SIAM Conference on Imaging Science

# A spectral method to detect invisibility in waveguides

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Bologna, 05/06/2018

# General setting

- We are interested in the propagation of waves in acoustic waveguides.

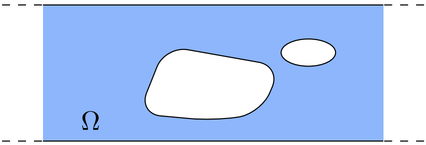

- In this talk, we study questions of invisibility.

> Can we find situations where waves go through like if there were no defect ?

- One can wish to have good energy transmission through the structure.
- One can wish to hide objects.

# Scattering problem in a waveguide

- Scattering in time-harmonic regime of an incident wave in the acoustic waveguide $\Omega$ coinciding with $\{(x, y) \in \mathbb{R} \times (0; 1)\}$ outside a compact region.




Find $v = v_i + v_s$ s. t.

$$\begin{array}{rcll} \Delta v + k^2 v & = & 0 & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega, \\ & & v_s \text{ is outgoing}. & \end{array}$$

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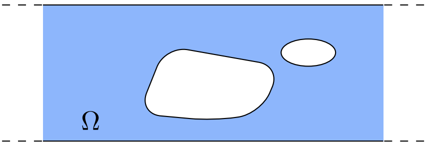


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- For this problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$, the modes are

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| Propagating | $w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2}$, $n \in [\![0, N-1]\!]$ |
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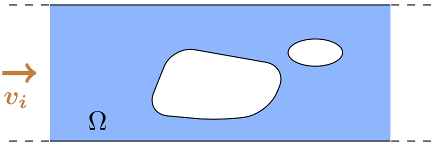


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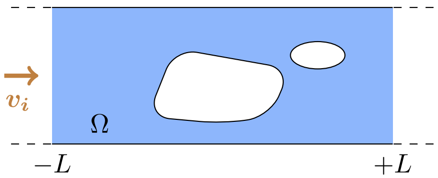


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- $v_s$ is outgoing $\Leftrightarrow$ $v_s = \sum_{n=0}^{+\infty} \gamma_n^{\pm} w_n^{\pm}$ for $\pm x \geq L$, with $(\gamma_n^{\pm}) \in \mathbb{C}^{\mathbb{N}}$.

# Goal of the talk

DEFINITION: $v$ is a non reflecting mode if $v_s$ is expo. decaying for $x \leq -L$
$\Leftrightarrow \quad \gamma_n^- = 0,\ n \in [\![0, N-1]\!] \quad \Leftrightarrow$ energy is completely transmitted.

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$\rightarrow$ Note that non reflection occurs for **particular** $\boldsymbol{v_i}$ to be computed.

# Outline of the talk

1. Introduction

2. Classical complex scaling

   We recall how to use classical complex scaling to compute trapped modes and complex resonances.

3. Conjugated complex scaling

   We explain how to use conjugated complex scaling to compute non reflecting modes.

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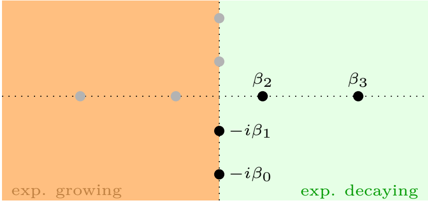
# Classical complex scaling to compute $v_s$ 1/2

REMINDER: $v_s = \sum_{n=0}^{N-1} \gamma_n^{\pm} e^{\pm i\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \gamma_n^{\pm} e^{\mp \beta_n x} \cos(n\pi y), \ \pm x \geq L.$

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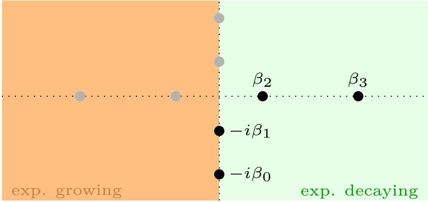


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- For $\theta \in (0;\pi/2)$, consider the complex change of variables

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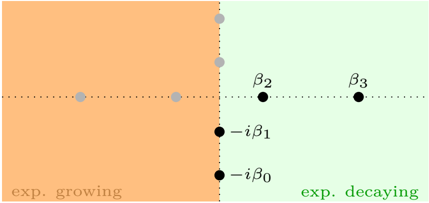


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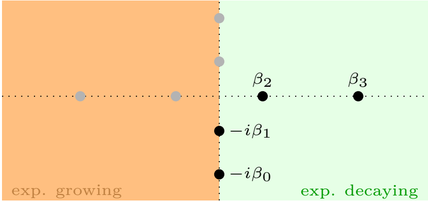


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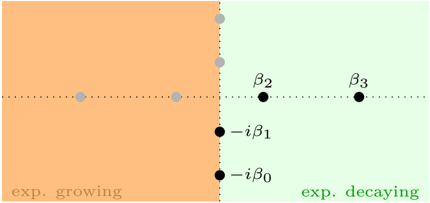


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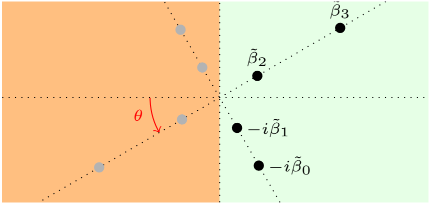


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## Classical complex scaling to compute $v_s$ 2/2

- $v_\theta$ solves $(*)\left|\begin{array}{rcll} \alpha_\theta \dfrac{\partial}{\partial x}\left(\alpha_\theta \dfrac{\partial v_\theta}{\partial x}\right) + \dfrac{\partial^2 v_\theta}{\partial y^2} + k^2 v_\theta & = & 0 & \text{in } \Omega \\ \partial_n v_\theta & = & -\partial_n v_i & \text{on } \partial\Omega. \end{array}\right.$

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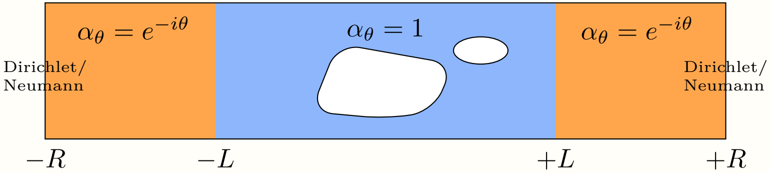
- Numerically we solve $(*)$ in the truncated domain




$\Rightarrow$ We obtain a good approximation of $v_s$ for $|x| < L$.

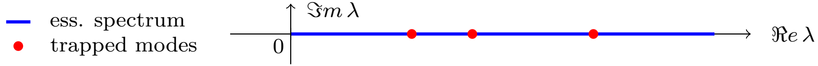
- This is the method of Perfectly Matched Layers (PMLs).

# Spectral analysis

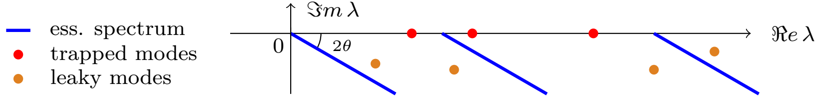
- Define the operators $A$, $A_\theta$ of $\mathrm{L}^2(\Omega)$ such that

$$Av = -\Delta v, \qquad A_\theta v = -\Big(\alpha_\theta \frac{\partial}{\partial x}\Big(\alpha_\theta \frac{\partial v}{\partial x}\Big) + \frac{\partial^2 v}{\partial y^2}\Big) \qquad + \partial_n v = 0 \text{ on } \partial\Omega.$$

- $A$ is selfadjoint and positive.
- $\sigma(A) = \sigma_{\mathrm{ess}}(A) = [0; +\infty)$.
- $\sigma(A)$ may contain embedded eigenvalues in the essential spectrum.

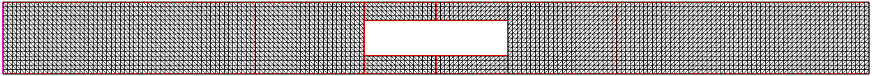



- $A_\theta$ is not selfadjoint. $\sigma(A_\theta) \subset \{\rho\, e^{i\gamma},\ \rho \geq 0,\ \gamma \in [-2\theta; 0]\}$.
- $\sigma_{\mathrm{ess}}(A_\theta) = \cup_{n\in\mathbb{N}}\{n^2\pi^2 + t\, e^{-2i\theta},\ t \geq 0\}$.
- real eigenvalues of $A_\theta$ = real eigenvalues of $A$.



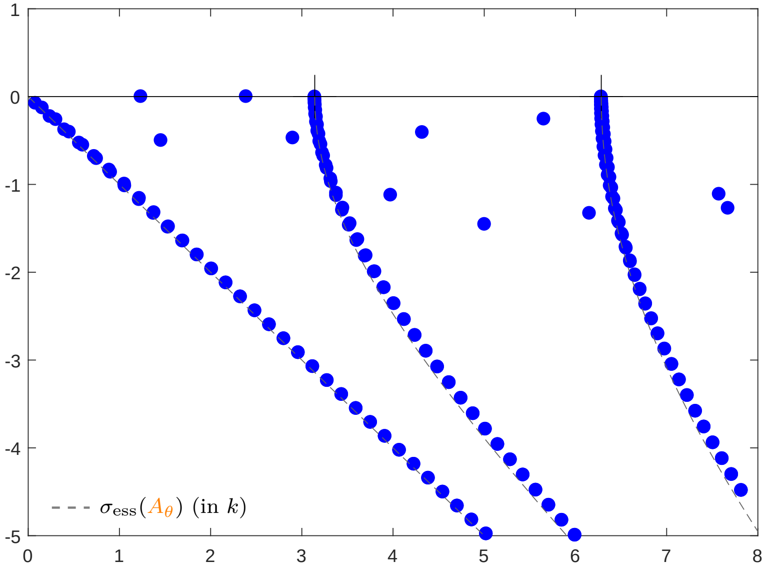
# Numerical results

- We work in the geometry



# Numerical results

- Discretized spectrum of $A_\theta$ in $k$ (not in $k^2$). We take $\theta = \pi/4$.



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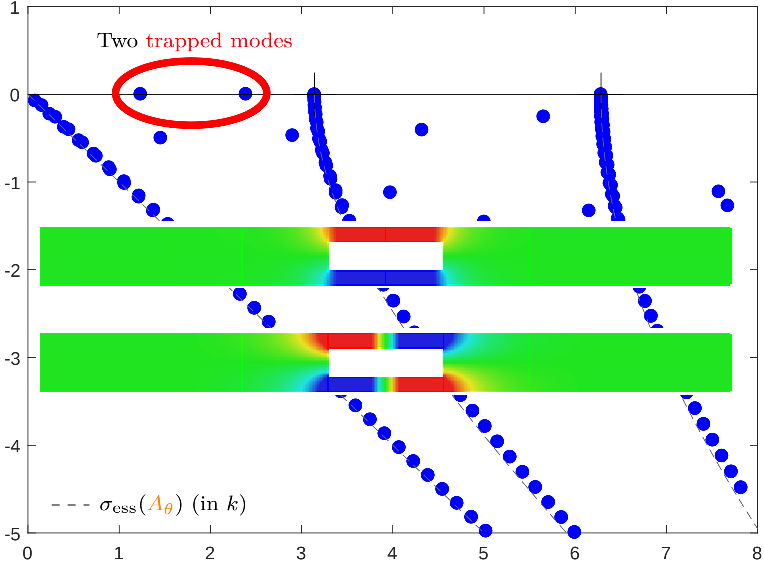

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We explain how to use conjugated complex scaling to compute non reflecting modes.

# A new complex spectrum for non reflecting $v$

- Usual complex scaling selects scattered fields which are

outgoing at $-\infty$ and outgoing at $+\infty$.

IMPORTANT REMARK: **general** $v$ decompose as

$$v = v_i + \sum_{n=0}^{N-1} \gamma_n^- \, w_n^- + \sum_{n=N}^{+\infty} \gamma_n^- \, w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ \, w_n^+ \quad x \geq L.$$

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Important remark: **non reflecting** $v$ decompose as

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IMPORTANT REMARK: **non reflecting** $v$ decompose as

$$v = \sum_{n=0}^{N-1} \boldsymbol{\alpha_n w_n^+} + \sum_{n=N}^{+\infty} \gamma_n^- w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ w_n^+ \quad x \geq L.$$

- In other words, **non reflecting** $v$ are

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Let us **change the sign** of the complex scaling at $-\infty$!

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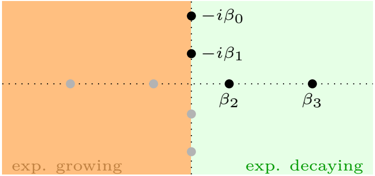


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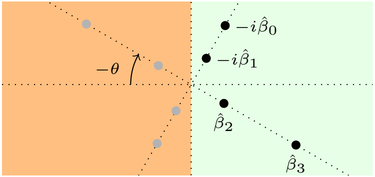


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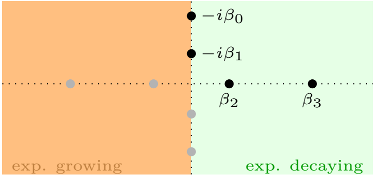


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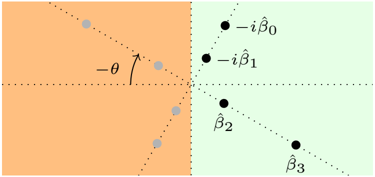


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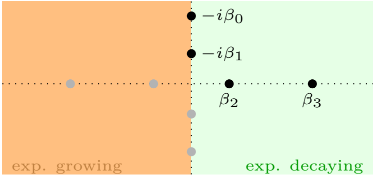


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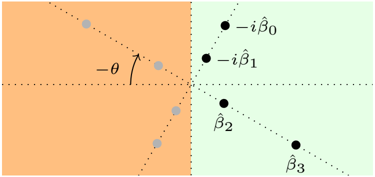


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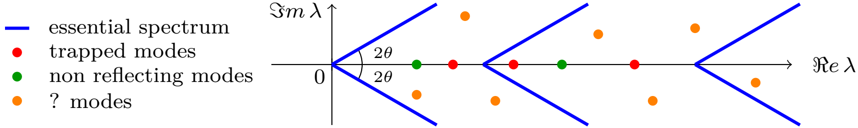
$\beta_\theta(x) = 1$ for $|x| < L$, $\beta_\theta(x) = e^{i\theta}$ for $x \le -L$, $\beta_\theta(x) = e^{-i\theta}$ for $x \ge L$

# Spectral analysis

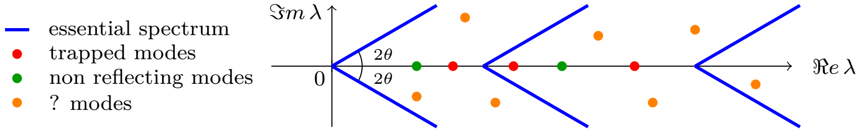
- Define the operator $B_\theta$ of $\mathrm{L}^2(\Omega)$ such that

$$B_\theta v = -\Big(\beta_\theta \frac{\partial}{\partial x}\Big(\beta_\theta \frac{\partial v}{\partial x}\Big) + \frac{\partial^2 v}{\partial y^2}\Big) \qquad + \partial_n v = 0 \text{ on } \partial\Omega.$$

- $B_\theta$ is not selfadjoint. $\sigma(B_\theta) \subset \{\rho\, e^{i\gamma},\ \rho \geq 0, \gamma \in [-2\theta; 2\theta]\}$.
- $\sigma_{\mathrm{ess}}(B_\theta) = \cup_{n\in\mathbb{N}}\{n^2\pi^2 + t\, e^{-2i\theta},\ t \geq 0\} \cup \{n^2\pi^2 + t\, e^{2i\theta},\ t \geq 0\}$.
- real eigenvalues of $B_\theta$ = real eigenvalues of $A$+non reflecting $k^2$.

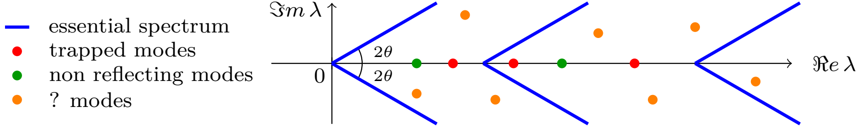


# Remark




- It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete.

# Remark

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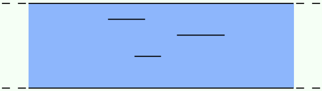
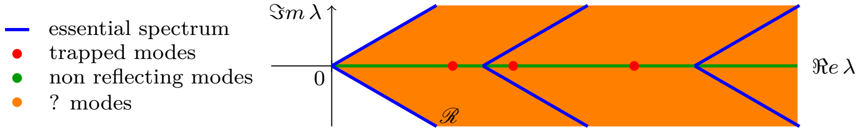


→ **Not true in general!**

$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

# Remark

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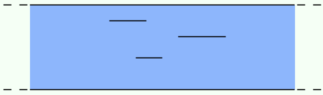

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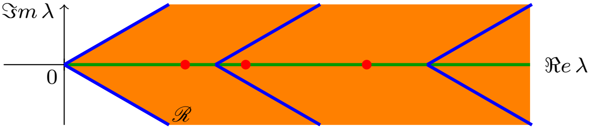
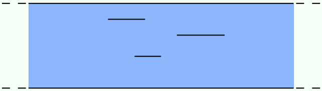
# Remark

- essential spectrum
- trapped modes
- non reflecting modes
- ? modes




- It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete.



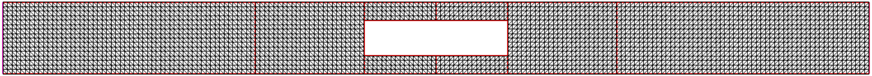
→ **Not true in general!**

$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

→ A compact perturbation can change drastically the spectrum ( $B_\theta$ is not selfadjoint ).
Numerical consequences?

# Numerical results

- Again we work in the geometry



- Define the operators $\mathcal{P}$ (Parity), $\mathcal{T}$ (Time reversal) such that

$$\mathcal{P}v(x,y) = v(-x,y) \quad \text{and} \quad \mathcal{T}v(x,y) = \overline{v(x,y)}.$$

PROP.: For symmetric $\Omega = \{(-x,y)\,|\,(x,y)\in\Omega\}$, $B_\theta$ is $\mathcal{PT}$ symmetric:

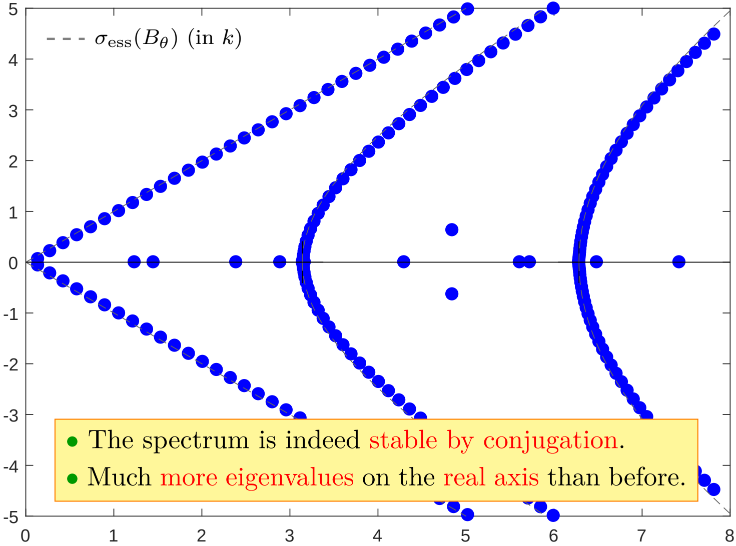
$$\mathcal{PT}B_\theta\mathcal{PT} = B_\theta.$$

As a consequence, $\sigma(B_\theta) = \overline{\sigma(B_\theta)}$.

$\Rightarrow$ If $\lambda$ is an "isolated" eigenvalue located close to the real axis, then $\lambda \in \mathbb{R}$!

# Numerical results

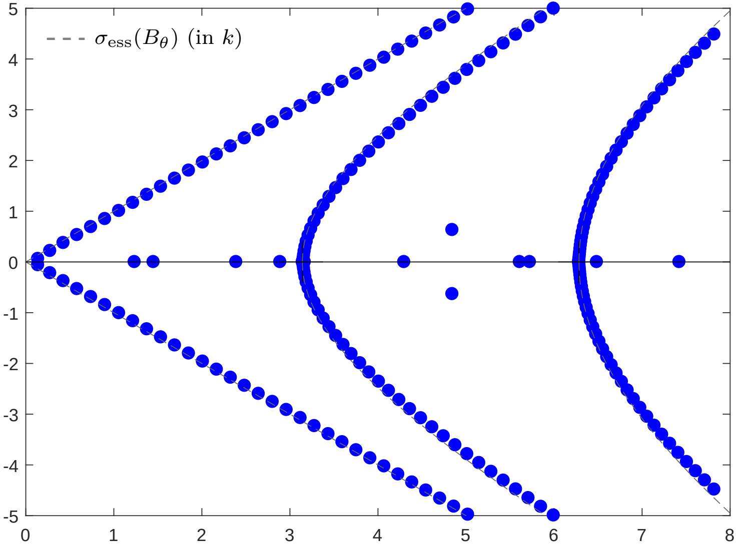
- Discretized spectrum in $k$ (not in $k^2$). We take $\theta = \pi/4$.




- The spectrum is indeed stable by conjugation.
- Much more eigenvalues on the real axis than before.

# Numerical results

- Discretized spectrum in $k$ (not in $k^2$). We take $\theta = \pi/4$.



# Numerical results

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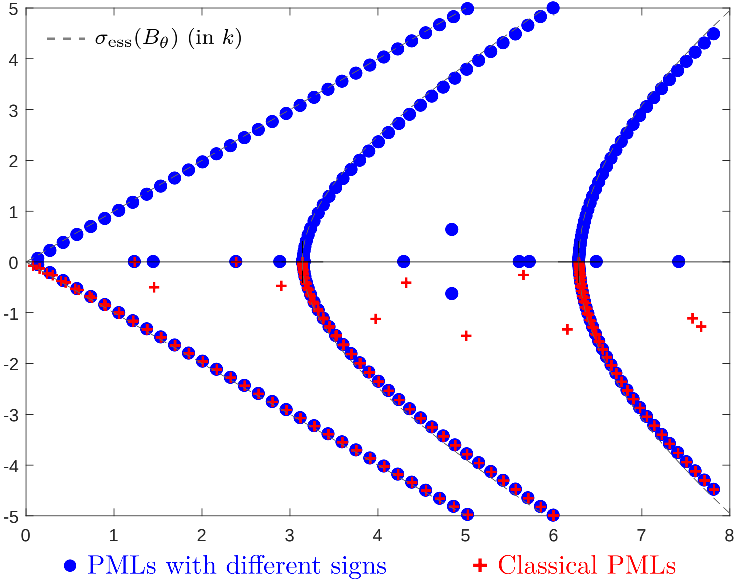
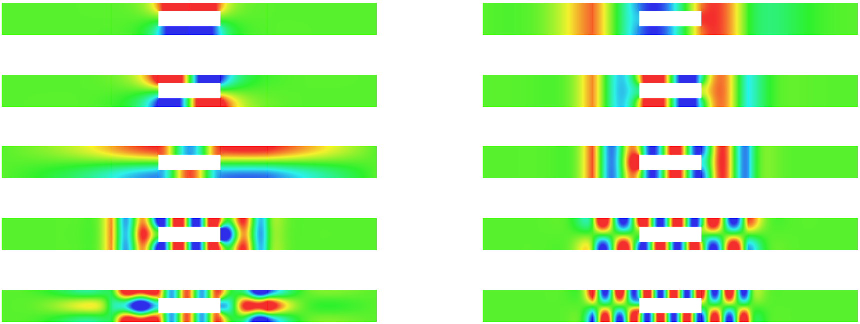


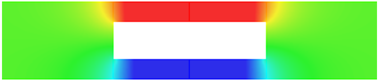
# Numerical results

► We display the eigenmodes for the ten first real eigenvalues in the whole computational domain (including PMLs).



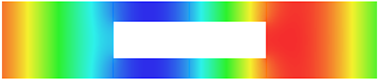
# Numerical results

- Let us focus on the eigenmodes such that $0 < k < \pi$.



First trapped mode
$k = 1.2355...$

Second trapped mode
$k = 2.3897...$

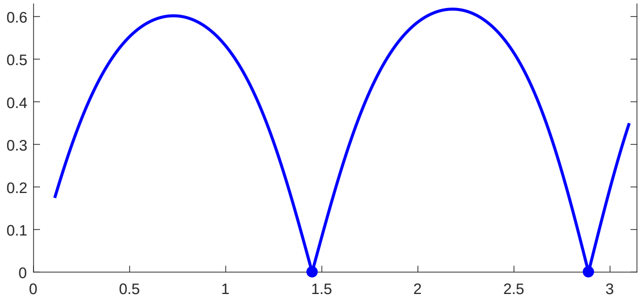


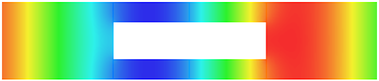
First non reflecting mode
$k = 1.4513...$

Second non reflecting mode
$k = 2.8896...$

# Numerical results

- To check our results, we compute $k \mapsto |R(k)|$ for $0 < k < \pi$.



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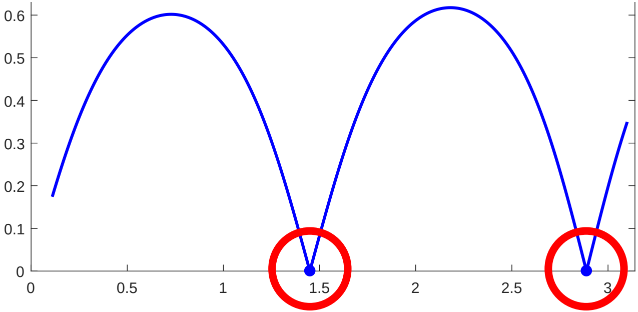
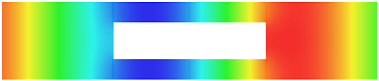


First non reflecting mode

$k = 1.4513...$

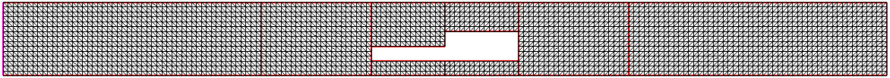
Second non reflecting mode

$k = 2.8896...$

There is perfect agreement!

# Numerical results

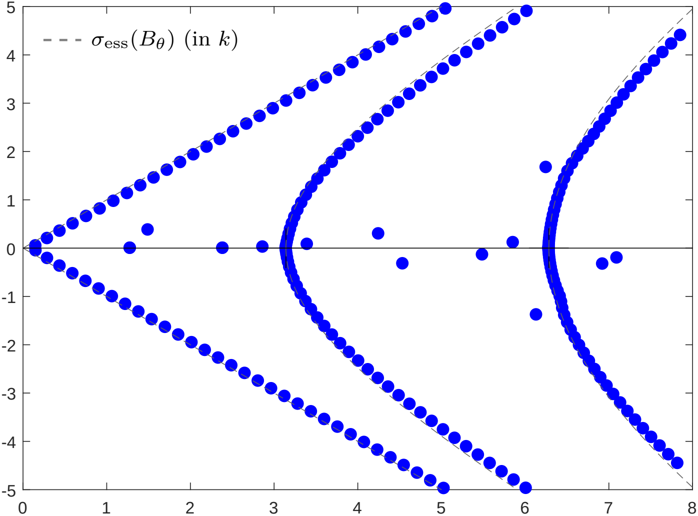
- Now the geometry is not symmetric in $x$ nor in $y$:



- The operator $B_\theta$ is no longer $\mathcal{PT}$-symmetric and we expect:
  - No trapped modes
  - No invariance of the spectrum by complex conjugation.

# Numerical results

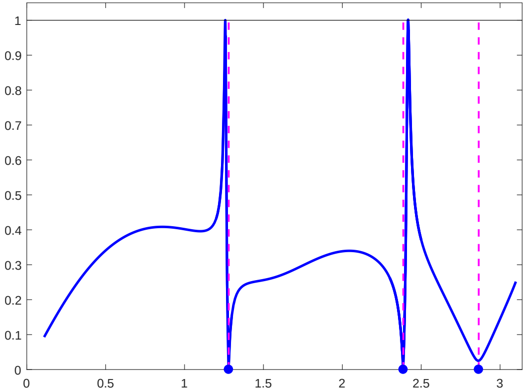
- Discretized spectrum of $B_\theta$ in $k$ (not in $k^2$). We take $\theta = \pi/4$.




- Indeed, the spectrum is not symmetric w.r.t. the real axis.

# Numerical results

- We compute $k \mapsto |R(k)|$ for $0 < k < \pi$.




$k = 1.28 + 0.0003i$

$k = 2.3866 + 0.0005i$

$k = 2.8647 + 0.0243i$

# Numerical results

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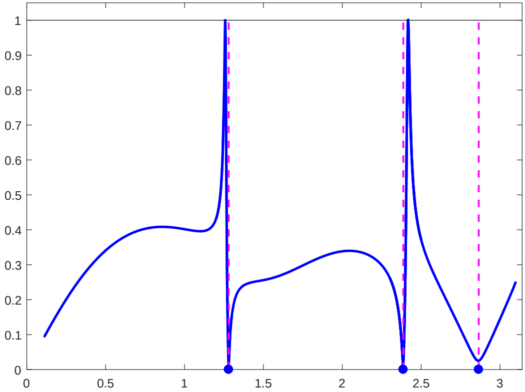


$k = 1.28 + 0.0003i$

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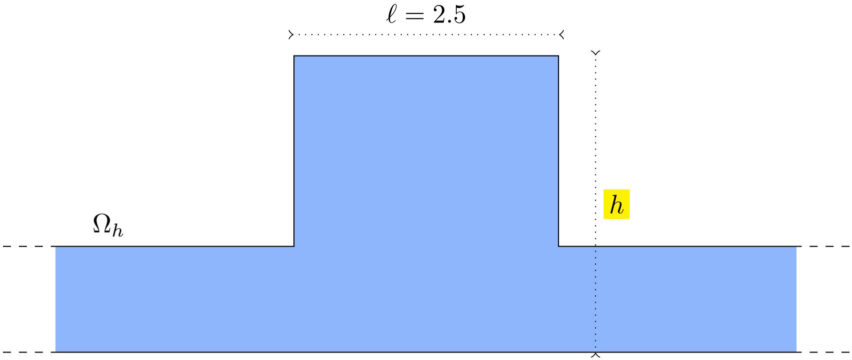
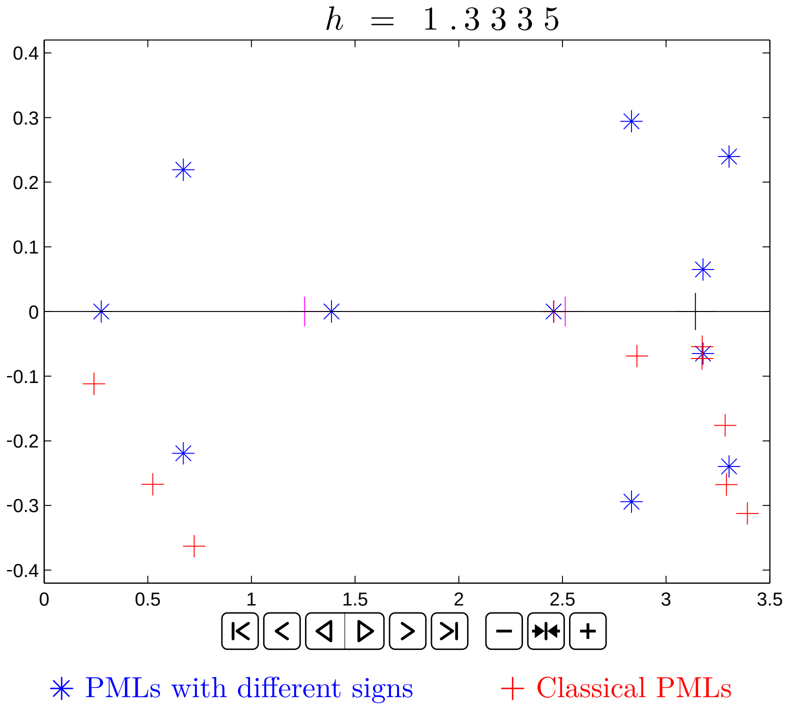
**Complex eigenvalues** also contain information on almost no reflection.

# Spectra for a changing geometry

- Two series of computations: one with PMLs with different sign, one with classical PMLs. We compute the spectra for $h \in (1.3; 8)$.




- The magenta marks on the real axis correspond to $k = \pi/\ell$ & $k = 2\pi/\ell$. For $k = 2\pi/\ell$, trapped modes and $T = 1$ should occur for certain $h$.
- We zoom at the region $0 < \Re e\, k < \pi$.


$h = 1.3335$
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
0.5
1
1.5
2
2.5
3
3.5
PMLs with different signs
Classical PMLs

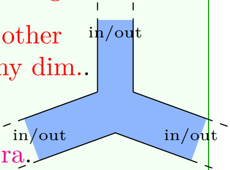
# Outline of the talk

# Conclusion

## What we did

- We presented an approach to compute non reflecting $k$ (values of $k$ s.t. there is a $v_i$ whose $v_s$ is exp. decaying at $-\infty$) for a given $\Omega$.
- The technique works with other B.C. (Dirichlet, ...), other kinds of perturbation (penetrable obstacles, ...), in any dim..

With $N$ leads, $2^N$ in/out spectra:
1 purely in, 1 purely out, $2^N - 2$ non reflecting spectra.



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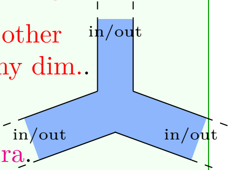
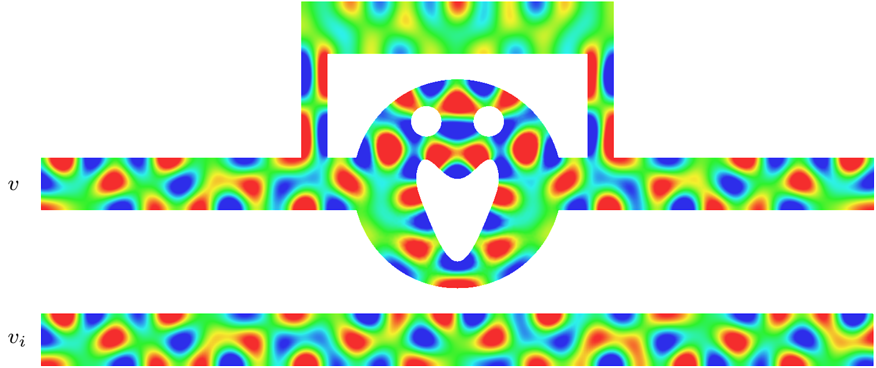



## Future work

1) How to justify the numerics? Absence of spectral pollution?
2) Can we find a spectral approach to compute completely reflecting or completely invisible $k$ for a given geometry?
3) Can we find a spectral approach to identify modal conversion?
4) Can we prove existence of non reflecting $k$ for the $\mathcal{PT}$-symmetric pb?




**Thank you for your attention!**