Exact zero transmission during the Fano resonance phenomenon in non symmetric waveguides

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Abstract

We investigate a time-harmonic wave problem in a waveguide. We work at low frequency so that only one mode can propagate. It is known that the scattering matrix exhibits a rapid variation for real frequencies in a vicinity of a complex resonance located close to the real axis. This is the so-called Fano resonance phenomenon. And when the geometry presents certain properties of symmetry, there are two different real frequencies such that $R = 0$ or $T = 0$, where R, T denote the reflection and transmission coefficients. In this work, we prove that without the assumption of symmetry of the geometry, quite surprisingly, there is always one real frequency such that $T = 0$. In this case, all the energy sent in the waveguide is reflected.

Keywords: waveguides, complex resonance, zero transmission, scattering matrix

1 Setting of the problem

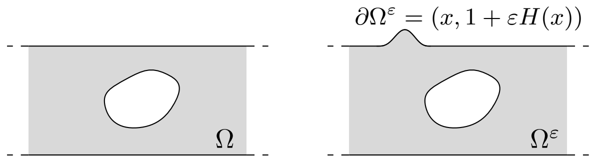

Figure 1: Original waveguide Ω (left) and perturbed geometry Ω^ε (right).

Let $\Omega \subset \mathbb{R}^2$ be a connected waveguide which coincides with the strip $\{(x,y) \in \mathbb{R} \times (0;1)\}$ for $|x| \geq d$ where $d > 0$ is given (see Figure 1 left). Propagation of acoustic waves in Ω with sound hard walls leads to study the problem

$$\left|\begin{array}{rcll} \Delta u + \lambda u & = & 0 & \text{in } \Omega \\ \partial_\nu u & = & 0 & \text{on } \partial\Omega. \end{array}\right. \qquad (1)$$

For $\lambda \in (0;\pi^2)$, only two waves $w^\pm(x,y) = e^{\pm i\sqrt{\lambda}x}$ can propagate in Ω. The scattering of the incident rightgoing wave w^+ yields a solution of (1) admitting the expansion

$$u_+ = \left|\begin{array}{rl} w^+ + R\,w^- + \dots, & \text{for } x < -d \\ T\,w^+ + \dots, & \text{for } x > d. \end{array}\right. \qquad (2)$$

Here $R \in \mathbb{C}$ is a reflection coefficient, $T \in \mathbb{C}$ is a transmission coefficient and the dots stand for terms which are exponentially decaying at infinity. Similarly, there is a solution u_- of (1) associated with the incident leftgoing wave w^-. We denote $\tilde{R}$, T the corresponding scattering coefficients (T is the same for u_+ and u_-). We define the scattering matrix

$$\mathfrak{s} := \begin{pmatrix} R & T \\ T & \tilde{R} \end{pmatrix} \in \mathbb{C}^{2\times 2},$$

which is unitary ($\mathfrak{s}\overline{\mathfrak{s}}^\top = \mathrm{Id}$). We assume that the geometry is such that the Neumann Laplacian in Ω admits a simple eigenvalue $\lambda^0 \in (0;\pi^2)$. In the sequel, we perturb a bit the geometry, so that this real eigenvalue becomes a complex resonance, and we study the behaviour of the scattering matrix for real frequencies in a neighbourhood of λ^0.

2 Perturbation of the frequency and of the geometry

We perturb the geometry from some smooth compactly supported profile function H with amplitude $\varepsilon \geq 0$ as in Figure 1 right. We denote Ω^ε the new waveguide and $\mathfrak{s}(\varepsilon,\lambda)$, $T(\varepsilon,\lambda)$, $R(\varepsilon,\lambda)$, $\tilde{R}(\varepsilon,\lambda)$ the quantities introduced above in the geometry Ω^ε at frequency λ. For short, we set $\mathfrak{s}^0 = \mathfrak{s}(0,\lambda^0)$, $T^0 = T(0,\lambda^0)$, $R^0 = R(0,\lambda^0)$, $\tilde{R}^0 = \tilde{R}(0,\lambda^0)$. Decomposition in Fourier series guarantees that the eigenfunctions associated with λ^0, the trapped modes, behave at infinity as $K_\pm e^{-\sqrt{\pi^2-\lambda^0}|x|}\cos(\pi y) + \dots$ where $K_\pm \in \mathbb{C}$. In [1], the following theorem is proved.

Theorem 1 *Assume that* $(K_+, K_-) \neq (0,0)$. *There is a quantity* $\ell(H) \in \mathbb{R}$, *which depends linearly on* H, *such that when* $\varepsilon \to 0$,

$$\mathfrak{s}(\varepsilon, \lambda^0 + \varepsilon\lambda') = \mathfrak{s}^0 + O(\varepsilon) \qquad \textit{for } \lambda' \neq \ell(H),$$

and, for any $\mu \in \mathbb{R}$,

$$\mathfrak{s}(\varepsilon, \lambda^0 + \varepsilon\ell(H) + \varepsilon^2\mu) = \mathfrak{s}^0 + \frac{\tau^\top\tau}{i\tilde{\mu} - |\tau|^2/2} + O(\varepsilon).$$

In this expression $\tau = (a, b) \in \mathbb{C} \times \mathbb{C}$ depends only on Ω and $\tilde{\mu} = A\mu + B$ for some unessential real constants A, B with $A \neq 0$.

Theorem 1 shows that the mapping $\mathfrak{s}(\cdot,\cdot)$ is not continuous at $(0, \lambda^0)$ (setting where trapped modes exist). And for ε_0 small fixed, the scattering matrix $\lambda \mapsto \mathfrak{s}(\varepsilon_0, \lambda)$ exhibits a quick change in a neighbourhood of $\lambda^0 + \varepsilon_0 \ell(H)$: this is the Fano resonance phenomenon. When $(K_+, K_-) = (0, 0)$ a faster Fano resonance phenomenon occurs. In the sequel, to simplify we denote $\mathfrak{s}^\varepsilon(\mu)$, $T^\varepsilon(\mu)$, $R^\varepsilon(\mu)$, $\tilde{R}^\varepsilon(\mu)$ the values of $\mathfrak{s}$, T, R, $\tilde{R}$ in Ω^ε at the frequency $\lambda = \lambda^0 + \varepsilon \ell(H) + \varepsilon^2 \mu$. When Ω^ε is symmetric with respect to an axis orthogonal to the direction of propagation of waves, one can deduce quite simply from Theorem 1 that the complex curves $\mu \mapsto T^\varepsilon(\mu)$ and $\mu \mapsto R^\varepsilon(\mu)$ pass through zero for ε small enough (see [1]). In the next section, we explain how to show that without assumption of symmetry, in Ω^ε, there is still a real frequency closed to λ^0 such that the transmission coefficient is zero. However in general $\mu \mapsto R^\varepsilon(\mu)$ does not pass through zero.

3 Exact zero transmission

Theorem 2 *Assume that $T^0 \neq 0$. Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0; \varepsilon_0]$, there is $\mu \in \mathbb{R}$ such that $T^\varepsilon(\mu) = 0$.*

PROOF. Theorem 1 provides the estimate

$$|T^\varepsilon(\mu) - T^{\text{asy}}(\mu)| \leq C\,\varepsilon \qquad (3)$$

with $\quad T^{\text{asy}}(\mu) = T^0 + \dfrac{ab}{i\tilde{\mu} - (|a|^2 + |b|^2)/2}.$

For any compact set $I \subset \mathbb{R}$, the constant $C > 0$ in (3) can be chosen independent of $\mu \in I$.

$\star$ First, we study the set $\{T^{\text{asy}}(\mu),\, \mu \in \mathbb{R}\}$. Classical results concerning the Möbius transform guarantee that $\{T^{\text{asy}}(\mu), \mu \in \mathbb{R}\}$ coincides with $\mathscr{C}^{\text{asy}} \setminus \{T^0\}$ where $\mathscr{C}^{\text{asy}}$ is a circle passing through T^0. Let us show that $\mathscr{C}^{\text{asy}}$ also passes through zero. One finds that $T^{\text{asy}}(\mu) = 0$ for some $\mu \in \mathbb{R}$ if and only if there holds

$$\frac{|a|^2 + |b|^2}{2} = \Re e \left(\frac{ab}{T^0} \right). \qquad (4)$$

An intermediate calculus of [1] implies $R^0\,\overline{a} + T^0\,\overline{b} = a$ and $T^0\,\overline{a} + \tilde{R}^0\,\overline{b} = b$. From this and the unitarity of $\mathfrak{s}^0$ which imposes $\tilde{R}^0 = -\overline{R^0}T^0/\overline{T^0}$, we can obtain (4). Denote $\mu_\star$ the value of μ such that $T^{\text{asy}}(\mu_\star) = 0$ and for $\varepsilon > 0$, define the interval $I^\varepsilon = (\mu_\star - \sqrt{\varepsilon}; \mu_\star + \sqrt{\varepsilon})$. From (3), for $\varepsilon > 0$ small, we know that the curve $\{T^\varepsilon(\mu),\, \mu \in I^\varepsilon\}$ passes close to zero. Now, using the unitary structure of $\mathfrak{s}^\varepsilon(\mu)$ as in [2], we show that this curves passes exactly through zero for ε small.

$\star$ Assume by contradiction that for all $\varepsilon > 0$, $\mu \mapsto T^\varepsilon(\mu)$ does not pass through zero in I_ε. Since $\mathfrak{s}^\varepsilon(\mu)$ is unitary, there holds $R^\varepsilon(\mu)\,\overline{T^\varepsilon(\mu)} + T^\varepsilon(\mu)\,\overline{\tilde{R}^\varepsilon(\mu)} = 0$ and so

$$-R^\varepsilon(\mu)/\overline{\tilde{R}^\varepsilon(\mu)} = T^\varepsilon(\mu)/\overline{T^\varepsilon(\mu)} \qquad \forall \mu \in I^\varepsilon.$$

But if $\mu \mapsto T^\varepsilon(\mu)$ does not pass through zero on I^ε, one can verify that the point $T^\varepsilon(\mu)/\overline{T^\varepsilon(\mu)} = e^{2i\arg(T^\varepsilon(\mu))}$ must run rapidly on the unit circle for $\mu \in I_\varepsilon$ as $\varepsilon \to 0$. On the other hand, $R^\varepsilon(\mu)/\overline{\tilde{R}^\varepsilon(\mu)}$ tends to a constant on I_ε as $\varepsilon \to 0$. This way we obtain a contradiction. □

Remark 3 *The fact that $\mathscr{C}^{\text{asy}}$ passes through zero is quite mysterious. Without assumption of symmetry, we do not have physical reason to explain this miracle.*

In the geometry of Figure 2, first we find that trapped modes exist for $\varepsilon = 0$ and $\sqrt{\lambda^0} \approx 1.2395$. Then we approximate (FEM) $T(\varepsilon, \lambda)$ ($\times$) and $R(\varepsilon, \lambda)$ ($\bullet$) for $\sqrt{\lambda} \in (1.2; 1.3)$ and $\varepsilon = 0.05$. As predicted, we observe that $\lambda \mapsto T(\varepsilon, \lambda)$ passes through zero around λ^0. Finally, we display the real part of u_+ in Ω^ε for $\varepsilon = 0.05$ and $\sqrt{\lambda} = 1.2449$. In this setting, we have $T(\varepsilon, \lambda) \approx 0$.

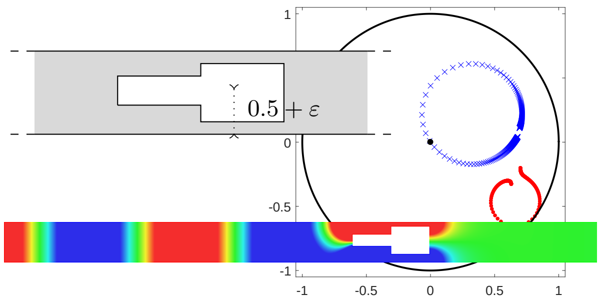

Figure 2: Zero transmission in a waveguide.

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