Scalar wave transmission problems with sign changing coefficients

ICIAM 2011

A.S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr.

POems team, Ensta, Paris, France

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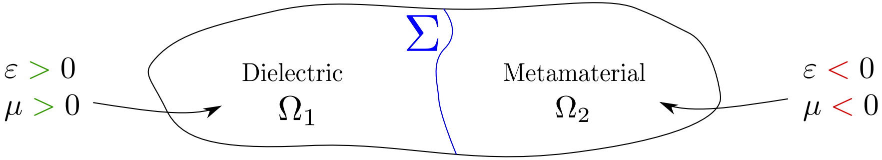
Introduction: setting of the problem

Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain:



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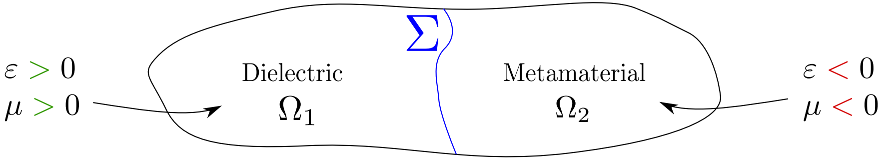

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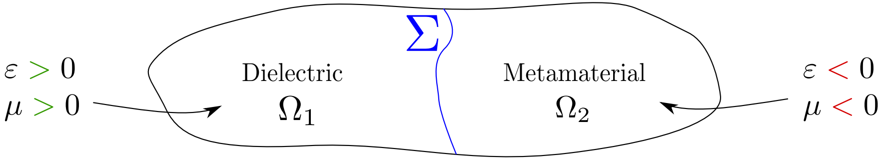

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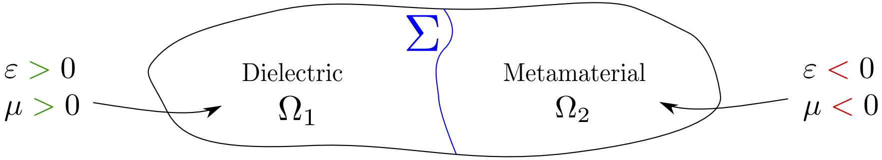

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Unusual transmission problem because the sign of the coefficients ε and μ changes.

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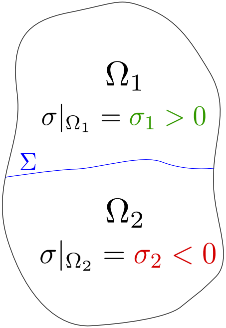

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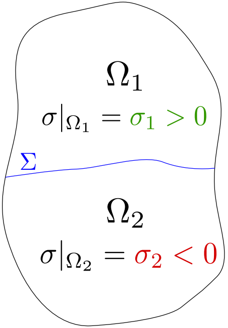

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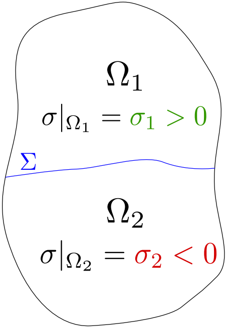

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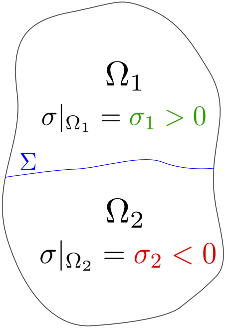

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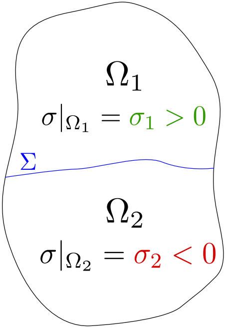

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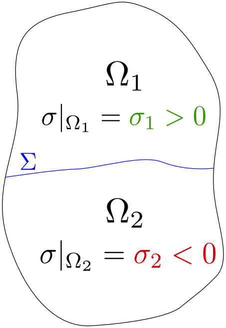

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Definition. Problem $(\mathscr{P})$ well-posed if for all $f \in H^{-1}(\Omega)$, it has one and only one solution with continuous dependence.

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1) A presentation of the T-coercivity method to derive a criterion on σ and on the geometry of the interface to ensure that problem $(\mathscr{P})$ is well-posed.

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1) A presentation of the T-coercivity method to derive a criterion on σ and on the geometry of the interface to ensure that problem $(\mathscr{P})$ is well-posed.

2) An use of this T-coercivity method to study an *a priori* different problem and emphasizing the generality of this approach.

1. Mathematical difficulty
2. The T-coercivity method for the dielectric/metamaterial transmission problem
3. The T-coercivity method for the Interior Transmission Eigenvalue Problem

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  - For a symmetric domain (w.r.t. Σ) with $\sigma_2 = -\sigma_1$, we can build a kernel of infinite dimension.

Idea of the T-coercivity 1/2

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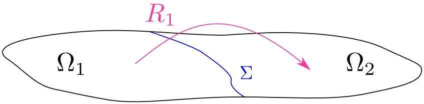

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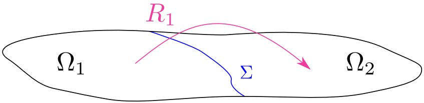

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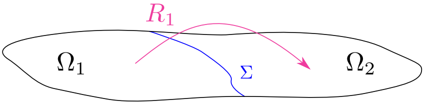

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5. Conclusion:

THEOREM. If the contrast $\kappa_\sigma = \sigma_2/\sigma_1 \notin [-\|R_2\|^2; -1/\|R_1\|^2]$ (critical interval) then $\operatorname{div}(\sigma \nabla \cdot)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

Choice of R_1,R_2?

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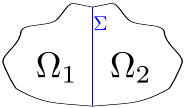

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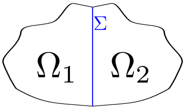

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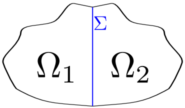

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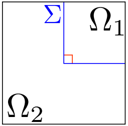

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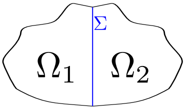

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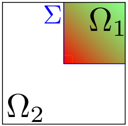

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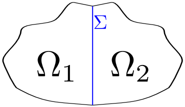

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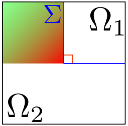

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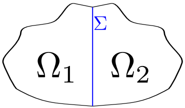

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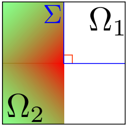

Action of R_1

Choice of R_1,R_2?

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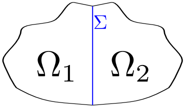

symmetry w.r.t. Σ

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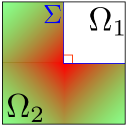

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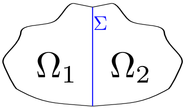

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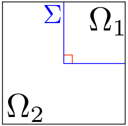

R_1 and R_2 built from
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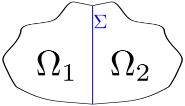

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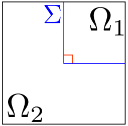

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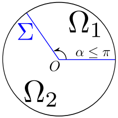

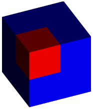

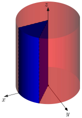

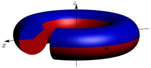

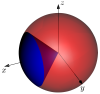

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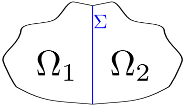

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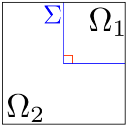

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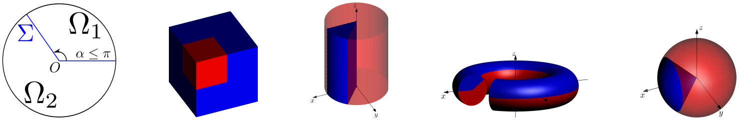
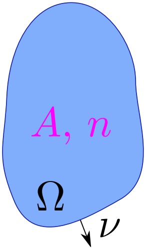


...

- General geometries: partition of unity and local inversion using particular cases.

Presentation of the ITEP

- Scattering by an inhomogeneous (A and n) media: we look for an incident wave that does not scatter.



Presentation of the ITEP

- Scattering by an inhomogeneous (A and n) media: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem.

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\text{Find } (u,w) \in H^1(\Omega) \times H^1(\Omega) \ \text{ s. t.:} & & \\
\operatorname{div}(A\nabla u) + k^2 n u & = & 0 \quad \text{in } \Omega \\
\Delta w + k^2 w & = & 0 \quad \text{in } \Omega \\
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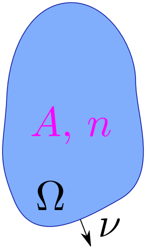

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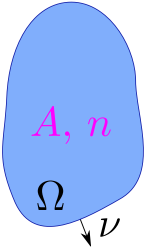

DEFINITION. Values of $k \in \mathbb{C}$ for which this problem has a nontrivial solution (u, w) are called transmission eigenvalues.

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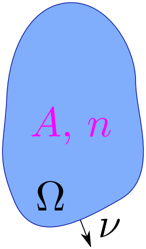

Definition. Values of $k \in \mathbb{C}$ for which this problem has a nontrivial solution (u, w) are called transmission eigenvalues.

- The goal is to prove that the set of transmission eigenvalues is discrete and countable.

Study of the ITEP

- Equivalent formulation:

> Find $(u, w) \in X$ s. t., $\forall (u', w') \in X$,
> $$a((u, w), (u', w')) = \int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} - k^2 (n u \overline{u'} - w \overline{w'})\, dx = 0,$$

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- Define the isomorphism $\mathtt{T}(u,w) = (u - 2w, -w)$.
- For $k \in \mathbb{R}i \backslash \{0\}$, $A > 1$ and $n > 1$, one finds

$$\Re e\, a((u,w), \mathtt{T}(u,w)) \geq C\,(\|u\|^2_{H^1(\Omega)} + \|w\|^2_{H^1(\Omega)}), \quad \forall (u,w) \in X.$$

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> PROPOSITION. Suppose that $A > 1$ and $n > 1$. Then the set of transmission eigenvalues is discrete and countable.

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> PROPOSITION. Suppose that $A>1$ and $n>1$. Then the set of transmission eigenvalues is discrete and countable.

- This result can extended to situations where $A-Id$ and $n-1$ change sign in Ω *(joint work with H. Haddar)*.

Generalizations

- ✔ $\mathtt{T}$-coercivity approach can be used for non-constant σ (L^∞) and other problems (Maxwell's equations, bilaplacian form of the ITEP ...).
- ✔ One can justify convergence of standard finite elements method for simple meshes.

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Ongoing work

♠ What happens in the critical interval ?
For $\kappa_\sigma \neq -1$, strong singularities appear at the corners of the interface $\Rightarrow$ ($\mathscr{P}$) is well-posed in a new functional framework *(joint work with X. Claeys)*.

♠ More generally, can we reconsider the homogenization process to take into account interfacial phenomena?
$\Rightarrow$*METAMATH project (ANR) directed by S. Fliss*.

Thank you for your attention.

A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., *Optimality of T-coercivity for scalar interface problems between dielectrics and metamaterials*, http://hal.archives-ouvertes.fr/hal-00564312_v1/, 2011.

A.-S. Bonnet-Ben Dhia, L. Chesnel, H. Haddar, *On the use of T-coercivity to study the interior transmission eigenvalue problem*, C. R. Acad. Sci. Paris, Ser. I, 349:647–651, 2011.

A.-S. Bonnet-Ben Dhia, P. Ciarlet Jr., C.M. Zwölf, *Time harmonic wave diffraction problems in materials with sign-shifting coefficients*, J. Comput. Appl. Math, 234:1912–1919, 2010, Corrigendum *J. Comput. Appl. Math.*, 234:2616, 2010.