Trapped modes in electromagnetic waveguides

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Collaboration with A.-S. Bonnet-Ben Dhia[2] and S. Fliss[2]

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CIRM, Marseille, 30/10/2025

Introduction

- For Ω a Lipschitz domain of $\mathbb{R}^d$, $d = 2, 3$, consider the **scalar** spectral pb

$$\left|\begin{array}{rcll} -\Delta u & = & \lambda u & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega. \end{array}\right.$$

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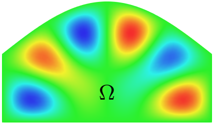

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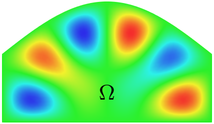
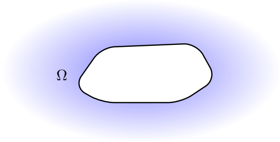


No, when Ω is the exterior of a bounded domain due to the **Rellich lemma**.



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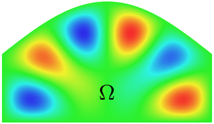

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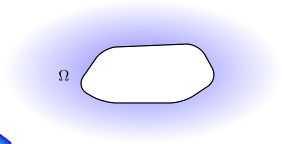

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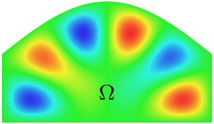

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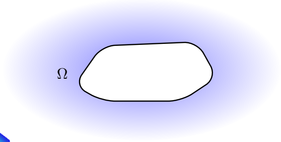

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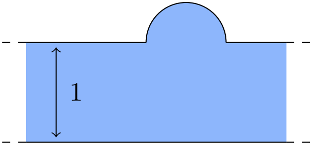
Corresponding eigenfunctions are called **trapped modes**.

Problem with Dirichlet BCs 1/2

- Assume Ω coincides with the strip $\mathbb{R} \times (0;1)$ outside of a bounded region



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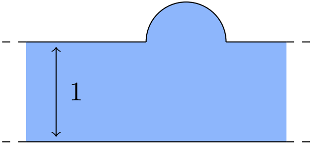

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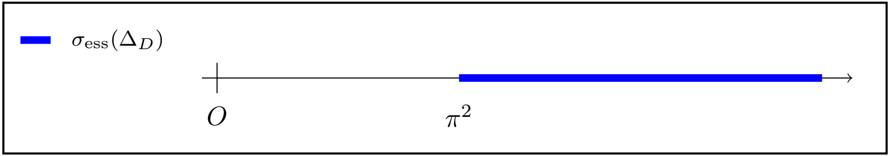

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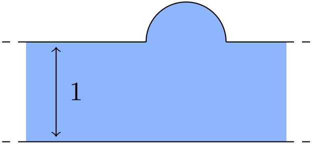

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- Depending on Ω, Δ_D may have discrete spectrum or punctual spectrum.

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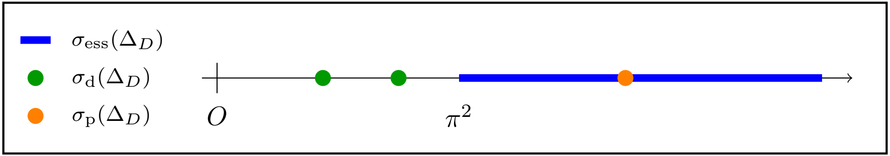

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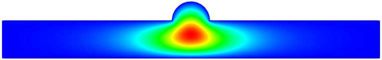
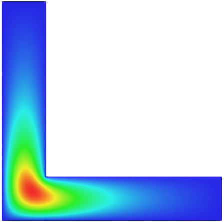
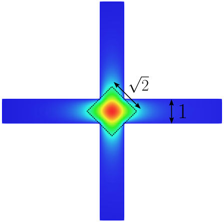
- Examples of trapped modes:

Bulgakov, Dauge, Davis, Duclos, Evans, Exner, Krejcirik, Kuznetsov, Levitin, Linton, Nazarov, Raymond, Sadreev,

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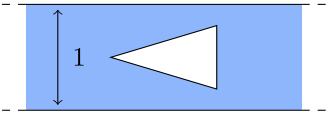

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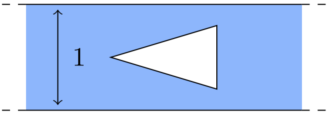

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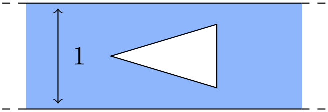

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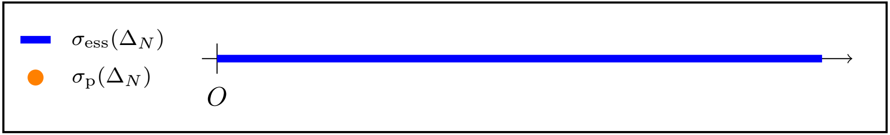

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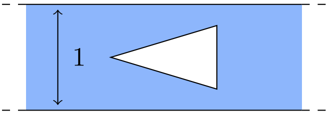

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  $\Rightarrow$ If trapped modes exist, eigenvalues are embedded in $\sigma_{\mathrm{ess}}(\Delta_N)$.

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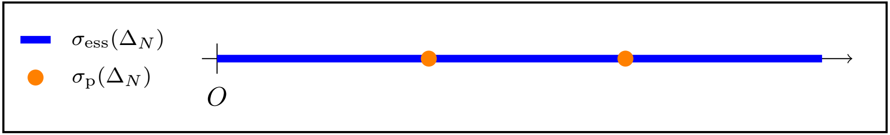

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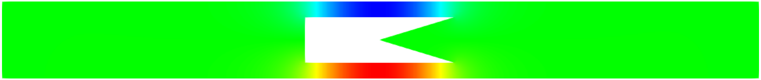

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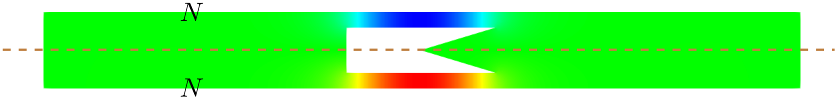

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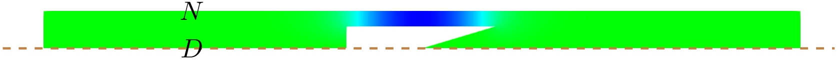

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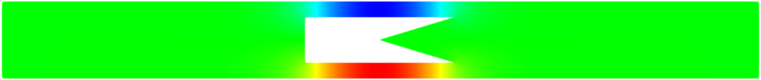

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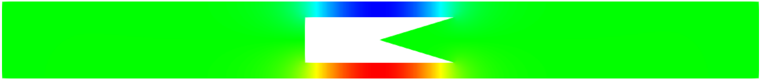

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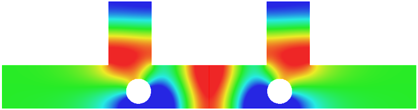
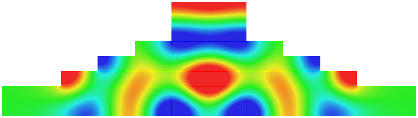
- Other examples of trapped modes:



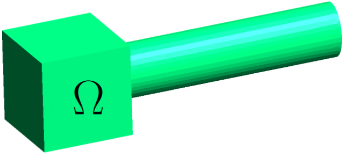


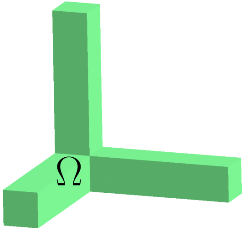
Chesnel, Evans, Koch, Kuznetsov, Levitin, Linton, McIver, Nazarov, Pagneux, Parnovski, Ursell, Vassiliev,...

- Note that symmetry is not necessary to get trapped modes.

Today

Do they exist **trapped modes** in electromagnetic waveguides?





- The connected waveguide $\Omega \subset \mathbb{R}^3$ is the union of a bounded resonator and one or several semi-infinite branches, with bounded cross-sections.
- The boundary $\partial\Omega$ is Lipschitz and we impose perfect conductor boundary conditions.
- We work with homogeneous materials ($\varepsilon = \mu \equiv 1$).

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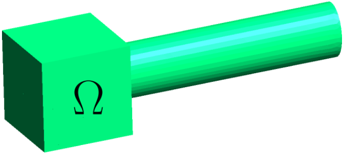

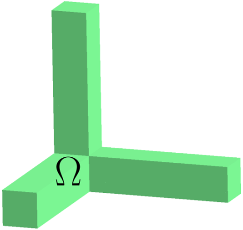

- While the literature is rich for scalar problems (acoustic, water waves, quantum mechanic, Maxwell independant of one variable)

Bonnet-Ben Dhia, Chesnel, Craster, Davies, Dauge, Duclos, Evans, Exner, Goldstone, Hein, Jaffe, Jones, Koch, Krejcirik, Kuznetsov, Levitin, Linton, Mercier, McIver, Nazarov, Pagneux, Parnovski, Raymond, Seba, Ursell, Vassiliev, Witsch,...

surprisingly, apart from the recent work Briet *et al.* 25 , almost no literature in electromagnetism.

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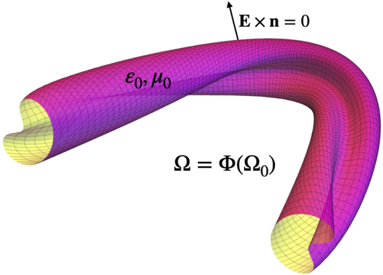
The effect of bending and twisting is studied in

P. Briet, M. Cassier, T. Ourmières-Bonafos and M. Zaccaron. Geometric spectral properties of electromagnetic waveguides. arXiv:2508.13591, 2025.



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Outline of the talk

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1. The Maxwell's operator
2. Trapped modes: complete separation of variables
3. Trapped modes: separation of variables in the resonator
4. Trapped modes: absence of separation of variables

The Maxwell's operator

- We consider the formulation for the electric field

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- Without the constraint div $\boldsymbol{E} = 0$ in Ω, the problem would have a kernel of infinite dimension containing $\{\nabla\varphi,\ \varphi \in \mathrm{H}^1_0(\Omega)\}$.

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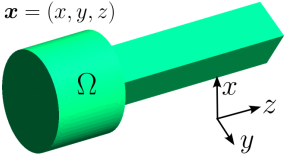
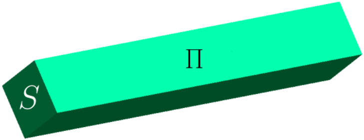
PROPOSITION. A is a positive selfadjoint operator and

$$(A\boldsymbol{E}, \boldsymbol{E}')_{\mathbf{L}^2(\Omega)} = \int_\Omega \operatorname{curl} \boldsymbol{E} \cdot \operatorname{curl} \boldsymbol{E}' \, d\boldsymbol{x}, \qquad \forall \boldsymbol{E}, \boldsymbol{E}' \in D(A).$$

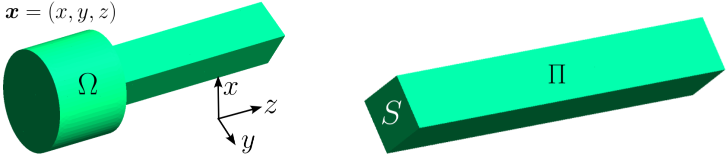
Essential spectrum





- Essential spectrum for A in Ω is due to propagating modes, *i.e.* solutions of the form $\boldsymbol{E}(\boldsymbol{x}) = \mathscr{E}(x, y)e^{i\beta z}$, with $\beta \in \mathbb{R}$, to

$$\left| \begin{array}{rcll} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} & = & \lambda \boldsymbol{E} & \text{in } \Pi \\ \operatorname{div} \boldsymbol{E} & = & 0 & \text{in } \Pi \\ \boldsymbol{E} \times \boldsymbol{\nu} & = & 0 & \text{on } \partial\Pi. \end{array} \right.$$



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1 Propagating Transverse Electric modes (TE, $E_z = 0$)

$$\boldsymbol{E}^{\mathrm{TE}}_{\pm}(\boldsymbol{x}) = \begin{pmatrix} \textbf{curl}_{\,2\mathrm{D}}\varphi_N(x,y) \\ 0 \end{pmatrix} e^{\pm i\sqrt{\lambda-\lambda_N}\,z},$$

exist for $\lambda > \lambda_N$. Here $\left|\begin{array}{l} \lambda_N \text{ is the first positive eigenvalue of } \Delta_N(S) \\ \varphi_N \text{ is a corresponding eigenfunction.} \end{array}\right.$

Essential spectrum

$\boxed{2}$ Propagating Transverse Magnetic modes (TM, $H_z = 0$)

$$\boldsymbol{E}_{\pm}^{\mathrm{TM}}(\boldsymbol{x}) = \begin{pmatrix} \nabla\varphi_D(x,y) \\ \mp i\beta_D^{-1}\lambda_D\varphi_D(x,y) \end{pmatrix} e^{\pm i\beta_D z}, \qquad \text{with } \beta_D := \sqrt{\lambda - \lambda_D},$$

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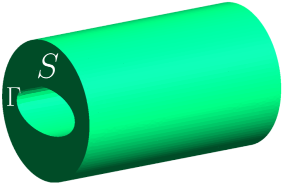

$\boxed{3}$ Propagating Transverse Electro-Magnetic modes (TEM, $E_z = H_z = 0$)

$$\boldsymbol{E}_{\pm}^{\mathrm{TEM}}(\boldsymbol{x}) = \begin{pmatrix} \nabla\varphi(x,y) \\ 0 \end{pmatrix} e^{\pm i\sqrt{\lambda}z},$$

exist for all $\lambda > 0$ iff S is not simply connected. Here $\left| \begin{array}{rcll} \Delta\varphi & = & 0 & \text{in } S \\ \varphi & = & 1 & \text{on } \Gamma \\ \varphi & = & 0 & \text{on } \partial S \setminus \Gamma. \end{array} \right.$

Essential spectrum 3/3

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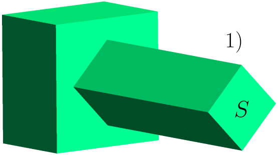

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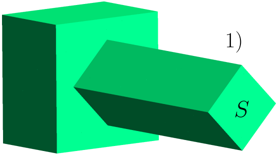

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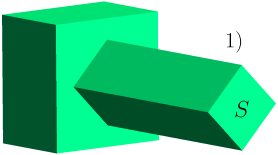

- We deduce the following statement:

THEOREM. Assume $\Omega \subset \mathbb{R}^3$ is a connected union of a bounded domain and several semi-infinite branches with the same section S.

1) If S is **simply connected**, then $\sigma_{\text{ess}}(A) = [\lambda_N; +\infty)$.

Proof. See Filonov 19 and Bogli, Ferraresso, Marletta, Tretter 23.

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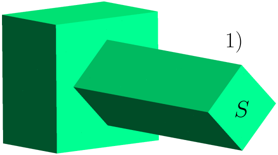

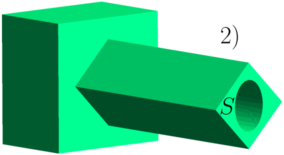

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2) If S is **not simply connected**, then $\sigma_{\mathrm{ess}}(A) = [0; +\infty)$.

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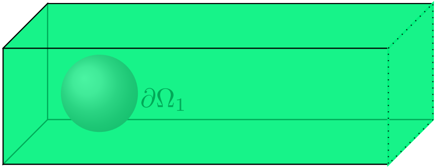
Remark

- Assume that $\partial\Omega$ is not connected and has one bounded component $\partial\Omega_1$.



- Define $\psi \in \mathrm{H}^1(\Omega)$ such that

$$\left|\begin{array}{rcll} \Delta\psi & = & 0 & \text{in } \Omega \\ \psi & = & 1 & \text{on } \partial\Omega_1 \\ \psi & = & 0 & \text{on } \partial\Omega \setminus \partial\Omega_1 \end{array}\right.$$

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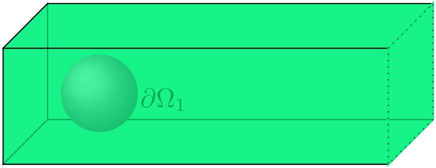

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- Then $\quad \nabla\psi \in \mathbf{X}_N(\Omega) \quad$ and $\quad \mathbf{curl}\,\mathbf{curl}\,\nabla\psi = 0.$

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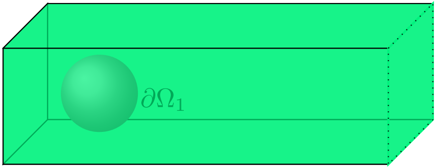

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☛ We wish to show existence of other trapped modes for A associated with positive eigenvalues.

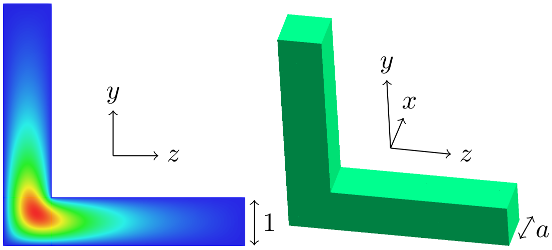
Outline of the talk

Complete separation of variables 1/2

Theorem. [Goldstone, Jaffe 92] If $\varphi \in \mathrm{H}^1_0(\Omega_{2D})$ is an eigenfunction of the Dirichlet Laplacian in Ω_{2D} associated to the eigenvalue $\lambda_\bullet$, then

$$\boldsymbol{E}(x,y,z) = \begin{pmatrix} \varphi(y,z) \\ 0 \\ 0 \end{pmatrix}$$

is an eigenfunction of A in $\Omega := (0;a) \times \Omega_{2D}$ associated to $\lambda_\bullet$ for any $a > 0$.



Example: there are trapped modes in the above 3D L-shape domain.

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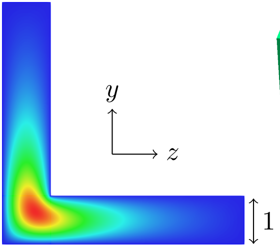

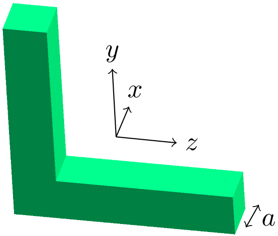

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$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} = -\boldsymbol{\Delta}\boldsymbol{E} = \lambda_\bullet \boldsymbol{E}$.
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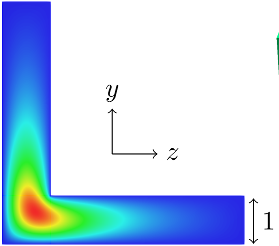

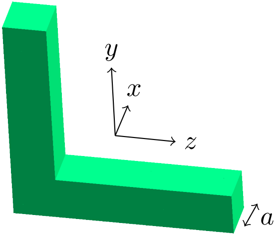

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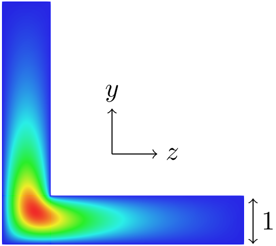

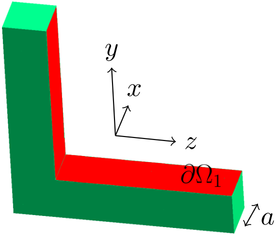

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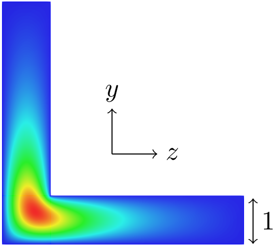

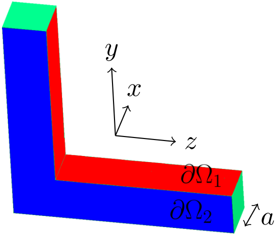

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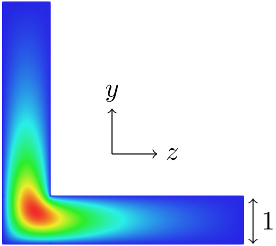

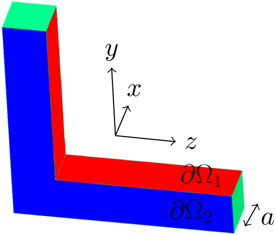

Since $\inf \sigma_{\text{ess}}(A) = \min(\pi^2, \pi^2/a^2)$, depend. on a, $\lambda_\bullet \approx 0.9293\pi^2$ may be embedded or not.

Complete separation of variables 1/2

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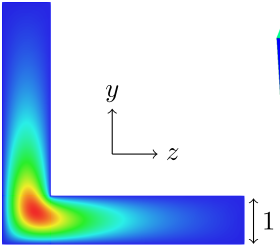

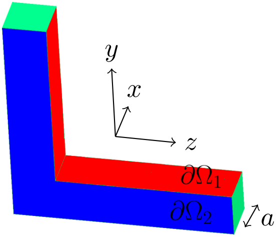

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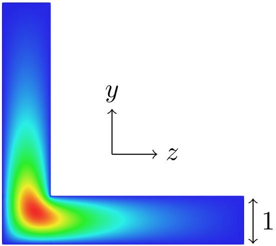

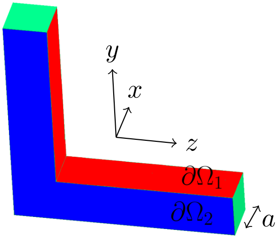

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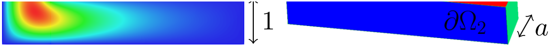
2) Thus

There is an unbounded sequence of embedded eigenvalues.

$\boldsymbol{E} = (\boldsymbol{E}\cdot\nu)\nu$ on $\partial\Omega_2$.



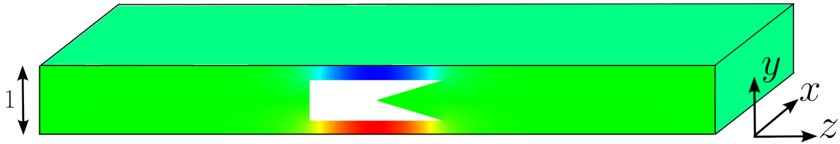
Since $\inf \sigma_{\mathrm{ess}}(A) = \min(\pi^2, \pi^2/a^2)$, $\lambda_\bullet + m^2\pi^2/a^2$ is embedded for all $m \geq 1$.

Complete separation of variables

THEOREM. If $\varphi \in \mathrm{H}^1(\Omega)$ is an eigenfunction of the Neumann Laplacian in $\Omega_{2\mathrm{D}}$ associated to the eigenvalue $\lambda_\bullet$, then for any $a > 0$, $m \in \mathbb{N}^*$,

$$\boldsymbol{E}(x, y, z) = \begin{pmatrix} 0 \\ \sin(m\pi/x)\,\mathbf{curl}_{\,2\mathrm{D}}\varphi(y, z) \end{pmatrix}$$

is an eigenfunction of A in $\Omega := (0; a) \times \Omega_{2\mathrm{D}}$ associated to $\lambda_\bullet + m^2\pi^2/a^2$.



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Complete separation of variables 2/2

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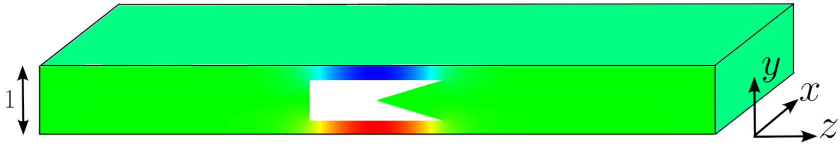


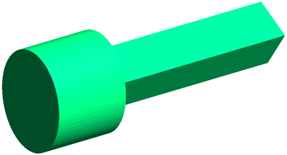
Since $\inf \sigma_{\mathrm{ess}}(A) = \min(\pi^2, \pi^2/a^2)$, $\lambda_\bullet + m^2\pi^2/a^2$ is embedded for all $m \geq 1$.

- This is similar to the results obtained in bounded domains in `Costabel, Dauge 19`.

Outline of the talk

With complete separation of variables, we were able to construct exactly eigenpairs for A. How to proceed without this assumption?

The min-max principle



- Assume that S is simply connected. Then we saw that

$$\sigma_{\text{ess}}(A) = [\lambda_N ; +\infty)$$

where λ_N is the first positive eigenvalue of the Neumann Laplacian in S.

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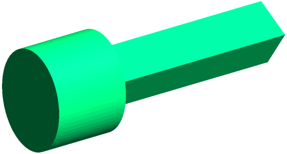

- Assume that S is simply connected. Then we saw that

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where λ_N is the first positive eigenvalue of the Neumann Laplacian in S.

According to the min-max principle, if there is $\boldsymbol{E}_p \not\equiv 0$ in

$$\mathbf{X}_N(\Omega) = \{\boldsymbol{E} \in \mathbf{L}^2(\Omega) \,|\, \mathbf{curl}\,\boldsymbol{E} \in \mathbf{L}^2(\Omega),\ \text{div}\ \boldsymbol{E} = 0 \text{ in } \Omega,\ \boldsymbol{E} \times \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}$$

such that

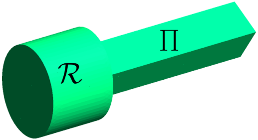
$$\frac{\displaystyle\int_\Omega |\mathbf{curl}\,\boldsymbol{E}_p|^2\, d\boldsymbol{x}}{\displaystyle\int_\Omega |\boldsymbol{E}_p|^2\, d\boldsymbol{x}} < \lambda_N,$$

then A has an eigenvalue below $\sigma_{\text{ess}}(A)$.

Building test fields

- Assume that $\quad \Omega = \mathcal{R} \cup \Pi$

where $\left| \begin{array}{l} \mathcal{R} \text{ is a bounded resonator} \\ \Pi = S \times [0; +\infty). \end{array} \right.$



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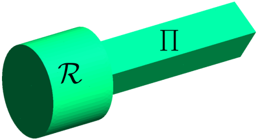

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- To construct test fields, a natural idea is to take

$$\boldsymbol{E}_p = \left| \begin{array}{ll} \boldsymbol{E}_{\mathcal{R}} & \text{in } \mathcal{R} \\ 0 & \text{in } \Pi \end{array} \right.$$

where $\boldsymbol{E}_{\mathcal{R}}$ is an eigenfunction of the resonator problem

$$\left| \begin{array}{rcll} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}_{\mathcal{R}} & = & \lambda_{\mathcal{R}} \boldsymbol{E}_{\mathcal{R}} & \text{in } \mathcal{R} \\ \boldsymbol{E}_{\mathcal{R}} \times \nu & = & 0 & \text{on } \partial\mathcal{R}. \end{array} \right.$$

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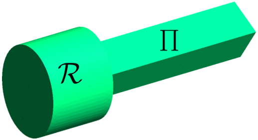

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- Then we would obtain

$$\frac{\displaystyle\int_{\Omega} |\textbf{curl}\,\boldsymbol{E}_p|^2\,d\boldsymbol{x}}{\displaystyle\int_{\Omega} |\boldsymbol{E}_p|^2\,d\boldsymbol{x}} = \frac{\displaystyle\int_{\mathcal{R}} |\textbf{curl}\,\boldsymbol{E}_{\mathcal{R}}|^2\,d\boldsymbol{x}}{\displaystyle\int_{\mathcal{R}} |\boldsymbol{E}_{\mathcal{R}}|^2\,d\boldsymbol{x}} = \lambda_{\mathcal{R}},$$

and if $\lambda_{\mathcal{R}} < \lambda_N$, this would prove that A has an eigenvalue below $\sigma_{\text{ess}}(A)$.

Building test fields

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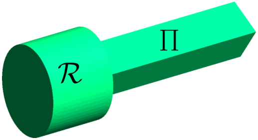

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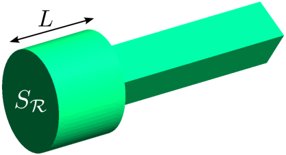
- We have $\mathbf{curl}\,\boldsymbol{E}_p \in \mathbf{L}^2(\Omega)$ but to get $\mathrm{div}\,\boldsymbol{E}_p = 0$ in Ω, we must have

$$\boldsymbol{E}_p \cdot \nu = 0 \quad \text{on } \partial\mathcal{R} \cap \partial\Pi,$$

which does not hold in general...

- Assume that $\mathcal{R} = S_{\mathcal{R}} \times (-L;0)$.



- Using TE modes in $\mathcal{R}$, we can create the eigenfunction

$$\boldsymbol{E}_{\mathcal{R}}(\boldsymbol{x}) = \begin{pmatrix} \mathbf{curl}_{\,2\mathrm{D}}\varphi_N(x,y) \\ 0 \end{pmatrix} \sin(\pi z/L)$$

such that

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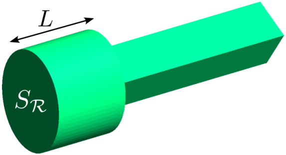

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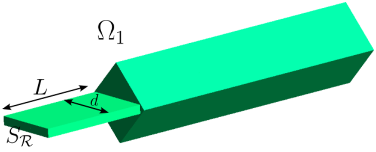
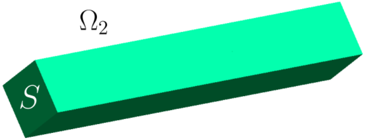
- From the previous analysis, we obtain the following statement:

THEOREM. For $\mathcal{R} = S_{\mathcal{R}} \times (-L; 0)$, there are trapped modes as soon as

$$\lambda_N(S_{\mathcal{R}}) + \frac{\pi^2}{L^2} < \lambda_N(S).$$

- This can be used to show the absence of monotonicity of the spectrum of A wrt to the geometry:





Since $\lambda_N(S_{\mathcal{R}}) = \pi^2/d^2 < \pi^2$, one has $\sigma_{\mathrm{d}}(A) \neq \emptyset$ for L large enough.

$\lambda_N(S) = \pi^2$ and $\sigma_{\mathrm{d}}(A) = \emptyset$

Though $\Omega_1 \subset \Omega_2$, we have $\inf \sigma(A(\Omega_1)) < \inf \sigma(A(\Omega_2))$.

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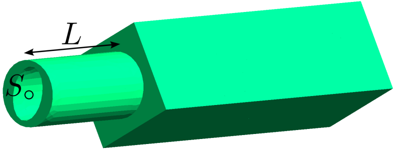
Resonators with separation of variables 2/2

- Assume now that $\mathcal{R} = S_\circ \times (-L; 0)$ where $S_\circ$ is not simply connected.



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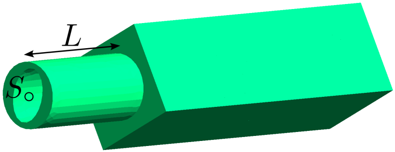

- Using the TEM modes in $\mathcal{R}$, we can create the eigenfunction

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such that
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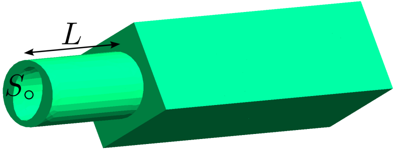

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Theorem. For $\mathcal{R} = S_\circ \times (-L;0)$, there are trapped modes as soon as
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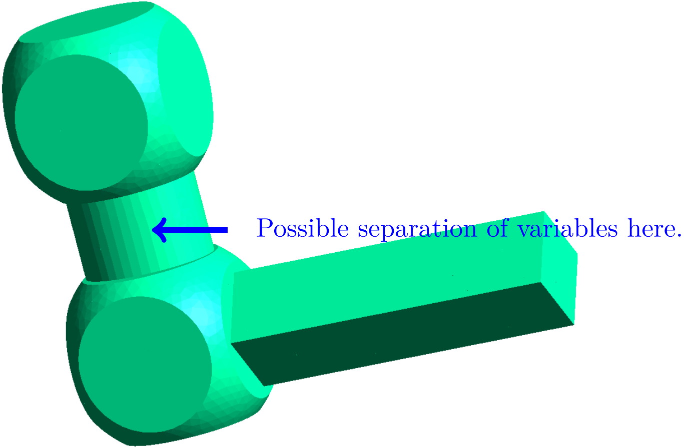
Resonators with separation of variables 2/2

Remark. Since we use extension by zero, it is sufficient to have separation of variables only in a part of the resonator.



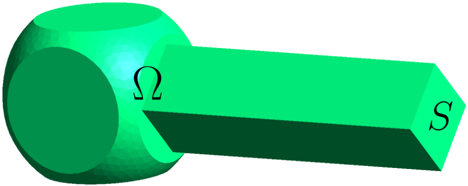
Outline of the talk

Resonators large enough

- Assume that $S \subset \mathbb{R}^2$ is simply connected so that $\sigma_{\text{ess}}(A) = [\lambda_N(S); +\infty)$.

Denote by $\Delta_D(\Omega)$ the Dirichlet Laplacian in $\Omega \subset \mathbb{R}^3$.

For a resonator large enough, $\Delta_D(\Omega)$ has a non-empty discrete spectrum.



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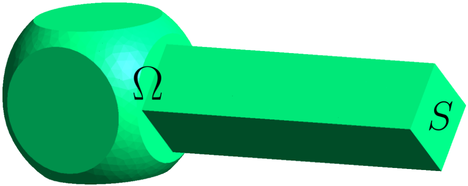

We wish to prove a similar result for A by adapting `Rohleder 25`:

THEOREM. Let $\Omega \subset \mathbb{R}^3$ be a bounded connected Lipschitz domain. The Maxwell's operator has at least two eigenvalues strictly less than $\lambda_D^1(\Omega)$.

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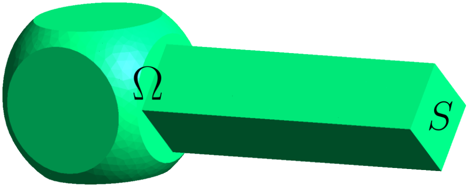

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Let us compare the eigenvalues of $\Delta_D(\Omega)$ and A.

Resonators large enough

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Denote by $\Delta_D(\Omega)$ the Dirichlet Laplacian in $\Omega \subset \mathbb{R}^3$.

THEOREM. Assume that Ω is such that $\Delta_D(\Omega)$ has an eigenvalue below $\lambda_N(S)$. Then A has an eigenvalue below $\lambda_N(S)$.

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PROOF. Let $\lambda_D(\Omega)$ denote the smallest eigenvalue of $\Delta_D(\Omega)$ and $\Phi \in \mathrm{H}^1_0(\Omega)$ be an associated eigenfunction.

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Theorem. Assume that Ω is such that $\Delta_D(\Omega)$ has an eigenvalue below $\lambda_N(S)$. Then A has an eigenvalue below $\lambda_N(S)$.

Proof. Let $\lambda_D(\Omega)$ denote the smallest eigenvalue of $\Delta_D(\Omega)$ and $\Phi \in \mathrm{H}^1_0(\Omega)$ be an associated eigenfunction. Set

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Resonators large enough

- Assume that $S \subset \mathbb{R}^2$ is simply connected so that $\sigma_{\text{ess}}(A) = [\lambda_N(S); +\infty)$.

Denote by $\Delta_D(\Omega)$ the Dirichlet Laplacian in $\Omega \subset \mathbb{R}^3$.

THEOREM. Assume that Ω is such that $\Delta_D(\Omega)$ has an eigenvalue below $\lambda_N(S)$. Then A has an eigenvalue below $\lambda_N(S)$.

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Finally the min-max principle ensures that A has an eigenvalue below $\lambda_N(S)$.□

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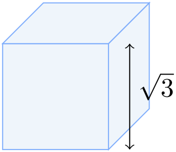
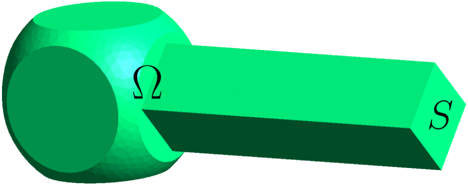
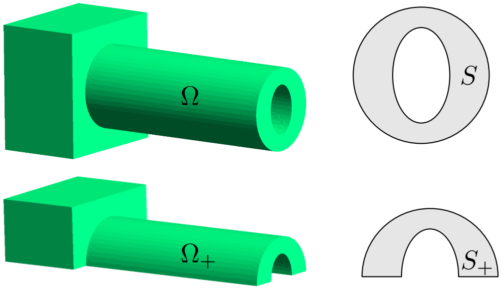
Application. Assume that $S = (0; 1)^2$. Then trapped modes exist for A as soon as Ω contains a cube of side $\sqrt{3}$.





From discrete to embedded eigenvalues

- Maxwell's equations offer an original way of playing with symmetries and topology to exhibit embedded eigenvalues.



$\sigma_{\mathrm{ess}}(A(\Omega)) = [0; +\infty)$

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Proposition. Discrete eigenvalues for $A(\Omega_+)$ are embedded for $A(\Omega)$.

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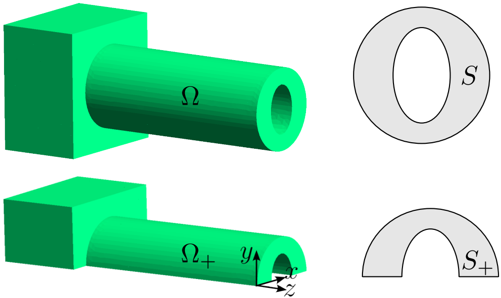

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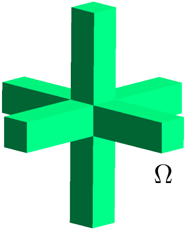
PROPOSITION. Discrete eigenvalues for $A(\Omega_+)$ are embedded for $A(\Omega)$.

PROOF. If $\boldsymbol{E}^+ \in \mathbf{X}_N(\Omega_+)$ is an eigenfunction of $A(\Omega_+)$, define $\boldsymbol{E}$ such that

$$\boldsymbol{E} = \boldsymbol{E}^+ \text{ in } \Omega_+, \qquad \boldsymbol{E}(x,y,z) = \left| \begin{array}{r} -\boldsymbol{E}_x^+(x,-y,z) \\ \boldsymbol{E}_y^+(x,-y,z) \\ -\boldsymbol{E}_z^+(x,-y,z) \end{array} \right. \text{ in } \Omega \setminus \Omega_+.$$

Then $\boldsymbol{E} \in \mathbf{X}_N(\Omega)$ is an eigenfunction of $A(\Omega)$. $\square$

Does this Ω support trapped modes?



- The section S of the branches of Ω is a square of size 1 so that

$$\sigma_{\text{ess}}(A) = [\pi^2; +\infty).$$

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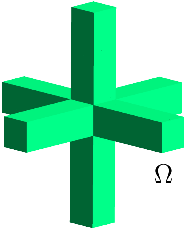

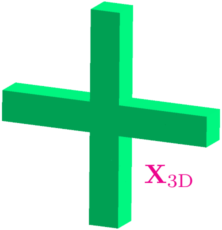

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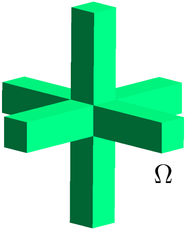
- The above results ensures that A in $\mathbf{X}_{3\mathrm{D}}$ admits a trapped mode with

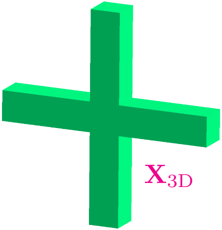
$$\lambda_{\bullet} \approx 0.6605\pi^2 \qquad \text{and} \qquad \boldsymbol{E}(x,y,z) = \begin{pmatrix} \varphi(y,z) \\ 0 \\ 0 \end{pmatrix},$$

The 6 legs animal $1/2$

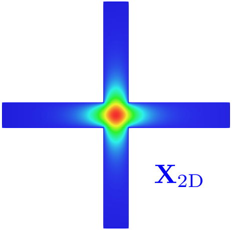
Does this Ω support trapped modes ❓



Ω



$\mathbf{X}_{3\mathrm{D}}$



$\mathbf{X}_{2\mathrm{D}}$

- The section S of the branches of Ω is a square of size 1 so that

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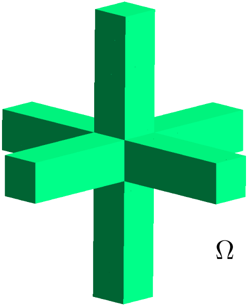
where $(\lambda_\bullet, \varphi)$ is a trapped mode of the Dirichlet Laplacian in $\mathbf{X}_{2\mathrm{D}}$.

THEOREM. The 6 legs geometry supports trapped modes.

PROOF.

- Set $\tilde{\boldsymbol{E}} := \left| \begin{array}{ll} \boldsymbol{E}_p & \text{in } \mathbf{X}_{3\text{D}} \\ 0 & \text{elsewhere.} \end{array} \right.$



Theorem. The 6 legs geometry supports trapped modes.

Proof.

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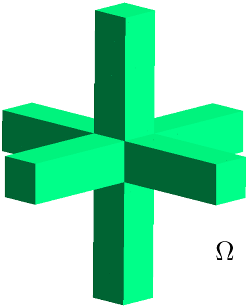

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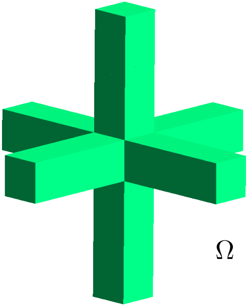

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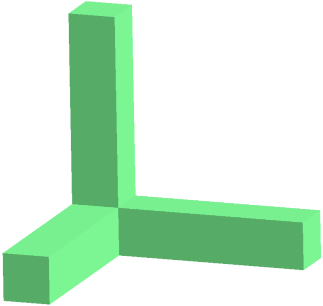
- With sharp estimates, one establishes

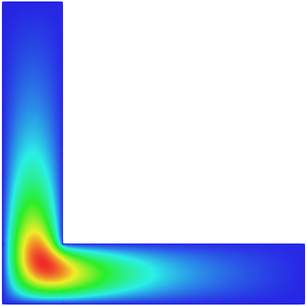
$$\int_\Omega |\mathbf{curl}\,\boldsymbol{E}|^2\,d\boldsymbol{x} < \pi^2 \int_\Omega |\boldsymbol{E}|^2\,d\boldsymbol{x}.$$

Finally, we conclude with the min-max principle. □

The 3 legs animal

Theorem. The 3 legs geometry supports trapped modes.





- The proof uses the trapped mode of the Dirichlet Laplacian in the 2D L-shaped domain.
- Estimates are surprisingly more difficult than for the 6 legs geometry....

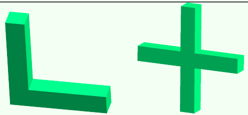
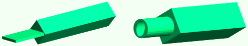
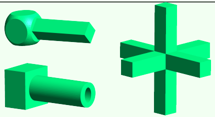
Outline of the talk

1. The Maxwell's operator
2. Trapped modes: complete separation of variables
3. Trapped modes: separation of variables in the resonator
4. Trapped modes: absence of separation of variables

Conclusion

What we did

- We presented examples of waveguides where the Maxwell's operator has eigenvalues.

		
Complete separation of variables	Separation of variables in $\mathcal{R}$	No separation of variables
Exact eigenv. of the scalar pb.	Min-max principle + *ad hoc* test field	

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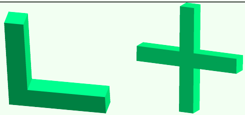	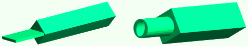	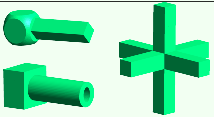
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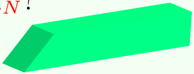
Future work

1) Below an eigenvalue of Δ_D, there is an eigenvalue of A. Is there anything to do with an (embedded) eigenvalue of Δ_N?

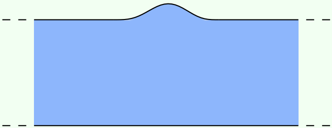
2) Can one show absence of eigenvalues in certain Ω?



Conclusion

Future work

3) In geometries with exterior bumps, Δ_D has discrete spectrum.



Can one prove an equivalent for A with the Piola transform?
Can one exploit the results we have concerning variable ε, μ?

Theorem. If S is simply connected, $\Omega = S \times \mathbb{R}$, $\varepsilon, \mu \geq 1$ with $\varepsilon > 1$ or $\mu > 1$ in a non-empty set, then $\sigma_{\mathrm{d}}(A^{\varepsilon,\mu}) \neq \emptyset$.

$\rightarrow$ post-doc of Michele Zaccaron.

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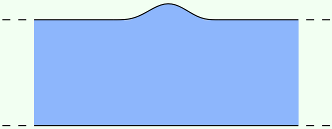

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4) Can one exploit embedded eigenvalues via the Fano resonance mechanism to achieve **invisibility** (zero reflection, perfect transmission,...)?

The Fano resonance phenomenon

- Generically, slight perturbations of a geometry supporting embedded eigenvalues give rise to complex resonances with small imaginary parts.

- Close to the complex resonances, one observes versatile scattering phenomena (the Fano resonance), which can be used to reach zero reflection.

The Fano resonance phenomenon

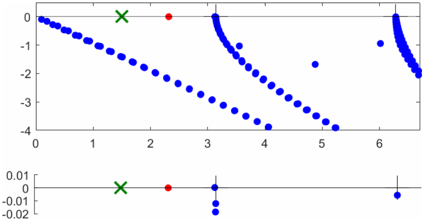
Sym. geom. | Slightly non sym. geom.

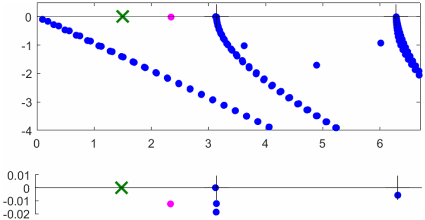
- $\omega \mapsto \Re e\, u(\omega)$ with $\omega = \sqrt{\lambda}$.

- **Complex spectrum** computed with PMLs (we zoom at the real axis).

• Trapped mode

• Complex resonance





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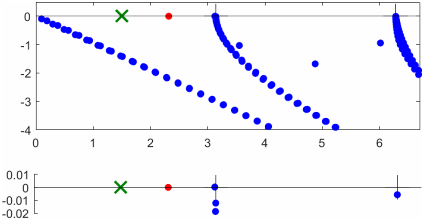

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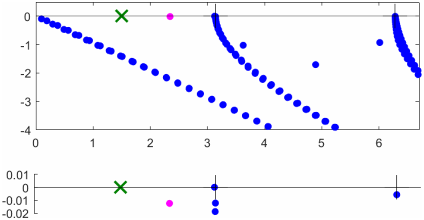

Thank you for your attention!

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