if they exist) converge to the transmission eigenvalues of the continuous framework when the number of directions tends to $+\infty$?

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Invisibility

For a given frequency, how to build an invisible obstacle?

- An important issue: can we reiterate the process to construct larger defects in the reference medium? *Work in progress...*

Thank you for your attention!!!