Spectrum for a small inclusion of negative material

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Coll. with X. Claeys[2] and S.A. Nazarov[3].

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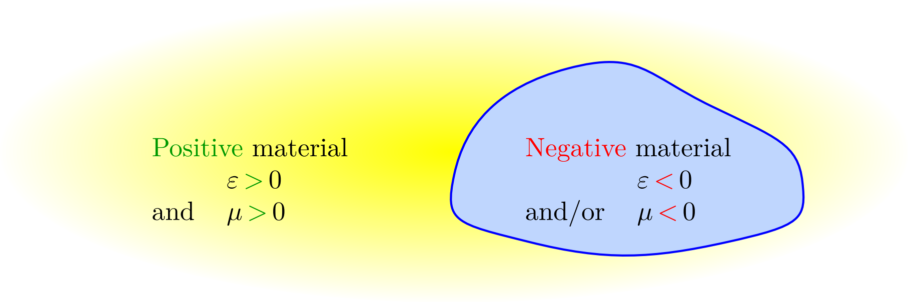
and Technology

Introduction: general setting

► Scattering by a negative material in electromagnetism in time-harmonic regime (at a given frequency):



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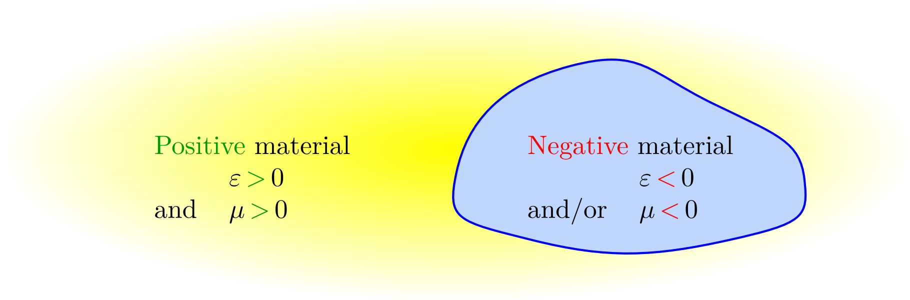

Do such negative materials occur in practice?

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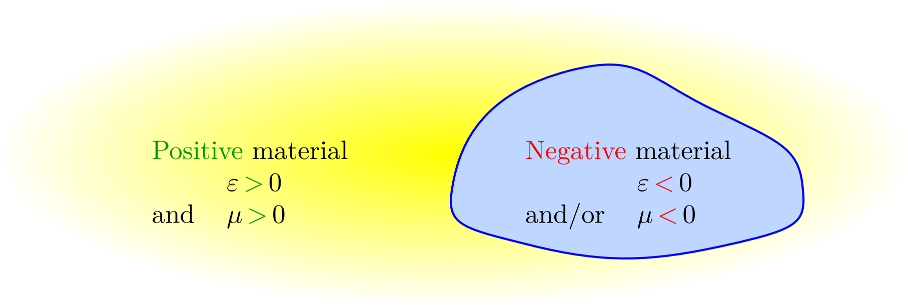

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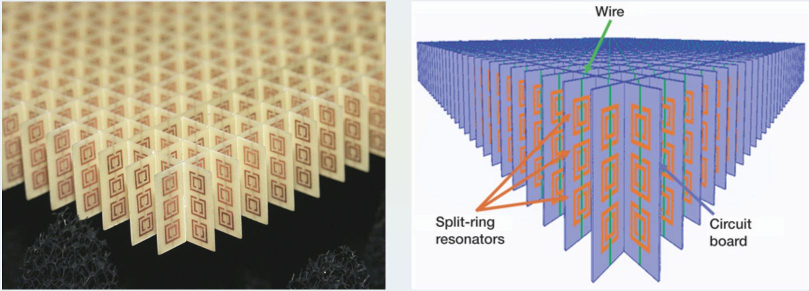
Zoom on a metamaterial: practical realizations of metamaterials are achieved by a periodic assembly of small resonators.



Example of metamaterial (NASA)

Mathematical justification of the homogenized model (`Bouchitté, Bourel, Felbacq 09`).

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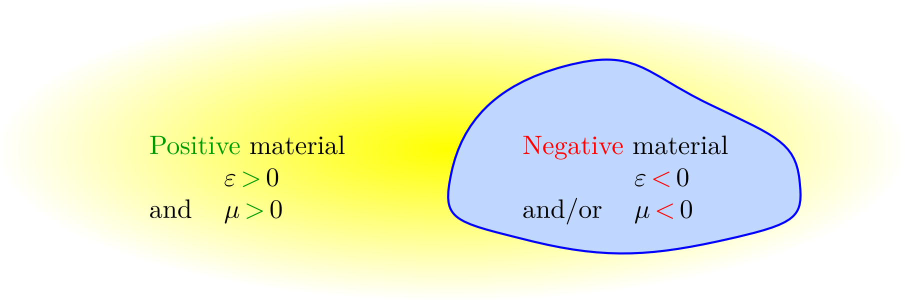

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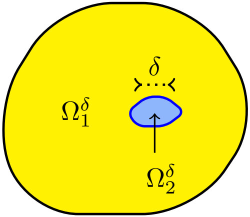
- In this talk, we investigate a Dirichlet spectral problem for a small inclusion of negative material in a bounded domain.

- Let Ω, ω be smooth domains of $\mathbb{R}^3$ such that $O \in \omega$, $\overline{\omega} \subset \Omega$. For $\delta \in (0; 1]$, we consider the problem

$$\left| \begin{array}{l} \text{Find } (\lambda^\delta, u^\delta) \in \mathbb{C} \times (\mathrm{H}^1_0(\Omega) \setminus \{0\}) \text{ s.t.:} \\ -\mathrm{div}(\sigma^\delta \nabla u^\delta) = \lambda^\delta u^\delta \quad \text{in } \Omega, \text{ with,} \end{array} \right.$$



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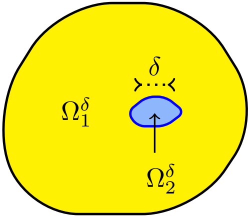

- $\mathrm{H}^1_0(\Omega) := \{u \in \mathrm{H}^1(\Omega) \,|\, u = 0 \text{ on } \partial\Omega\}$

- $\sigma^\delta = \left|\begin{array}{lll}\sigma_1 > 0 & \text{in} & \Omega_1^\delta := \Omega \setminus \overline{\delta\,\omega}\\ \sigma_2 < 0 & \text{in} & \Omega_2^\delta := \delta\,\omega.\end{array}\right.$

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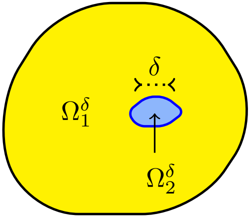

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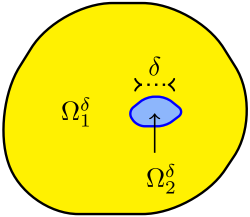

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► We define the operator $\mathrm{A}^\delta : D(\mathrm{A}^\delta) \to \mathrm{L}^2(\Omega)$ such that

$$\left|\begin{array}{l}D(\mathrm{A}^\delta) = \{u \in \mathrm{H}^1_0(\Omega) \,|\, \mathrm{div}(\sigma^\delta \nabla u) \in \mathrm{L}^2(\Omega)\}\\ \mathrm{A}^\delta u = -\mathrm{div}(\sigma^\delta \nabla u).\end{array}\right.$$

Introduction: main question of the talk

- Using boundary integral equations (see `Costabel and Stephan 85, Dauge and Texier 97`) or the T-coercivity approach (see `Bonnet-Ben Dhia` *`et al.`* `99,10,12,13`), we can prove the :

Proposition. Assume that $\sigma_2/\sigma_1 \neq -1$. For $\delta > 0$, the operator A^δ is selfadjoint and has compact resolvent. Its spectrum $\mathfrak{S}(\mathrm{A}^\delta)$ consists in two sequences of isolated eigenvalues:

$$-\infty \underset{n\to+\infty}{\leftarrow} \dots \lambda^\delta_{-n} \le \dots \le \lambda^\delta_{-1} < 0 \le \lambda^\delta_1 \le \lambda^\delta_2 \le \dots \le \lambda^\delta_n \dots \underset{n\to+\infty}{\rightarrow} +\infty.$$

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> ? What happens to the negative spectrum when δ tends to zero?

Outline of the talk

1. Limit operators

   We introduce the two natural limit operators which appear when $\delta \to 0$.

2. Results

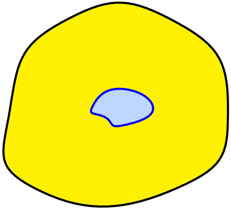
   We state the main results concerning the asymptotic behaviour of the eigenvalues when $\delta \to 0$.

3. Numerical experiments

   We illustrate the theoretical results with numerical experiments.

Far field operator

- As $\delta \to 0$, the small inclusion of negative material disappears.



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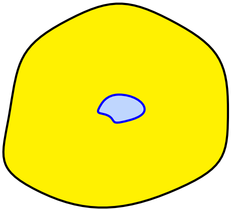

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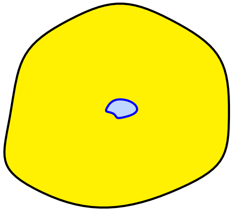

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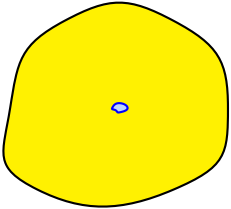

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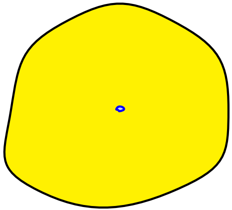

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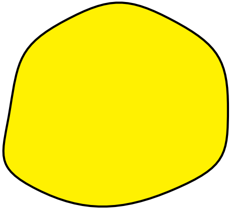

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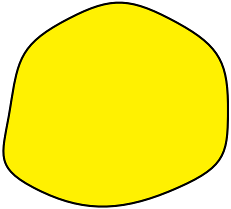


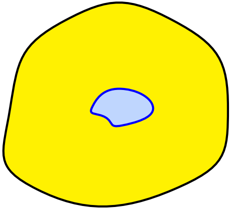
- We introduce the far field operator A^0 such that

$$\left|\begin{array}{l} D(\mathrm{A}^0) = \{v \in \mathrm{H}^1_0(\Omega) \,|\, \Delta v \in \mathrm{L}^2(\Omega)\} \\ \mathrm{A}^0 v = -\sigma_1 \Delta v. \end{array}\right.$$

PROPOSITION. There holds $\mathfrak{S}(\mathrm{A}^0) = \{\mu_n\}_{n\geq 1}$ with $0 < \mu_1 < \mu_2 \leq \dots \leq \mu_n \dots \underset{n\to+\infty}{\to} +\infty$.

Near field operator

- Introduce the rapid coordinate $\boldsymbol{\xi} := \delta^{-1}\boldsymbol{x}$ and let $\delta \to 0$.



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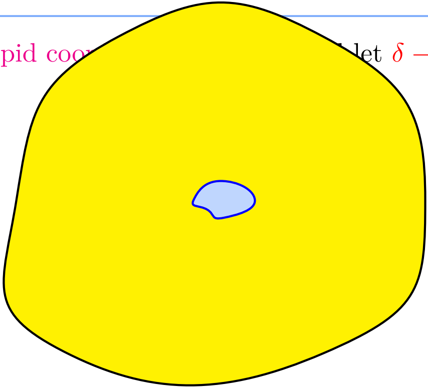

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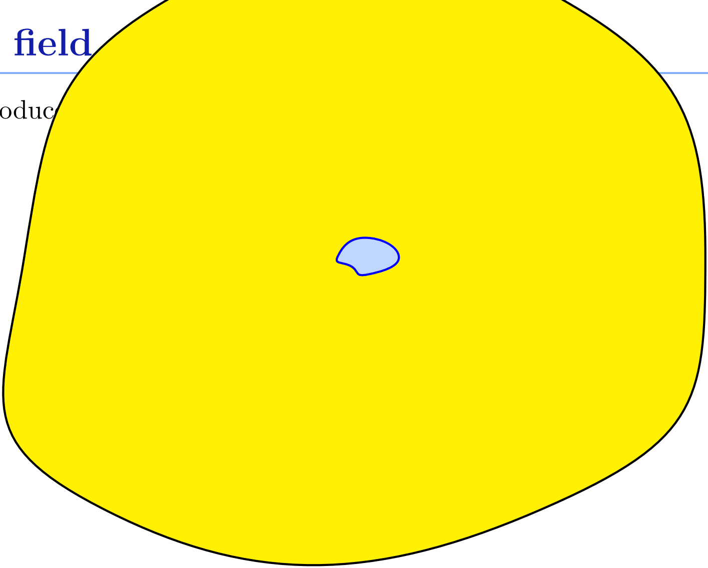

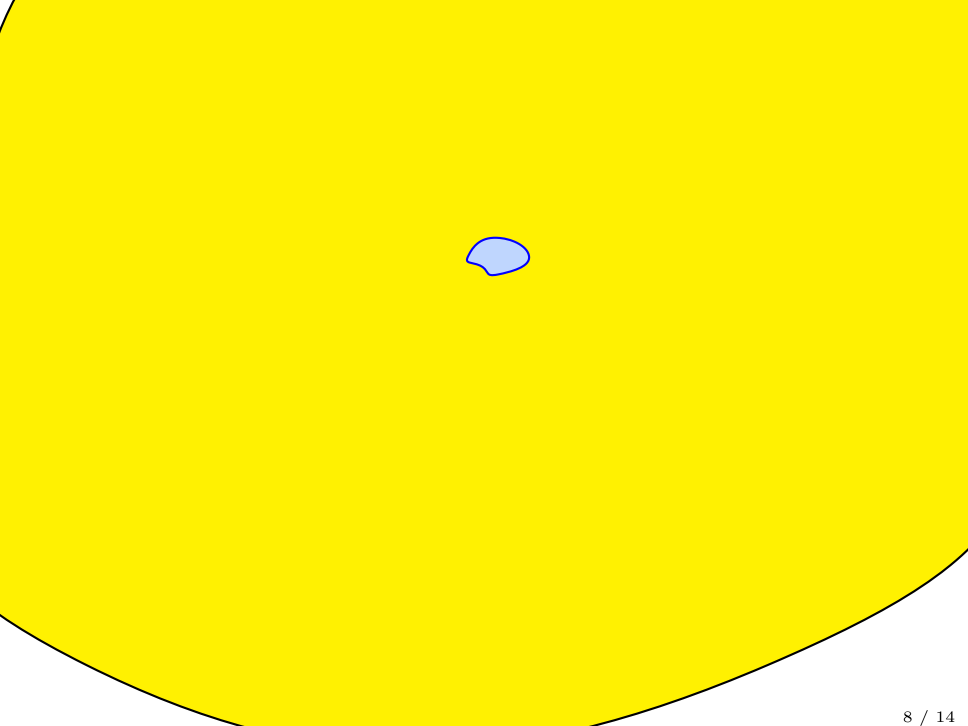

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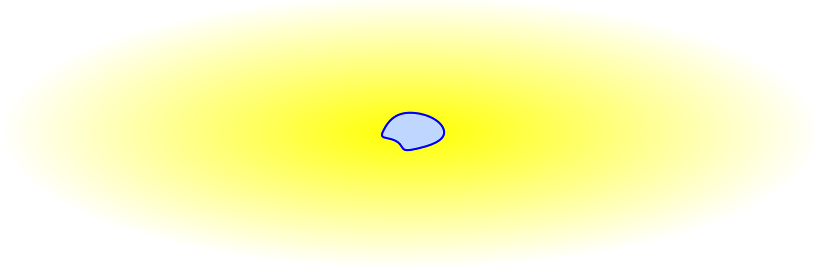

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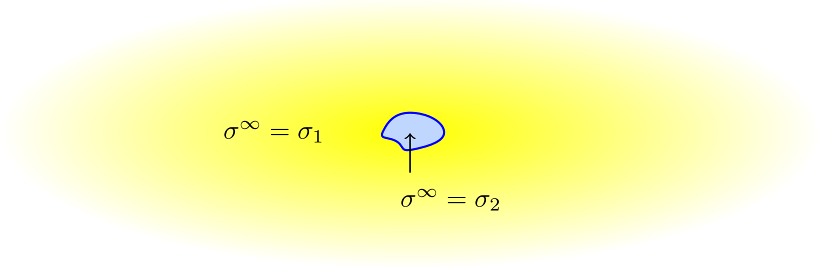

- Define the near field operator B^∞ such that

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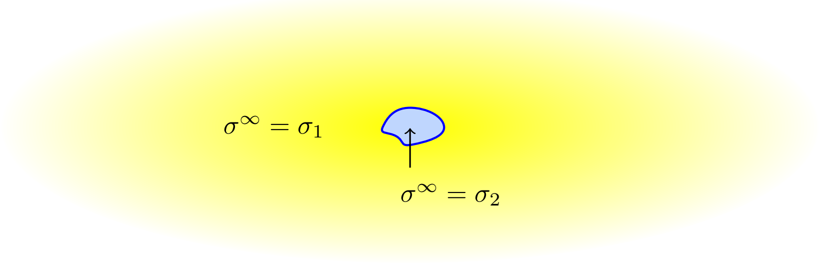

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PROPOSITION. Assume that $\sigma_2/\sigma_1 \neq -1$. The continuous spectrum of B^∞ is equal to $[0;+\infty)$ while its discrete spectrum is a sequence of eigenvalues:

$$\mathfrak{S}(\mathrm{B}^\infty) \setminus \overline{\mathbb{R}_+} = \{\mu_{-n}\}_{n\geq 1} \quad \text{with} \quad 0 > \mu_{-1} \geq \cdots \geq \mu_{-n} \ldots \underset{n\to+\infty}{\to} -\infty.$$

Spectrum for a small inclusion: results

Assume that $\sigma_2/\sigma_1 \neq -1$ and that B^∞ is injective. For $n \in \mathbb{N}^*$, we denote $\lambda^\delta_{\pm n}$, μ^δ_n, μ^δ_{-n} the eigenvalues of A^δ, A^0, B^∞ as in the previous slides.

Theorem. (Positive spectrum) For all $n \in \mathbb{N}^*$, $\varepsilon \in (0;1)$, there exist constants $C, \delta_0 > 0$ depending on n, ε but independent of δ, such that

$$|\lambda^\delta_n - \mu_n| \leq C\,\delta^{3/2-\varepsilon}, \qquad \forall \delta \in (0;\delta_0].$$

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Idea of the proof:

1. We prove the *a priori* estimate $\|u^\delta\|_{\mathrm{H}^1_0(\Omega)} \le c\,\|\mathrm{A}^\delta u^\delta\|_\Omega$ for δ small enough (♠ hard part of the proof: weighted Sobolev spaces+overlapping cut-off functions +`Nazarov's` technique).

2. If (μ_n, v_n) is an eigenpair of A^0, we construct u such that

$$\|\mathrm{A}^\delta u - \mu_n u\|_\Omega \le c\,\delta^\beta \|u\|_\Omega, \qquad \text{for some } \beta > 0.$$

3. If $(\lambda_n^\delta, u_n^\delta)$ is an eigenpair of A^δ, we construct v such that

$$\|\mathrm{A}^0 v - \lambda_n^\delta v\|_\Omega \le c\,\delta^\beta \|v\|_\Omega, \qquad \text{for some } \beta > 0.$$

4. We conclude with a classical lemma on quasi eigenvalues.

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Why is it a δ^{-2}?

- If $(\lambda^\delta_{-n}, u^\delta_{-n})$ is an eigenpair of A^δ, there holds

$$\int_\Omega \sigma^\delta \nabla_x u^\delta \cdot \nabla_x v\, dx = \lambda^\delta \int_\Omega u^\delta v\, dx, \quad \forall v \in \mathrm{H}^1_0(\Omega).$$

- $x = \delta\xi \Rightarrow \nabla_x = \delta^{-1}\nabla_\xi$. Denoting $U^\delta(\xi) = u^\delta(\delta\xi)$, we deduce

$$\int_{\delta^{-1}\Omega} \sigma^\infty \nabla_\xi U^\delta \cdot \nabla_\xi V\, d\xi = \delta^2\lambda^\delta \int_{\delta^{-1}\Omega} U^\delta V\, d\xi, \quad \forall V \in \mathrm{H}^1_0(\delta^{-1}\Omega).$$

Why the convergence is exponential?

- If (μ_{-n}, v_{-n}) is an eigenpair of B^∞, v_{-n} is exponentially decaying at ∞.

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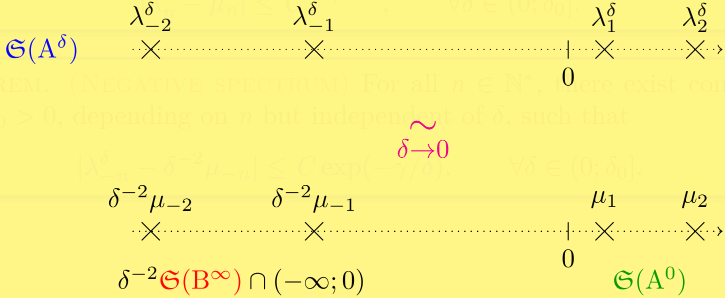
Schematically, we have:



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THEOREM. (POSITIVE SPECTRUM) For all $n \in \mathbb{N}^*$, $\varepsilon \in (0;1)$, there exist constants $C, \delta_0 > 0$ depending on n, ε but independent of δ, such that

$$|\lambda^\delta_n - \mu_n| \leq C\,\delta^{3/2-\varepsilon}, \qquad \forall \delta \in (0;\delta_0].$$

THEOREM. (NEGATIVE SPECTRUM) For all $n \in \mathbb{N}^*$, there exist constants $C, \gamma, \delta_0 > 0$, depending on n but independent of δ, such that

$$|\lambda^\delta_{-n} - \delta^{-2}\mu_{-n}| \leq C\exp(-\gamma/\delta), \qquad \forall \delta \in (0;\delta_0].$$

Spectrum for a small inclusion: results

Assume that $\sigma_2/\sigma_1 \neq -1$ and that B^∞ is injective. For $n \in \mathbb{N}^*$, we denote $\lambda^\delta_{\pm n}$, μ^δ_n, μ^δ_{-n} the eigenvalues of A^δ, A^0, B^∞ as in the previous slides.

Theorem. (Positive spectrum) For all $n \in \mathbb{N}^*$, $\varepsilon \in (0;1)$, there exist constants $C, \delta_0 > 0$ depending on n, ε but independent of δ, such that

$$|\lambda^\delta_n - \mu_n| \leq C\,\delta^{3/2-\varepsilon}, \qquad \forall \delta \in (0;\delta_0].$$

Theorem. (Negative spectrum) For all $n \in \mathbb{N}^*$, there exist constants $C, \gamma, \delta_0 > 0$, depending on n but independent of δ, such that

$$|\lambda^\delta_{-n} - \delta^{-2}\mu_{-n}| \leq C\exp(-\gamma/\delta), \qquad \forall \delta \in (0;\delta_0].$$

Proposition. (Localization effect) For all $n \in \mathbb{N}^*$, let u^δ_{-n} be an eigenfunction corresponding to the negative eigenvalue λ^δ_{-n}. There exist constants $C, \gamma, \delta_0 > 0$, depending on n but independent of δ, such that

$$\int_\Omega (|u^\delta_{-n}|^2 + |\nabla u^\delta_{-n}|^2)\,e^{\gamma x/\delta}\,d\boldsymbol{x} \leq C\,\|u^\delta_{-n}\|_\Omega, \qquad \forall \delta \in (0;\delta_0].$$

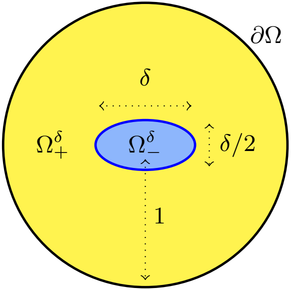
Numerical experiments for the small inclusion

- Using *FreeFem++*, we approximate numerically the spectrum of A^δ using a usual P1 Finite Element Method. We solve the problem

$$\left|\begin{array}{l}\text{Find } (\lambda_h^\delta, u_h^\delta) \in \mathbb{C} \times (\mathrm{V}_h \setminus \{0\}) \text{ s.t.:}\\ \displaystyle\int_\Omega \sigma_h^\delta \nabla u_h^\delta \cdot \nabla v_h = \lambda_h^\delta \int_\Omega u_h^\delta\, v_h, \quad \forall v_h \in \mathrm{V}_h,\end{array}\right.$$

where V_h approximates $\mathrm{H}_0^1(\Omega)$ as $h \to 0$ (h is the mesh size).

- We consider the following 2D geometry:



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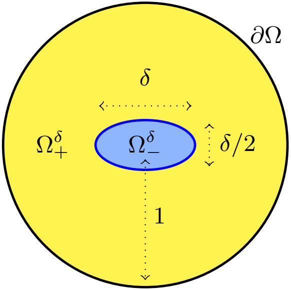
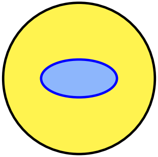
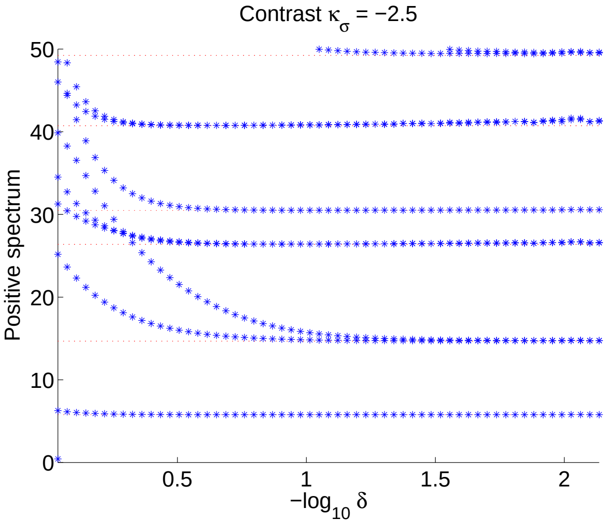
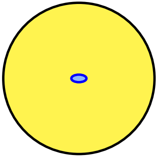


We display the spectrum as $\delta \to 0$ (h is more or less fixed).

Numerical experiments for the small inclusion

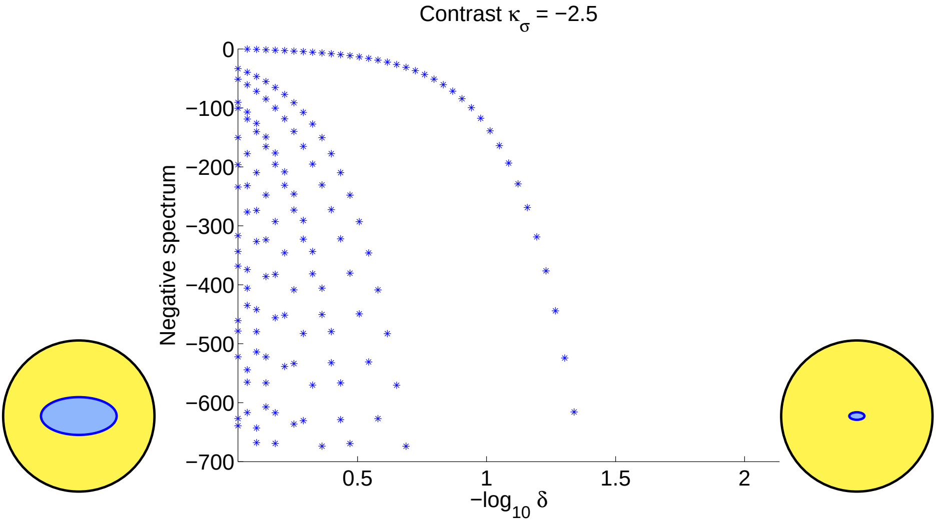






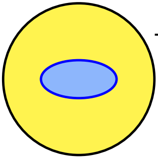
- The positive part of $\mathfrak{S}(\mathrm{A}^\delta)$ converges to $\mathfrak{S}(\mathrm{A}^0)$ when $\delta \to 0$.

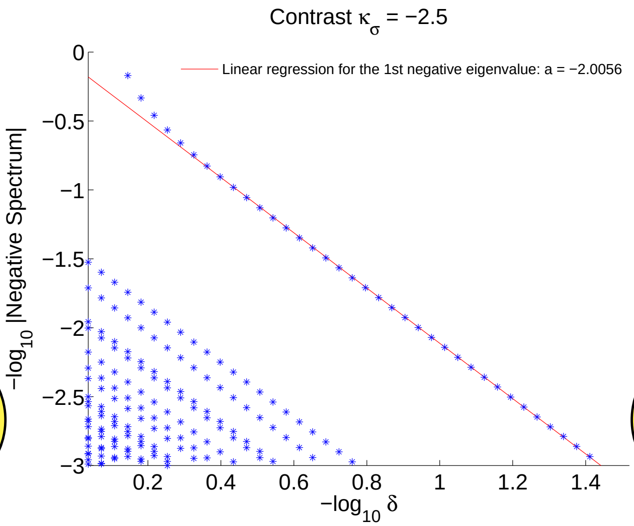
Numerical experiments for the small inclusion

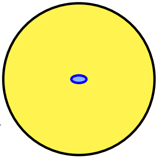


- The negative part of $\mathfrak{S}(\mathrm{A}^\delta)$ is asymptotically equivalent to the negative part of $\delta^{-2}\mathfrak{S}(\mathrm{B}^\infty)$ when $\delta \to 0$.

Numerical experiments for the small inclusion





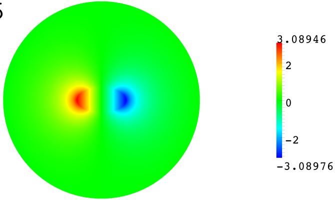


- The negative part of $\mathfrak{S}(\mathrm{A}^\delta)$ is asymptotically equivalent to the negative part of $\delta^{-2}\mathfrak{S}(\mathrm{B}^\infty)$ when $\delta \to 0$.

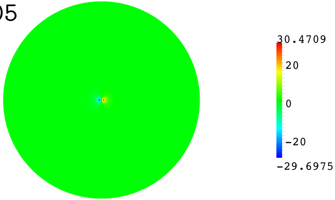
Localization effect

Eigenfunction associated to the first negative eigenvalue

δ=0.5

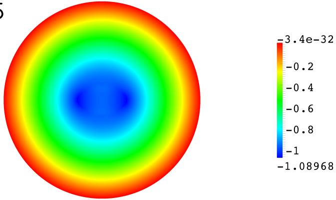


δ=0.05

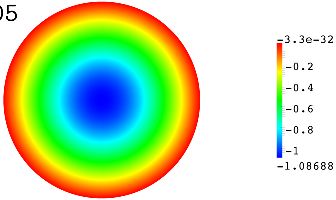


Eigenfunction associated to the first positive eigenvalue

δ=0.5



δ=0.05



- The eigenfunctions corresponding to the negative eigenvalues are localized around the small inclusion. Here, $\sigma_2/\sigma_1 = -2.5$.

Thank you for your attention!!!

A.-S. Bonnet-Ben Dhia, K. Ramdani, *A non elliptic spectral problem related to the analysis of superconducting micro-strip lines*, Math. Mod. Num. Anal., 36(3):461–487, 2002.

A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., *T-coercivity for scalar interface problems between dielectrics and metamaterials*, M2AN, 46, 1363–1387, 2012.

L. Chesnel, X. Claeys, S.A. Nazarov, *Spectrum for a small inclusion of negative material*, preprint, 2014.