Séminaire de Mathématiques Appliquées

Invisibilité en champ lointain pour un problème de diffraction acoustique

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Coll. with A.-S. Bonnet-Ben Dhia[2], H. Haddar[1] and S.A. Nazarov[3].

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General setting

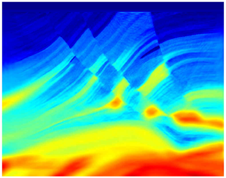
► We are interested in methods based on the propagation of waves to determine the shape, the physical properties of objects, in an exact or qualitative manner, from given measurements.

► General principle of the methods:

i) send waves in the medium;

ii) measure the scattered field;

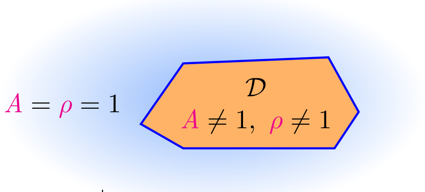
iii) deduce information on the structure.



- Many techniques: Xray, ultrasound imaging, seismic tomography, ...
- Many applications: biomedical imaging, non destructive testing of materials, geophysics, ...

Model problem

- Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients A, ρ) in $\mathbb{R}^2$.

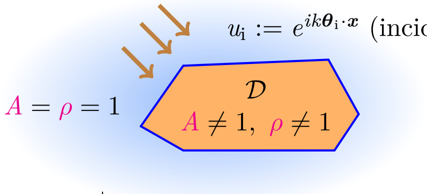


$$\left| \begin{array}{rcll} \multicolumn{4}{l}{\text{Find } u \text{ such that}} \\ -\mathrm{div}\,(A \nabla u) & = & k^2 \rho \, u & \text{in } \mathbb{R}^2, \\ u & = & u_{\mathrm{i}} + u_{\mathrm{s}} & \text{in } \mathbb{R}^2, \\ \multicolumn{4}{c}{\displaystyle \lim_{r \to +\infty} \sqrt{r} \left(\frac{\partial u_{\mathrm{s}}}{\partial r} - ik u_{\mathrm{s}} \right) = 0.} \end{array} \right. \qquad (1)$$

Model problem

- Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients A, ρ) in $\mathbb{R}^2$.

$u_{\mathrm{i}} := e^{ik\boldsymbol{\theta}_{\mathrm{i}}\cdot\boldsymbol{x}}$ (incident dir. $\boldsymbol{\theta}_{\mathrm{i}} \in \mathbb{S}^1$)



Find u such that

$$\begin{aligned} -\operatorname{div}(A\nabla u) &= k^2\rho\, u \quad \text{in } \mathbb{R}^2,\\ u &= u_{\mathrm{i}} + u_{\mathrm{s}} \quad \text{in } \mathbb{R}^2,\\ \lim_{r\to+\infty}\sqrt{r}\left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}}\right) &= 0. \end{aligned} \tag{1}$$

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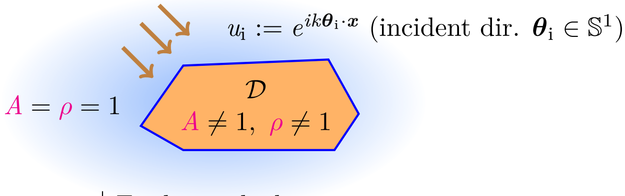

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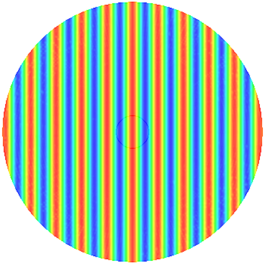
Definition:
$u_{\rm i}$ = incident field (datum)
u = total field (uniquely defined by (1))
$u_{\rm s}$ = scattered field (uniquely defined by (1)).

Illustration of the scattering of a plane wave

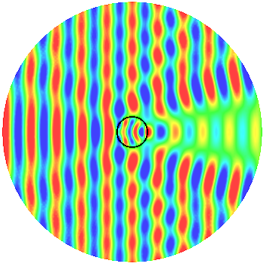
► Below, the movies represent a numerical approximation of the solution of the previous problem.

Incident field



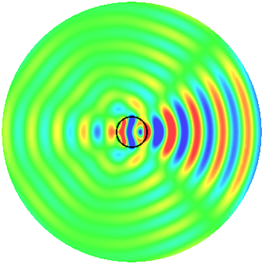
$t \mapsto \Re e\,(e^{-i\omega t} u_{\mathrm{i}}(\boldsymbol{x}))$

Total field



$t \mapsto \Re e\,(e^{-i\omega t} u(\boldsymbol{x}))$

Scattered field



$t \mapsto \Re e\,(e^{-i\omega t} u_{\mathrm{s}}(\boldsymbol{x}))$

► The pulsation ω is defined by $\omega = k/c$ where $c = 1$ is the celerity of the waves in the homogeneous medium.

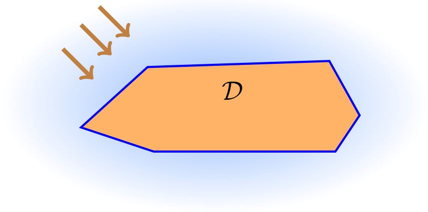
Outline of the talk

We are interested by defects that cannot be detected and by invisibility.

- For a given obstacle, is there an incident field that does not scatter?
- And when there is only a finite number of emitters/receivers?
- For a given obstacle and a finite number of emitters/receivers, how to build invisible obstacles?

Presentation of the ITEP

- We look for an incident field that does not scatter.



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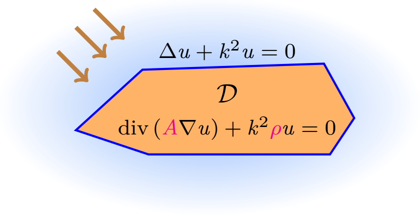

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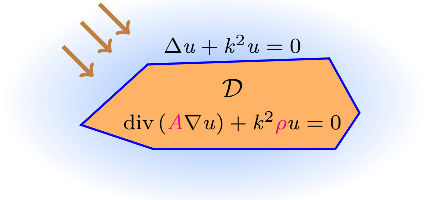

- This leads to study the interior transmission eigenvalue problem (Kirsch 86, Colton & Monk 88):

  ☛ u is the total field in $\mathcal{D}$

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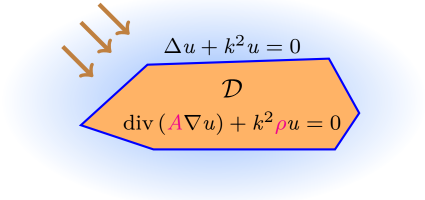

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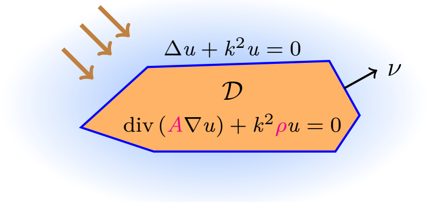

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BCs? $\left|\begin{array}{ll} [u] = 0 & \text{on } \partial\mathcal{D} \\ [\nu\cdot A\nabla u] = 0 & \text{on } \partial\mathcal{D} \end{array}\right. \quad + \quad u = w + 0 \text{ in } \mathbb{R}^2 \setminus \mathcal{D}.$

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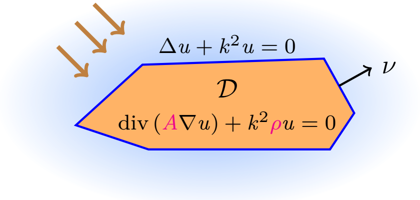

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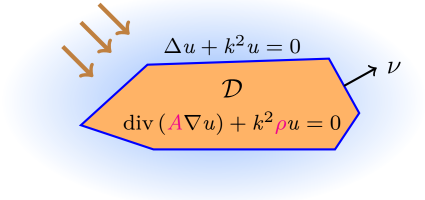

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TRANS. COND. ON $\partial\mathcal{D}$

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What framework for the ITEP?

- Generalized combination of incident plane waves:

$$w(\boldsymbol{x}) = \sum_{n=1}^{N} \alpha_n e^{ik\boldsymbol{\theta}_i^n \cdot \boldsymbol{x}} \quad \Rightarrow \quad w(\boldsymbol{x}) = \underbrace{\int_{\mathbb{S}^1} g(\boldsymbol{\theta}_i) e^{ik\boldsymbol{\theta}_i \cdot x} \, d\boldsymbol{\theta}_i,}_{\text{Herglotz wave function}} \qquad g \in \mathrm{L}^2(\mathbb{S}^1).$$

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$$\left|\begin{array}{lcll} \multicolumn{4}{l}{\text{Find } (u, w) \in \mathrm{H}^1(\mathcal{D}) \times \mathrm{H}^1(\mathcal{D}) \text{ such that}} \\ \operatorname{div}(A\nabla u) + k^2 \rho u & = & 0 & \text{in } \mathcal{D} \\ \Delta w + k^2 w & = & 0 & \text{in } \mathcal{D} \\ u - w & = & 0 & \text{in } \partial\mathcal{D} \\ \nu \cdot A\nabla u - \nu \cdot \nabla w & = & 0 & \text{in } \partial\mathcal{D}. \end{array}\right.$$

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- For transmission eigenvalues, there are generalized combination of incident planes waves which produce arbitrarily small scattered fields.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X}\backslash\{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\int_{\mathcal{D}} A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'}\, d\boldsymbol{x} \;=\; k^2 \int_{\mathcal{D}} (\rho u \overline{u'} - w \overline{w'})\, d\boldsymbol{x},$$

with $\mathrm{X} = \{(u, w) \in \mathrm{H}^1(\mathcal{D}) \times \mathrm{H}^1(\mathcal{D}) \,|\, u - w \in \mathrm{H}^1_0(\mathcal{D})\}$.

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- Define on $\mathrm{X} \times \mathrm{X}$ the sesquilinear form

$$a_k((u, w), (u', w')) = \int_{\mathcal{D}} A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} - k^2(\rho u\overline{u'} - w\overline{w'})\, d\boldsymbol{x}.$$

Study of ITEP 1/2

- First, we consider the source term problem.
- Let $\mathtt{T}$ be an isomorphism of X. For $\ell \in \mathrm{X}'$, we have

$$[\quad (\mathscr{P}_V) \quad a_k((u,w),(u',w')) = \ell(u',w'), \quad \forall (u',w') \in \mathrm{X}]$$

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Study of ITEP 1/2

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Proposition. Suppose that $A > 1$ and $\rho > 1$. Then the set of transmission eigenvalues is discrete and countable.

- This approach also works only assuming that $A - 1$ and $n - 1$ have a constant sign in a neighbourhood of $\partial\mathcal{D}$.

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Find $(u,w) \in \mathrm{X}$ such that:

$$
\int_{\mathcal{D}} A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'}\, d\boldsymbol{x} = k^2 \int_{\mathcal{D}} (\rho u\overline{u'} - w\overline{w'})\, d\boldsymbol{x}, \quad \forall (u',w') \in \mathrm{X}.
$$

When $A = 1$, the principal symbol vanishes. It is necessary to modify the functional framework (also sometimes when $A - 1$ changes sign on $\partial\mathcal{D}$).

Many other interesting questions

- Recently, other topics have been considered:
  - existence of real and complex transmission eigenvalues;
  - localization of transmission eigenvalues in the complex plane;
  - Weyl laws for the transmission eigenvalues;
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but many questions remain open (see the recent review `F. Cakoni, H. Haddar, Transmission Eigenvalues in Inverse Scattering Theory, Inside Out II, 60, MSRI Publi., 527-578, 2012`).

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By the way, why such a detailed study ?

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By the way, why such a detailed study ?

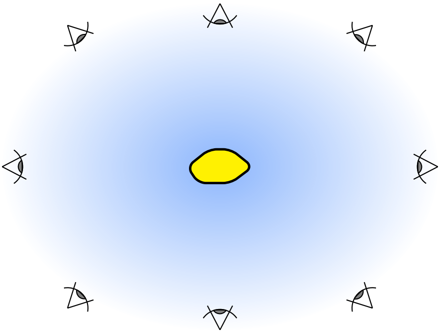
- Certain reconstruction methods need to avoid transmission eigenvalues.
- But transmission eigenvalues can also be determined from measurements and they carry information about the inclusion properties.

$\Rightarrow$ *They can be used to find qualitative properties of the object.*

Problematic

- In the previous section, it was assumed that one can produce incident plane waves and measure the resulted scattered fields in all directions of $\mathbb{S}^1$.

- In practice, one has always a finite number of emitters/receivers.



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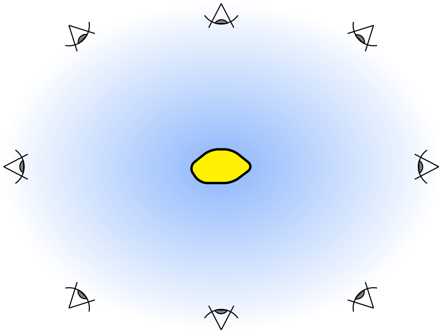

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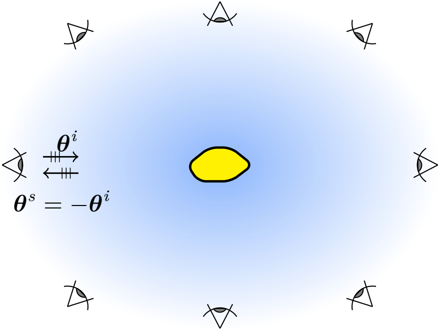

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$\Rightarrow$ Multistatic backscattering measurements: emitters & receivers coincide.

Far field pattern

- For a given incident direction $\boldsymbol{\theta}_{\mathrm{i}}$, the scattered field $u_{\mathrm{s}}(\cdot, \boldsymbol{\theta}_{\mathrm{i}})$ admits the asymptotic expansion

$$u_{\mathrm{s}}(\boldsymbol{x}, \boldsymbol{\theta}_{\mathrm{i}}) = \frac{e^{ikr}}{\sqrt{r}} \left(\; u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_{\mathrm{s}}, \boldsymbol{\theta}_{\mathrm{i}}) \; + O(1/r) \right)$$

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The far field pattern is the quantity one can measure at infinity (the other terms are too small).

- In practice, the goal of imaging techniques is to find features of the inclusion from the knowledge of $u_{\mathrm{s}}^{\infty}(\cdot, \cdot)$ on a finite subset of $\mathbb{S}^1 \times \mathbb{S}^1$.

Relative scattering matrix

- Let $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ be given directions of $\mathbb{S}^1$. We introduce the relative scattering matrix $\mathscr{S}(k) \in \mathbb{C}^{N\times N}$ defined via

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  - these incident field can be constructed exactly (no density argument);
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- Unlike in the continuous setting, this problem does not reduce to a problem set on the (compact) support of the inclusion.

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- Open questions:

- How to proceed to prove discreteness of transmission eigenvalues for situations other than multistatic backscattering measurements?

- Can we relax assumptions on A and ρ?

- Can we prove existence of transmission eigenvalues in this setting?

- Do transmission eigenvalues in the discrete setting (if they exist) converge to the transmission eigenvalues of the continuous framework when the number of directions tends to $+\infty$?

- ...

Invisible inclusions: setting

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We explain how to construct non trivial inclusions such that $\mathscr{S}(k) = 0$. These inclusions cannot be detected from far field measurements.

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FORMULATION OF THE PROBLEM:

Find a real valued function $\rho \not\equiv 1$, with $\rho - 1$ supported in $\mathcal{D}$, such that the solution of the problem

$$\left|\begin{array}{l} \text{Find } u = u_{\mathrm{s}} + e^{ik\boldsymbol{\theta}_{\mathrm{i}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u \;=\; k^2\rho\, u \quad \text{in } \mathbb{R}^2, \\ \displaystyle\lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}}\right) = 0 \end{array}\right.$$

verifies $u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_1) = \dots = u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_N) = 0$.

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Origin of the method:

- The idea we will use has been introduced in Nazarov 11 to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.

- It has been adapted in Bonnet-Ben Dhia & Nazarov 13 to build invisible perturbations of waveguides (see also Bonnet-Ben Dhia, Nazarov & Taskinen 14 for an application to a water-waves problem).

General scheme: step 1

When there is no inclusion, there is no scattered field. Let us look for ρ as a perturbation of the reference coefficient:

$$\boxed{\rho^\varepsilon = 1 + \varepsilon\mu}, \qquad \text{with } \mu \text{ compactly supported.}$$

- We denote u^ε, u_s^ε the functions satisfying

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with $u_{\mathrm{s}}^{\varepsilon\,\infty}(\boldsymbol{\theta}_{\mathrm{s}}) = c\,k^2\displaystyle\int_{\mathcal{D}}(\rho^\varepsilon - 1)\,(u_{\mathrm{i}} + u_{\mathrm{s}}^\varepsilon)\,e^{-ik\boldsymbol{\theta}_{\mathrm{s}}\cdot\boldsymbol{x}}\,d\boldsymbol{x}$ $\qquad\left(c = \dfrac{e^{i\pi/4}}{\sqrt{8\pi k}}\right)$.

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- With this choice, we obtain the expansion, for small ε

$$u_{\mathrm{s}}^{\varepsilon\,\infty}(\boldsymbol{\theta}_{\mathrm{s}}) = 0 + \varepsilon\, c\, k^2 \int_{\mathcal{D}} \mu\, e^{ik(\boldsymbol{\theta}_{\mathrm{i}} - \boldsymbol{\theta}_{\mathrm{s}})\cdot\boldsymbol{x}}\, d\boldsymbol{x} + O(\varepsilon^2).$$

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General scheme: step 2

- In the expression $\rho^\varepsilon = 1 + \varepsilon\mu$, we redecompose μ as

$$\boxed{\mu = \mu_0 + \sum_{m=1}^{N} \tau_{1,m}\,\mu_{1,m} + \sum_{m=1}^{N} \tau_{2,m}\,\mu_{2,m}}$$

where | $\tau_{1,m}$, $\tau_{2,m}$ are real parameters that we will tune

| $\mu_0 \not\equiv 0$, $\mu_{1,m}$, $\mu_{2,m}$ are given real valued functions supp. on $\overline{\mathcal{D}}$ s.t.

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$$\int_{\mathcal{D}} \mu_{1,m} \cos(k(\boldsymbol{\theta}_{\mathrm{i}} - \boldsymbol{\theta}_n)\cdot \boldsymbol{x})\, d\boldsymbol{x} = \delta^{mn}, \qquad \int_{\mathcal{D}} \mu_{1,m} \sin(k(\boldsymbol{\theta}_{\mathrm{i}} - \boldsymbol{\theta}_n)\cdot \boldsymbol{x})\, d\boldsymbol{x} = 0$$

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\int_{\mathcal{D}} \mu_{2,m} \cos(k(\boldsymbol{\theta}_{\mathrm{i}} - \boldsymbol{\theta}_n)\cdot \boldsymbol{x})\, d\boldsymbol{x} &= 0, & \int_{\mathcal{D}} \mu_{2,m} \sin(k(\boldsymbol{\theta}_{\mathrm{i}} - \boldsymbol{\theta}_n)\cdot \boldsymbol{x})\, d\boldsymbol{x} &= \delta^{mn}.
\end{aligned}$$

We introduce $2N$ real parameters because we want to cancel N complex coefficients.

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Why this choice ? ...

General scheme: step 3

- ... because then we find

$$u_{\mathrm{s}}^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = \varepsilon\, c\, k^2\, (\tau_{1,n} + i\tau_{2,n}) \ + \varepsilon^2 c\, k^2\, (F^{\varepsilon}_{1,n}(\vec{\tau}) + iF^{\varepsilon}_{2,n}(\vec{\tau})),$$

where $F^{\varepsilon}_{1,n}$, $F^{\varepsilon}_{2,n}$ are real-valued functions depending (non linearly) on ε, $\vec{\tau} := (\tau_{1,1}, \ldots, \tau_{1,N}, \tau_{2,1}, \ldots, \tau_{2,N})^{\top}$.

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Use the term at order ε whose dependence with respect to ρ is simple to control and cancel the whole expansion.

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- Now, we can impose $\boxed{u_s^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = 0}$ solving the fixed point problem:

$$\boxed{\text{Find } \vec{\tau} \in \mathbb{R}^{2N} \text{ such that } \vec{\tau} = F^{\varepsilon}(\vec{\tau}),} \tag{2}$$

with $F^{\varepsilon}(\vec{\tau}) := -\varepsilon\,(F^{\varepsilon}_{1,1}(\vec{\tau}), \dots, F^{\varepsilon}_{1,N}(\vec{\tau}), F^{\varepsilon}_{2,1}(\vec{\tau}), \dots, F^{\varepsilon}_{2,N}(\vec{\tau}))^{\top}$.

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- We can prove that the map $F^{\varepsilon} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ verifies the estimate $|F^{\varepsilon}(\vec{\tau}) - F^{\varepsilon}(\vec{\tau}')| \le C\,\varepsilon\, |\vec{\tau} - \vec{\tau}'|$. Therefore F^{ε} is a contraction for ε small enough and (2) has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.

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Proposition: For ε small enough, define $\rho^{\text{sol}} = 1 + \varepsilon\mu^{\text{sol}}$ with

$$\mu^{\text{sol}} = \mu_0 + \sum_{m=1}^{N} \tau_{1,m}^{\text{sol}}\, \mu_{1,m} + \sum_{m=1}^{N} \tau_{2,m}^{\text{sol}}\, \mu_{2,m}.$$

Then the solution of the scattering problem

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u_{\text{s}}^\varepsilon + e^{ik\boldsymbol{\theta}_{\text{i}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u \quad = \quad k^2 \rho^{\text{sol}}\, u \quad \text{ in } \mathbb{R}^2, \\ \displaystyle \lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u_{\text{s}}}{\partial r} - iku_{\text{s}}\right) = 0 \end{array}\right.$$

verifies $u_{\text{s}}^\infty(\boldsymbol{\theta}_1) = \cdots = u_{\text{s}}^\infty(\boldsymbol{\theta}_N) = 0$.

with $F^\varepsilon(\vec{\tau}) := -\varepsilon\,(F_{1,1}(\vec{\tau}), \dots, F_{1,N}(\vec{\tau}), F_{2,1}(\vec{\tau}), \dots, F_{2,N}(\vec{\tau}))^\top$.

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How to build the shape functions?

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$$\int_{\mathcal{D}} \mu_{1,m} \cos(k(\boldsymbol{\theta}_{\mathrm{i}} - \boldsymbol{\theta}_n) \cdot \boldsymbol{x}) \, d\boldsymbol{x} = \delta^{mn}, \quad \int_{\mathcal{D}} \mu_{1,m} \sin(k(\boldsymbol{\theta}_{\mathrm{i}} - \boldsymbol{\theta}_n) \cdot \boldsymbol{x}) \, d\boldsymbol{x} = 0$$

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$$e_n(\boldsymbol{x}) = \cos(k(\boldsymbol{\theta}_{\mathrm{i}} - \boldsymbol{\theta}_n) \cdot \boldsymbol{x}) \qquad \text{and} \qquad e_{N+n}(\boldsymbol{x}) = \sin(k(\boldsymbol{\theta}_{\mathrm{i}} - \boldsymbol{\theta}_n) \cdot \boldsymbol{x}).$$

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On $\mathcal{D}$, $\{e_n\}_{n=1}^{2N}$ is linearly independent $\Rightarrow$ the matrix $\mathbb{B} \in \mathbb{R}^{2N \times 2N}$ s.t.

$$\mathbb{B}_{mn} = \int_{\mathcal{D}} e_m(\boldsymbol{x}) e_n(\boldsymbol{x}) \, d\boldsymbol{x}$$

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$$\boxed{\mu_{1,m} = \sum_{n=1}^{2N} \mathbb{D}_{mn} \, e_n} \qquad \text{and} \qquad \boxed{\mu_{2,m} = \sum_{n=1}^{2N} \mathbb{D}_{(N+m)n} \, e_n}$$

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where $\mu_0^{\#} \notin \operatorname{span}\{\mu_{1,1}, \ldots, \mu_{1,N}, \mu_{2,1}, \ldots, \mu_{2,N}\}$.

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- The inclusion can be built and does not involve singular materials ($\neq$ cloaking techniques). Moreover, μ^{sol} is just a small perturbation of μ_0:

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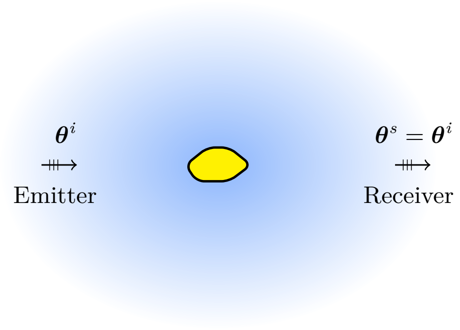

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- No solution if $\mathcal{D}$ has corners and under certain assumptions on ρ.

- Corners always scatter, E. Blåsten, L. Päivärinta, J. Sylvester, 2014
- Corners and edges always scatter, J. Elschner, G. Hu, 2015

- And if $\mathcal{D}$ is smooth? $\Rightarrow$ The problem seems open.

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Data and algorithm

- We can solve the fixed point problem using an iterative procedure: we set $\vec{\tau}^{\,0} = (0, \dots, 0)^\top$ then define

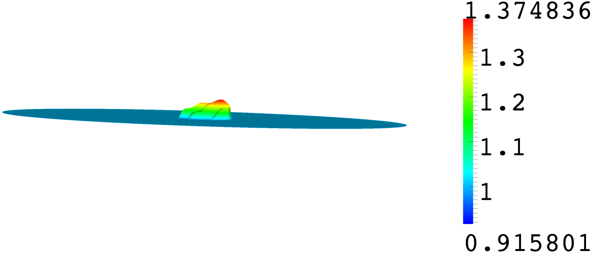
$$\boxed{\vec{\tau}^{\,n+1} = F^{\varepsilon}(\vec{\tau}^{\,n}).}$$

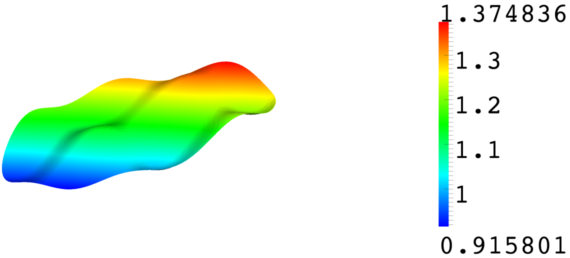
- At each step, we solve a scattering problem. We use a P2 finite element method set on the ball B_8. On $\partial\mathrm{B}_8$, a truncated Dirichlet-to-Neumann map with 13 harmonics serves as a transparent boundary condition.

- For the numerical experiments, we take $\mathcal{D} = \mathrm{B}_1$, $M = 3$ (3 directions of observation) and

$$\left|\begin{array}{lll} \boldsymbol{\theta}_{\mathrm{i}} &= (\cos(\psi_{\mathrm{i}}), \sin(\psi_{\mathrm{i}})), & \psi_{\mathrm{i}} = 0^\circ \\ \boldsymbol{\theta}_{\mathrm{s}}^1 &= (\cos(\psi_{\mathrm{s}}^1), \sin(\psi_{\mathrm{s}}^1)), & \psi_{\mathrm{s}}^1 = 90^\circ \\ \boldsymbol{\theta}_{\mathrm{s}}^2 &= (\cos(\psi_{\mathrm{s}}^2), \sin(\psi_{\mathrm{s}}^2)), & \psi_{\mathrm{s}}^2 = 180^\circ \\ \boldsymbol{\theta}_{\mathrm{s}}^1 &= (\cos(\psi_{\mathrm{s}}^3), \sin(\psi_{\mathrm{s}}^3)), & \psi_{\mathrm{s}}^3 = 225^\circ \end{array}\right.$$

Results: coefficient ρ at the end of the process





Results: scattered field

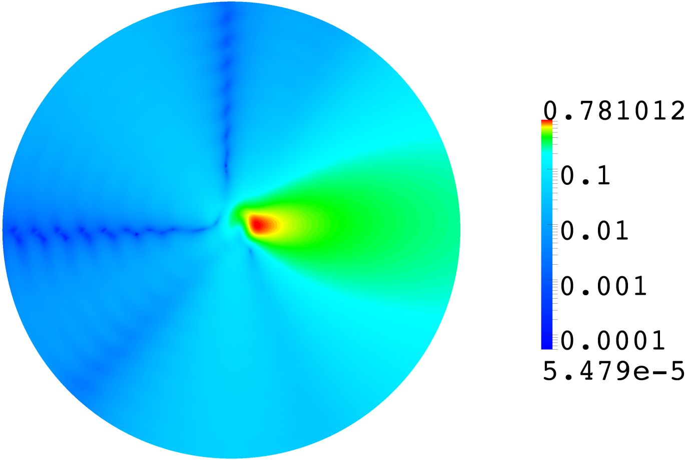


Figure: $|u_\mathrm{s}|$ at the end of the fixed point procedure in logarithmic scale. As desired, we see it is very small far from $\mathcal{D}$ in the directions corresponding to the angles 90°, 180° and 225°. The domain is equal to B_8.

Results: far field pattern

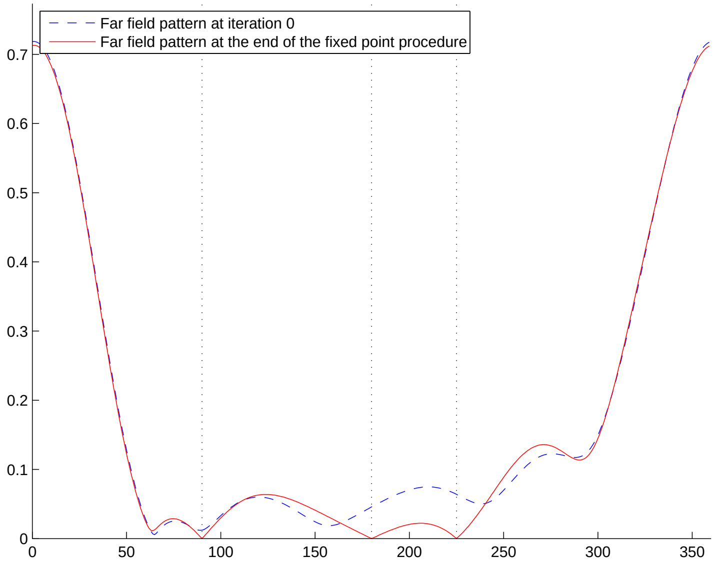


Figure: The dotted lines show the directions where we want u_s^∞ to vanish.

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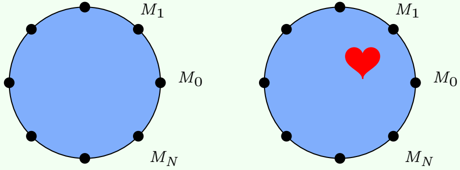

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Find $\sigma \not\equiv 1$, with $\operatorname{supp}\sigma \subset \Omega$, s.t.

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for $m, n = 1, \ldots, N$.

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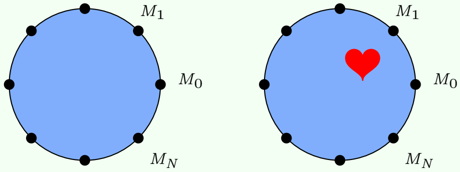

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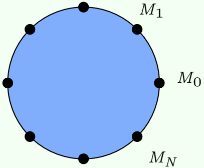

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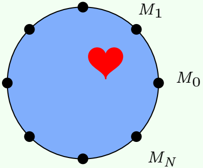

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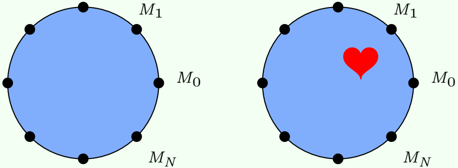

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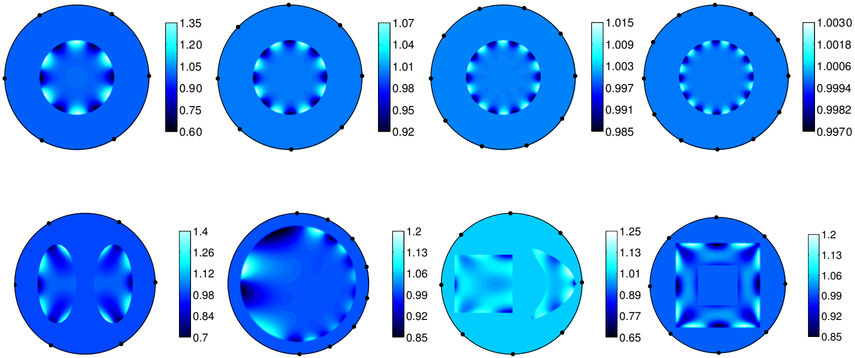
Numerical results

Examples of conductivities which provide the same measurements as the reference conductivity $\sigma = 1$.



- The dots corresponds to the positions of the electrodes.

Conclusion

Interior transmission eigenvalue problem

For a given obstacle, is there an incident field that does not scatter?

- Continuous setting: non classical spectral problem.
  $\Rightarrow$ Abundant literature but still many open questions.
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Invisibility

For a given frequency, how to build an invisible obstacle?

♠ Continuous setting: impossible! The knowledge of the far field operator on $\mathbb{S}^1 \times \mathbb{S}^1$ uniquely determine the parameter of the inclusion (`Sylvester & Uhlmann 87`, `Bukhgeim 08`).

♠ Finite number of emitters/receivers: we presented a method.
⇒ An important issue: can we reiterate the process to construct larger defects in the reference medium? *Work in progress...*

Thank you for your attention.

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