Time harmonic Maxwell's equations with sign changing coefficients

Workshop : Around scattering by obstacles and billiards

A.-S. Bonnet-Ben Dhia[†], L. Chesnel[†], P. Ciarlet[†], X. Claeys[‡]

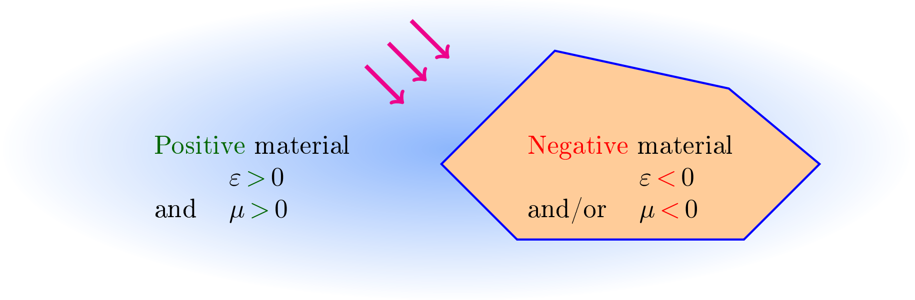
[†]POems team, Ensta, Paris, France

[‡]DMIA, ISAE, Toulouse, France

University of Aveiro, Aveiro, Portugal, March 31, 2012

Introduction: objective

Scattering by a negative material in electromagnetism in 3D in time-harmonic regime (at a given frequency):



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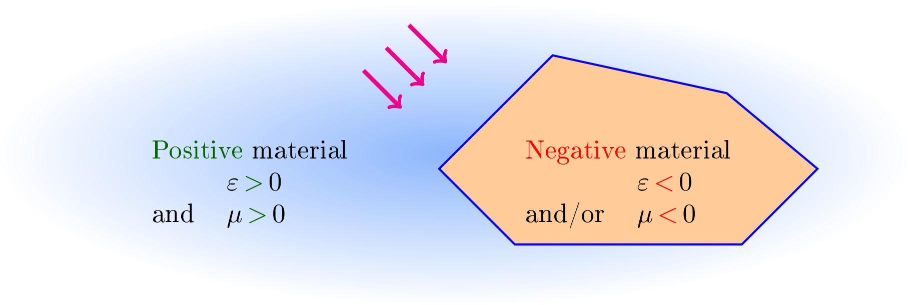

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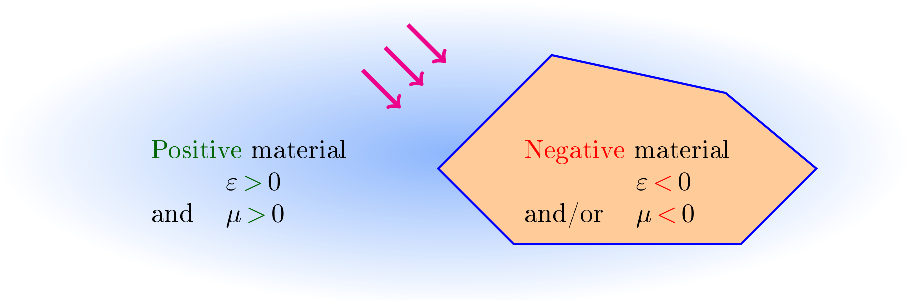

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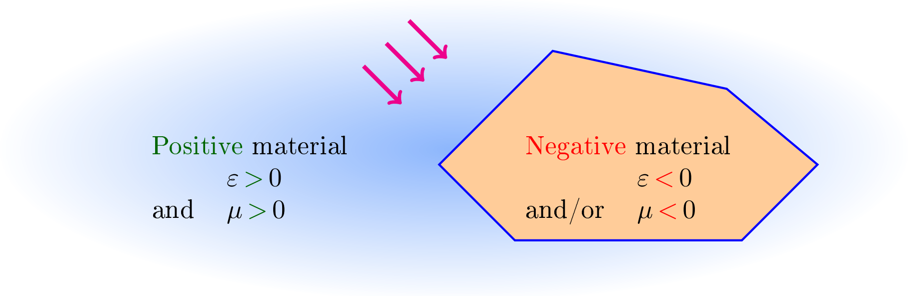

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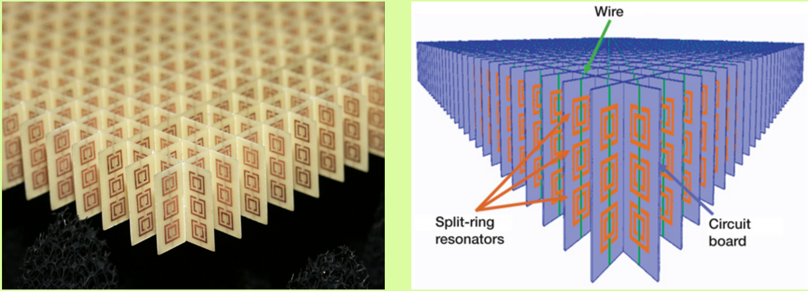
Zoom on a metamaterial: practical realizations of metamaterials are achieved by a periodic assembly of small resonators.



Example of metamaterial (NASA)

Mathematical justification of the homogenized model: Bouchitté, Bourel and Felbacq, 09.

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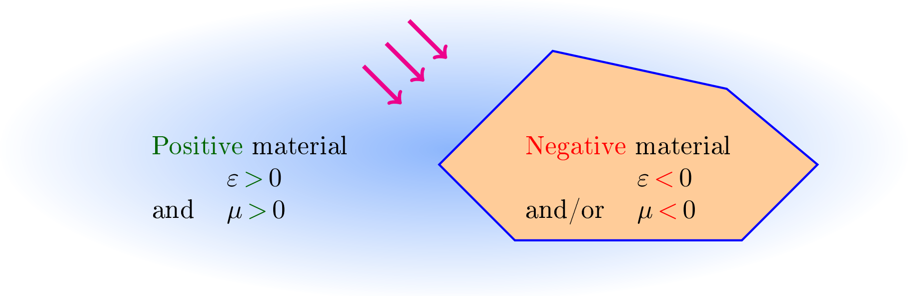

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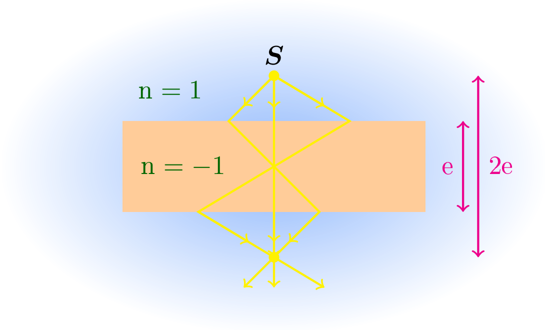
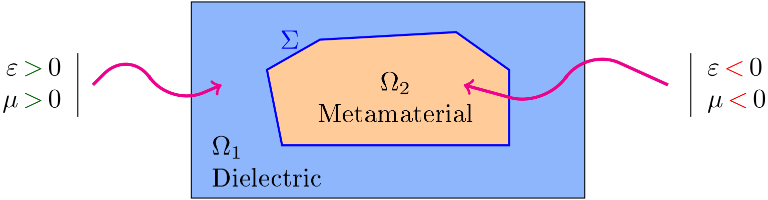


- The negative refraction at the interface metamaterial/dielectric could allow the realization of perfect lenses (Pendry, 00), photonic traps ...

Introduction: in this talk

Problem set in a bounded domain $\Omega \subset \mathbb{R}^3$:



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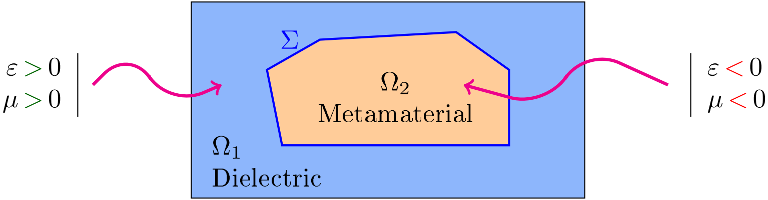

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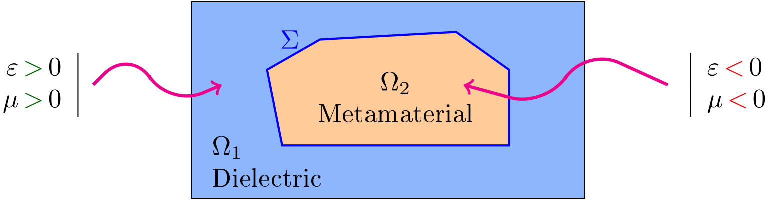

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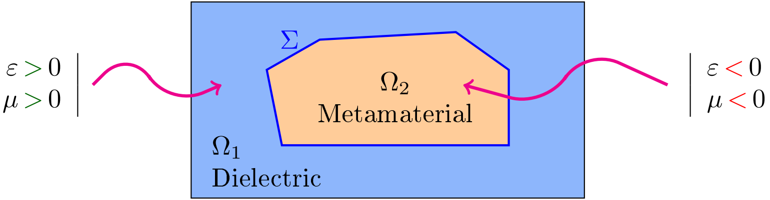

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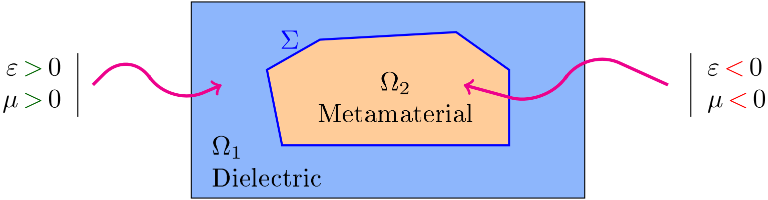

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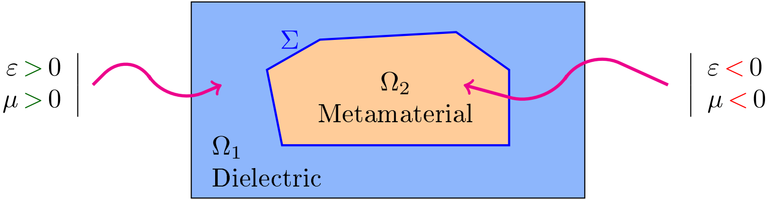

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Does well-posedness still hold ? What is the appropriate functional framework ? What about the convergence of approximation methods ? ...

Classical results for positive materials

Suppose we want to solve the following problem, for the electric field:

Find $\boldsymbol{E} \in \mathbf{H}(\mathbf{curl}\,; \Omega)$ such that:
$\mathbf{curl}\, \mu^{-1} \mathbf{curl}\, \boldsymbol{E} - \omega^2 \varepsilon \boldsymbol{E} = \boldsymbol{F}$ in Ω and $\boldsymbol{E} \times \boldsymbol{n} = 0$ on $\partial\Omega$.

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What happens in presence of sign-changing ε and μ ?

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4. Conclusion

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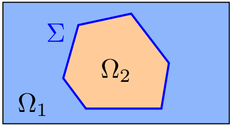

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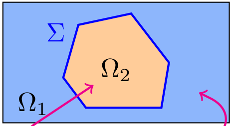

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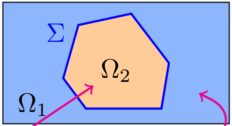

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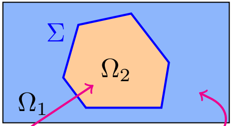

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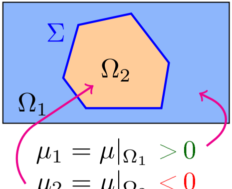

$\mu_1 = \mu|_{\Omega_1} > 0$
$\mu_2 = \mu|_{\Omega_2} < 0$
(constant to simplify)

- $\mathrm{H}^1_0(\Omega) = \{v \in \mathrm{L}^2(\Omega)\,|\,\nabla v \in \mathrm{L}^2(\Omega);\, v|_{\partial\Omega} = 0\}$
- f is the source term in $\mathrm{H}^{-1}(\Omega)$

Since $\mathrm{H}^1_0(\Omega) \subset\subset \mathrm{L}^2(\Omega)$, we focus on the principal part.

$$(\mathscr{P})\ \left|\begin{array}{l}\text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:}\\ \operatorname{div}(\mu^{-1}\,\nabla u) = -f \text{ in } \Omega.\end{array}\right. \quad \Leftrightarrow \quad (\mathscr{P}_V)\ \left|\begin{array}{l}\text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:}\\ a(u,v) = l(v),\ \forall v \in \mathrm{H}^1_0(\Omega).\end{array}\right.$$

with $a(u,v) = \displaystyle\int_\Omega \mu^{-1}\,\nabla u \cdot \nabla v$ and $l(v) = \langle f, v\rangle_\Omega$.

Definition. We will say that the problem $(\mathscr{P})$ is well-posed if the operator $A = \operatorname{div}(\mu^{-1}\nabla\cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$.

Mathematical difficulty

- Classical case $\mu > 0$ everywhere :

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  - For a symmetric domain (w.r.t. Σ) with $\mu_2 = -\mu_1$, we can build a kernel of infinite dimension.

Idea of the $\mathtt{T}$-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$$(\mathscr{P}) \Leftrightarrow (\mathscr{P}_V) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(u,v) = l(v),\ \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

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Let T be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

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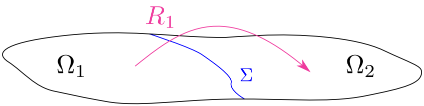
R_1 transfer/extension operator



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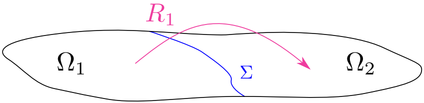

$$\begin{array}{ll} R_1 u_1 = u_1 & \text{on } \Sigma \\ R_1 u_1 = 0 & \text{on } \partial\Omega_2 \setminus \Sigma \end{array}$$

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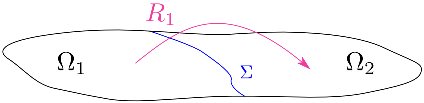

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2. $\mathtt{T}_1 \circ \mathtt{T}_1 = Id$ so $\mathtt{T}_1$ is an isomorphism of $\mathrm{H}^1_0(\Omega)$

Idea of the T-coercivity 2/2

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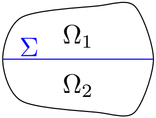
5 Conclusion:

THEOREM. If the contrast $\kappa_\mu = \mu_2/\mu_1 \notin [-\|R_1\|^2; -1/\|R_2\|^2]$ (critical interval) then $\mathrm{div}\,(\mu^{-1}\,\nabla\cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$.

Choice of R_1,R_2?

- A simple case: symmetric domain



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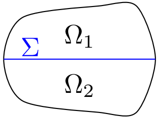

$$R_1 = R_2 = S_{\Sigma}$$
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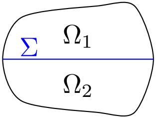

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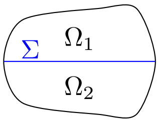

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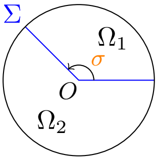

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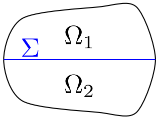

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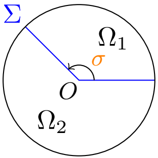

Action of R_1:

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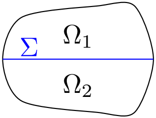

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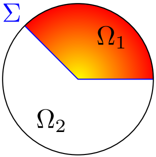

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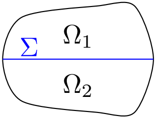

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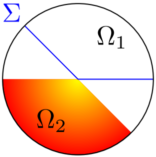

Action of R_1: symmetry w.r.t θ

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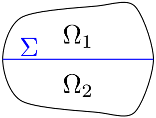

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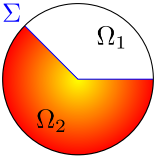

Action of R_1: symmetry + dilatation w.r.t θ

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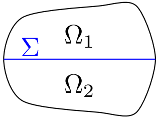

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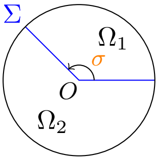

Action of R_1: symmetry + dilatation w.r.t θ

$$\|R_1\|^2 \qquad = \mathcal{R}_\sigma = (2\pi - \sigma)/\sigma$$

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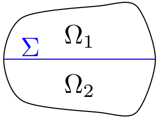

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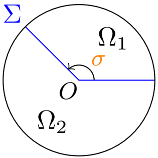

$$\text{Action of } R_1\text{: symmetry + dilatation w.r.t } \theta$$
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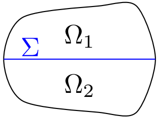

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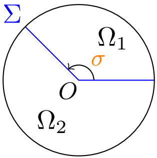

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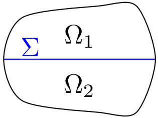

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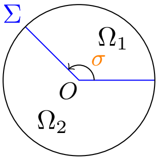
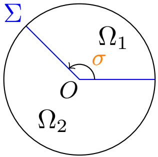


Action of R_1: symmetry + dilatation w.r.t θ
Action of R_2: symmetry + dilatation w.r.t θ
$$\|R_1\|^2 = \|R_2\|^2 = \mathcal{R}_\sigma = (2\pi - \sigma)/\sigma$$
$(\mathscr{P})$ well-posed $\Leftarrow \kappa_\mu \notin [-\mathcal{R}_\sigma; -1/\mathcal{R}_\sigma]$

PROPOSITION. The problem $(\mathscr{P})$ is well-posed in the Fredholm sense for a polygonal interface if $\kappa_\mu \notin [-\mathcal{R}_\sigma; -1/\mathcal{R}_\sigma]$ where σ is the smallest angle.

The case of an interface with corners



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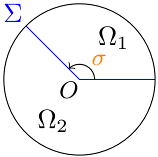

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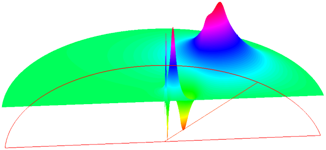
- There exists a propagative singularity $\varphi(\theta) r^{\pm i\eta} = \varphi(\theta) e^{\pm i\eta \ln r}$ with $\eta \in \mathbb{R}^*$ which belongs to $\mathrm{H}^{1-\delta}(\Omega)$ but not to $\mathrm{H}^1(\Omega)$.



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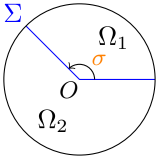

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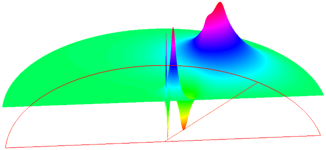

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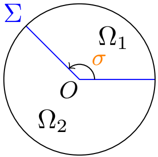

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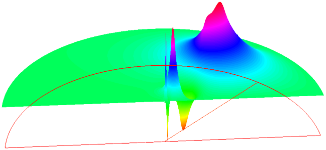
- Due to this singularity, the problem is not Fredholm in $\mathrm{H}^1(\Omega)$.
- We have justified a new functional framework in which Fredholm property is recovered, by selecting the outgoing singularity.
  Bonnet-Ben Dhia, Chesnel and Claeys (submitted)



Extensions for the scalar case

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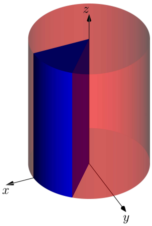

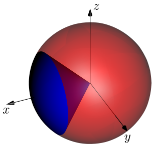

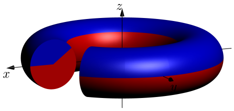

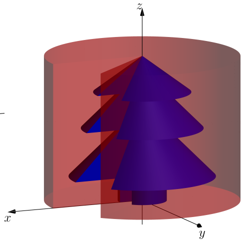

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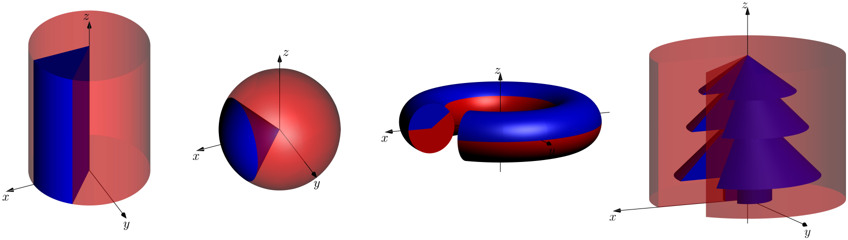
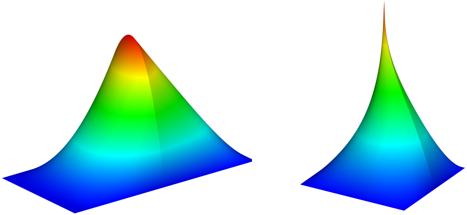


- The T-coercivity technique allows to justify convergence of standard finite element method for simple meshes *(joint work with P. Ciarlet Jr.)*.



$\mathbb{T}$-coercivity in the vector case 1/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

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Not possible!

$\mathbb{T}$-coercivity in the vector case 2/3

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♠ Impossible because $\mathrm{div}\,(\varepsilon\,\mathbf{curl}\,\boldsymbol{H}) \neq 0$.

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♠ Impossible because $\mathrm{div}\,(\varepsilon\, \mathbf{curl}\, \boldsymbol{H}) \neq 0$. *The solution: add a gradient...*

$\mathbb{T}$-coercivity in the vector case 3/3

Third attempt

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu; \Omega)$.

$\mathbb{T}$-coercivity in the vector case 3/3

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Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu; \Omega)$.

1. Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $\operatorname{div}(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)) = 0$.

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✔ Ok if $(\varphi, \varphi') \mapsto \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi'$ is T-coercive on $\mathrm{H}^1_0(\Omega)$. $(\mathcal{A}_\varepsilon)$

$\mathbb{T}$-coercivity in the vector case 3/3

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☛ Use the results of the previous section to check $(\mathcal{A}_\varepsilon)$ and $(\mathcal{A}_\mu)$.

$\mathbb{T}$-coercivity in the vector case 3/3

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Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega)$.

LEMMA. Suppose

$$(\varphi,\varphi') \mapsto \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' \text{ is } \mathtt{T\text{-}coercive} \text{ on } \mathrm{H}^1_0(\Omega); \qquad (\mathcal{A}_\varepsilon)$$

$$(\varphi,\varphi') \mapsto \int_\Omega \mu \nabla\varphi \cdot \nabla\varphi' \text{ is } \mathtt{T\text{-}coercive} \text{ on } \mathrm{H}^1(\Omega)/\mathbb{R}. \qquad (\mathcal{A}_\mu)$$

Then, there exists $\mathbb{T} \in \mathcal{L}(\mathbf{V}_T(\mu;\Omega))$ such that, for all $\boldsymbol{H}$, $\boldsymbol{H}'$

$$a(\boldsymbol{H},\mathbb{T}\boldsymbol{H}') = a(\mathbb{T}\boldsymbol{H},\boldsymbol{H}') = \int_\Omega \mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\boldsymbol{H}'.$$

$$a(\boldsymbol{H},\mathbb{T}\boldsymbol{H}) = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\boldsymbol{u} = \int_\Omega |\mathbf{curl}\,\boldsymbol{H}|^2.$$

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A result of compact embedding

- Using the analogue of the previous result in $\mathbf{V}_N(1;\Omega)$, we can prove the

THEOREM. Suppose

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Then, the embedding of

$$\mathbf{V}_T(\mu;\Omega) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\Omega) \,|\, \mathrm{div}\,(\mu\,\boldsymbol{u}) = 0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$$

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- Since for all $\boldsymbol{H}$, $\boldsymbol{H}'$

$$a(\boldsymbol{H},\mathbb{T}\boldsymbol{H}') = a(\mathbb{T}\boldsymbol{H},\boldsymbol{H}') = \int_\Omega \mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{H}',$$

we deduce *a posteriori* that $\mathbb{T}$ is an isomorphism of $\mathbf{V}_T(\mu;\Omega)$.

The result for the magnetic field

Consider $\boldsymbol{F} \in \mathbf{L}^2(\Omega)$ such that $\operatorname{div} \boldsymbol{F} \in \mathbf{L}^2(\Omega)$.

THEOREM. Suppose

$$(\varphi, \varphi') \mapsto \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' \text{ is T-coercive on } \mathrm{H}^1_0(\Omega); \qquad (\mathcal{A}_\varepsilon)$$

$$(\varphi, \varphi') \mapsto \int_\Omega \mu \nabla\varphi \cdot \nabla\varphi' \text{ is T-coercive on } \mathrm{H}^1(\Omega)/\mathbb{R}. \qquad (\mathcal{A}_\mu)$$

Then, the problem for the magnetic field

$$\left|\begin{array}{ll} \text{Find } \boldsymbol{H} \in \mathbf{H}(\mathbf{curl}\,;\Omega) \text{ such that:} & \\ \mathbf{curl}\,\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H} - \omega^2\mu\boldsymbol{H} = \boldsymbol{F} & \text{in } \Omega \\ \mu\boldsymbol{H}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{array}\right.$$

is well-posed for all $\omega \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

… and the result for the electric field

Consider $\boldsymbol{F} \in \mathbf{L}^2(\Omega)$ such that div $\boldsymbol{F} \in \mathbf{L}^2(\Omega)$.

THEOREM. Suppose

$$(\varphi, \varphi') \mapsto \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' \text{ is T-coercive on } \mathrm{H}^1_0(\Omega); \qquad (\mathcal{A}_\varepsilon)$$

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Then, the problem for the electric field

$$\left| \begin{array}{ll} \text{Find } \boldsymbol{E} \in \mathbf{H}(\mathbf{curl}\,; \Omega) \text{ such that:} & \\ \mathbf{curl}\, \mu^{-1} \mathbf{curl}\, \boldsymbol{E} - \omega^2 \varepsilon \boldsymbol{E} = \boldsymbol{F} & \text{in } \Omega \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{array} \right.$$

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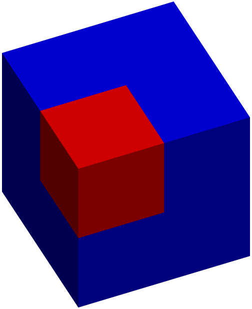
Application to the Fichera's corner



PROPOSITION. Suppose

$$\kappa_\varepsilon \notin \left[-7; -\frac{1}{7}\right] \qquad \text{and} \qquad \kappa_\mu \notin \left[-7; -\frac{1}{7}\right]. \quad ✻$$

Then, the problems for the electric and magnetic fields are well-posed for all $\omega \in \mathbb{C} \backslash \mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

✻ *Note that 7 is the ratio of the blue volume over the red volume...*

Conclusions for the bounded domain

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For the continuous problem

- ✔ We have studied the scalar problem with a geometric $\mathtt{T}$-coercivity.
- ✔ We have proved that the Maxwell's problems for $\boldsymbol{E}$ and $\boldsymbol{H}$ are well-posed as soon as the two scalar problems are well-posed.

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For the numerical methods

- ✔ In the scalar case: in practice, usual methods converge. Partial proofs are available Bonnet-Ben Dhia *et al.*, 10, Nicaise-Venel, 11, Chesnel-Ciarlet, submitted.
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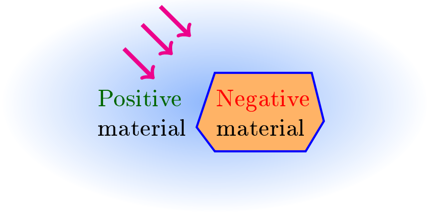
Inside the critical interval

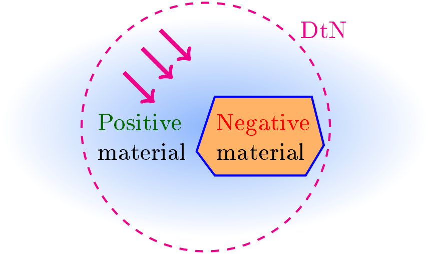
- ✔ In the scalar case, we have developed a functional framework to take into account the propagative singularity Bonnet-Ben Dhia *et al.*, submitted.
- ♠ In the vector case: a theoretical study is in progress.
- ♠ Our new model raises a lot of questions, related to the physics of plasmonics and metamaterials.

Perspectives for the scattering problem



✔ We hope to be able to deal with the scattering problem for a bounded negative obstacle using a Dtn operator and the results presented in this talk.

Perspectives for the scattering problem



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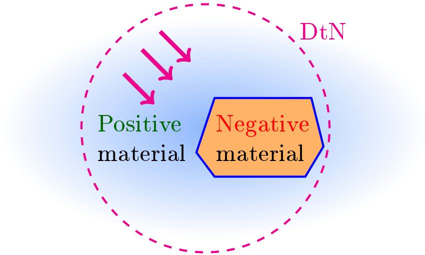

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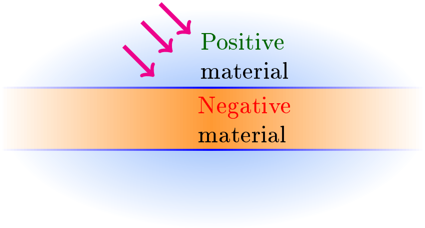

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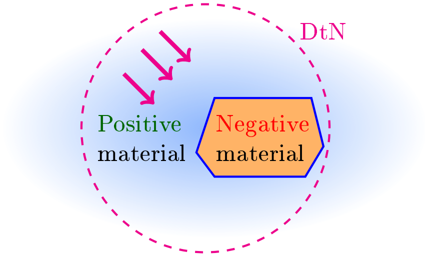

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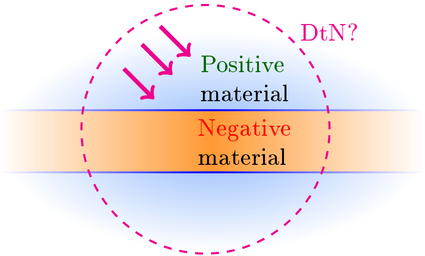

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Thank you for your attention.

- A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., *T-coercivity for scalar interface problems between dielectrics and metamaterials*, M2AN, to appear, 2012.
- A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., *T-coercivity for the Maxwell transmission problem between a dielectric and a negative material*, on going work.
- L. Chesnel, P. Ciarlet Jr., *T-coercivity and continuous Galerkin methods: application to transmission problems with sign changing coefficients*, submitted.
- A.-S. Bonnet-Ben Dhia, L. Chesnel, X. Claeys, *Radiation condition for a non-smooth interface between a dielectric and a metamaterial*, submitted.