Playing with thin resonant ligaments in acoustic waveguides

Lucas Chesnel[1]

Coll. with J. Heleine[2] and S.A. Nazarov[3].

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MARSEILLE, 10/12/2023

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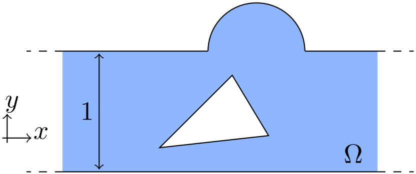

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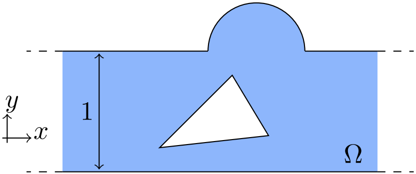

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- For this problem, the modes are

Propagating	$w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2},\ n \in [\![0, N-1]\!]$
Evanescent	$w_n^{\pm}(x,y) = e^{\mp \beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2},\ n \geq N.$

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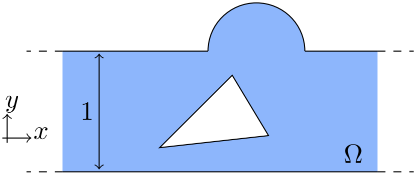

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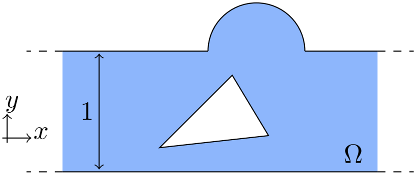

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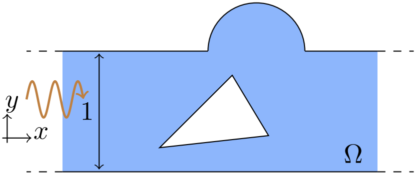

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► We fix $k \in (0;\pi)$ so that only the plane waves $e^{\pm ikx}$ can propagate.

► The scattering of the wave e^{ikx} leads us to consider the solutions of $(\mathscr{P})$ with the decomposition

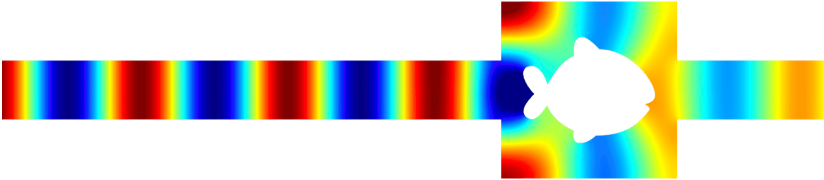
$$u = \left| \begin{array}{lr} e^{ikx} + R\, e^{-ikx} + \dots & x \to -\infty \\ T\, e^{+ikx} + \dots & x \to +\infty \end{array} \right.$$

R, $T \in \mathbb{C}$ are the scattering coefficients , the $\dots$ are expon. decaying terms.

► We have the relation of conservation of energy $|R|^2 + |T|^2 = 1$.

- Without obstacle, $u = e^{ikx}$ so that $(R, T) = (0, 1)$.

- With an obstacle, in general $(R, T) \neq (0, 1)$.



Introduction

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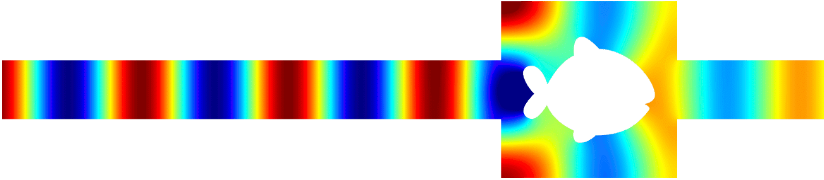
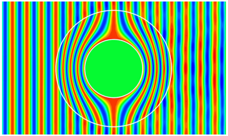


Initial goal

We wish to identify situations (geometries, k) where $R = 0$ (zero reflection) or $T = 1$ (perfect invisibility) $\Rightarrow$ cloaking at "infinity".

Introduction 3/3

Difficulty: the scattering coefficients have a non explicit and non linear dependence wrt the geometry and k.
→ Optimization techniques fail due to local minima.



Remark: different from the usual cloaking picture (Pendry *et al.* 06, Leonhardt 06, Greenleaf *et al.* 09) because we wish to control only the scattering coef..

→ Less ambitious but doable without fancy materials (and relevant in practice).

Outline of the talk

1. Construction of small invisible perturbations

2. Cloaking of given large obstacles with resonant ligaments

3. Playing with resonant ligaments for other applications

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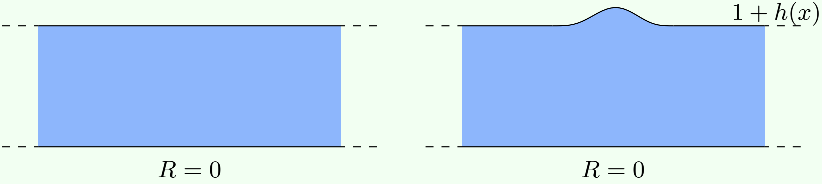
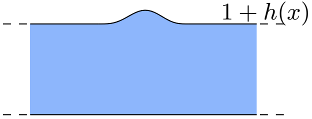
Perturbative techniques: general picture

- We can construct small invisible defects using variants of the implicit functions theorem.



Sketch of the method

- For $h \in \mathscr{C}_0^\infty(\mathbb{R})$, denote $R(h) \in \mathbb{C}$ the reflection coef. in the geometry:



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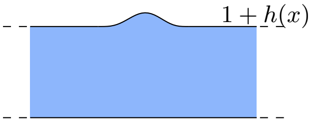
  > Note that $R(0) = 0$
  > (no obstacle leads to null measurements).



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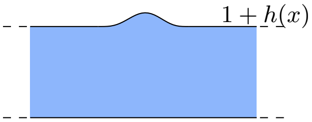

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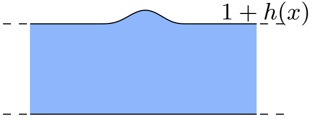

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- We look for h of the form $h = \varepsilon\mu$ with $\varepsilon > 0$ small and μ to determine.

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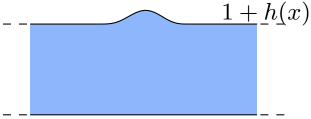

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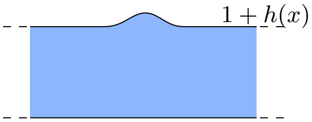

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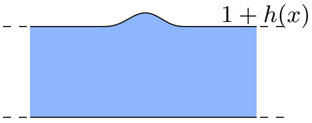

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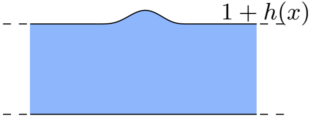

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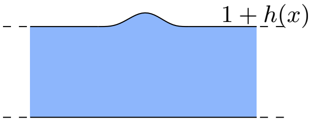

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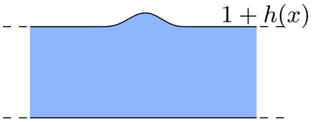

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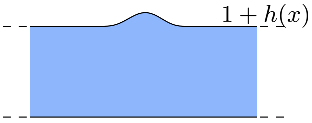

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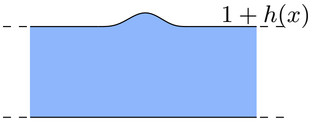

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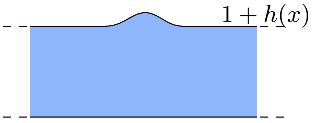

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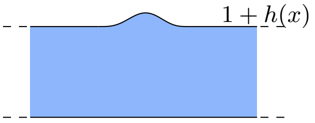

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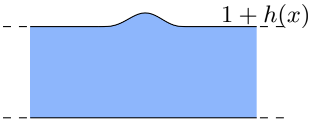

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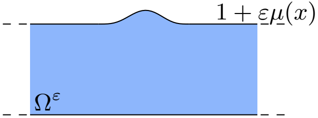
Set $h^{\text{sol}} := \varepsilon\mu^{\text{sol}}$. We have $R(h^{\text{sol}}) = 0$ (non reflecting perturbation).

Calculus of the differential



- Using classical results of asymptotic analysis, we obtain

$$R(\varepsilon\mu) = 0 + \varepsilon\left(-\frac{1}{2}\int_{-\ell}^{\ell} \partial_x \mu(x) e^{2ikx}\, dx\right) + O(\varepsilon^2).$$

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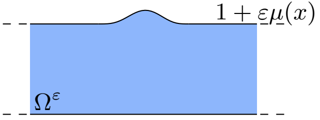

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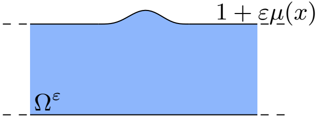

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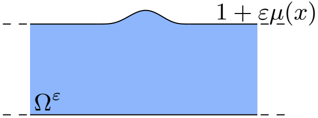

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$$T(\varepsilon\mu) - 1 = 0 + \varepsilon\ 0\ + O(\varepsilon^2).$$

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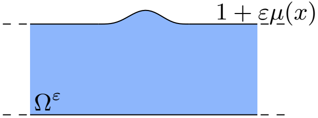

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$$R(\varepsilon\mu) = 0 + \varepsilon \underbrace{\left(-\frac{1}{2} \int_{-\ell}^{\ell} \partial_x \mu(x) e^{2ikx}\, dx \right)}_{dR(0)(\mu)} + O(\varepsilon^2).$$

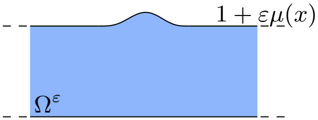
$dR(0) : \mathscr{C}_0^{\infty}(\mathbb{R}) \to \mathbb{C}$ is onto $\Rightarrow$ we can get non trivial Ω s.t. $R = 0$.

- Can we use the technique to construct Ω such that $T = 1$? We obtain

$$T(\varepsilon\mu) - 1 = 0 + \varepsilon\, \boxed{0} + O(\varepsilon^2).$$

$dT(0)$ is not onto $\Rightarrow$ the approach fails to impose $T = 1$.

Calculus of the differential



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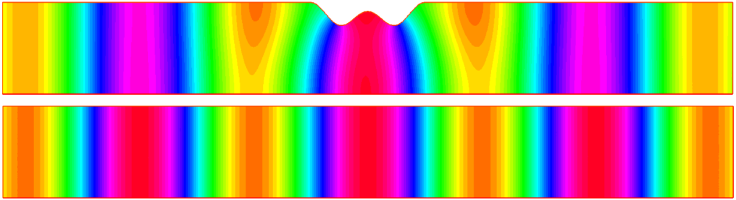
However other types of perturbations allow one to get $T = 1$.

Numerical results

- The fixed point problem can be solved iteratively: $\vec{\tau}^{\,n+1} = G^{\varepsilon}(\vec{\tau}^{\,n})$.



Numerics done by a group of students of École Polytechnique with the ***`Freefem++`*** *library* $\rightarrow$ P2 FEM + Dirichlet-to-Neumann to truncate Ω.

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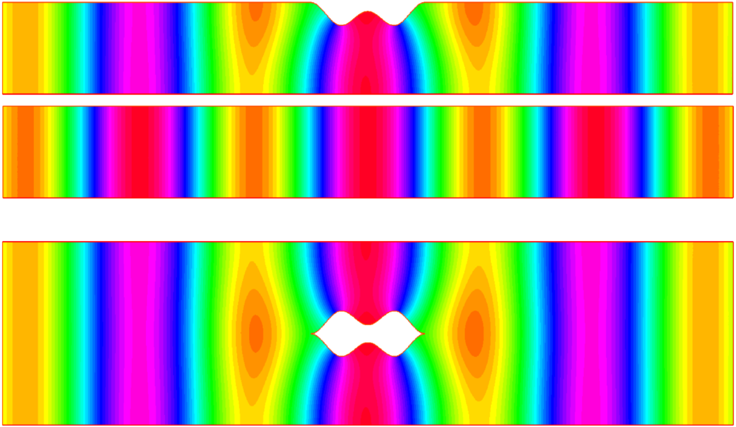

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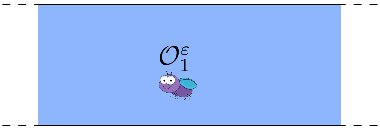
Non reflecting clouds of small obstacles 1/2

Can one hide a small Dirichlet obstacle centered at M_1 ?



Find $u = u_i + u_s$ s. t.

$$\begin{array}{rl} \Delta u + k^2 u = 0 & \text{in } \Omega^\varepsilon := \Omega \setminus \overline{\mathcal{O}_1^\varepsilon}, \\ u = 0 & \text{on } \partial\Omega^\varepsilon, \\ u_s \text{ is outgoing.} & \end{array}$$

► With Dirichlet B.C., the modes are not the same as previously but this not important. Denote by $w^{\pm}$ the first propagating modes.

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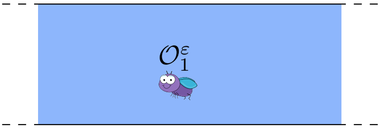

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$$\begin{aligned} R &= 0 + \varepsilon\ (4i\pi \operatorname{cap}(\mathcal{O}) w^+(M_1)^2)\ + O(\varepsilon^2) \\ T &= 1 + \varepsilon\ (4i\pi \operatorname{cap}(\mathcal{O}) |w^+(M_1)|^2)\ + O(\varepsilon^2). \end{aligned}$$

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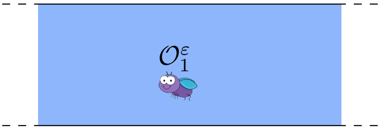

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Non zero terms! ($\mathrm{cap}(\mathcal{O}) > 0$)

Non reflecting clouds of small obstacles 1/2

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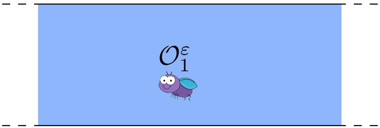

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Non zero terms!
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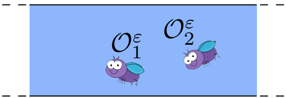
$\Rightarrow$ One single small obstacle cannot even be non reflecting.



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- We obtain
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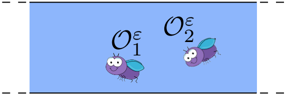

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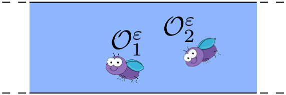
Non reflecting clouds of small obstacles 2/2



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We can find M_1, M_2 such that $R = O(\varepsilon^2)$.

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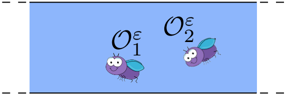

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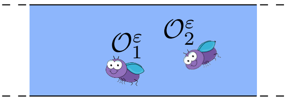

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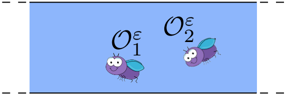
COMMENTS:

→ Hard part is to justify the asymptotics for the fixed point problem.

→ We cannot impose $T = 1$ with this strategy.

→ When there are more propagative waves, we need more obstacles.



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COMMENTS:

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Acting as a team, flies can become invisible!

Outline of the talk

1. Construction of small invisible perturbations

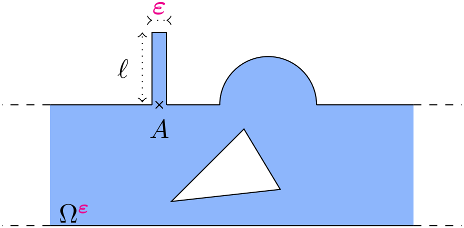
2. Cloaking of given large obstacles with resonant ligaments

We constructed invisible defects.
How to hide given large obstacles ?

3. Playing with resonant ligaments for other applications

Setting

Main ingredient of our approach: outer resonators of width $\varepsilon \ll 1$.



$$(\mathscr{P}^{\varepsilon}) \left| \begin{array}{rl} \Delta u + k^2 u = 0 & \text{in } \Omega^{\varepsilon}, \\ \partial_n u = 0 & \text{on } \partial\Omega^{\varepsilon} \end{array} \right.$$

- In this geometry, we have the scattering solutions

$$u_+^{\varepsilon} = \left| \begin{array}{r} e^{ikx} + R_+^{\varepsilon}\, e^{-ikx} + \dots \\ T^{\varepsilon}\, e^{+ikx} + \dots \end{array} \right. \qquad u_-^{\varepsilon} = \left| \begin{array}{r} T^{\varepsilon}\, e^{-ikx} + \dots \\ e^{-ikx} + R_-^{\varepsilon}\, e^{+ikx} + \dots \end{array} \right. \qquad \begin{array}{l} x \to -\infty \\ x \to +\infty \end{array}$$

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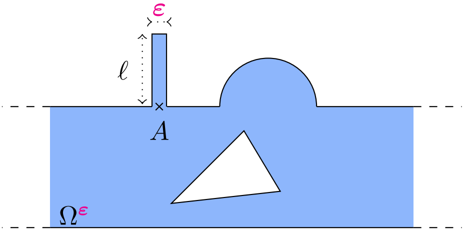

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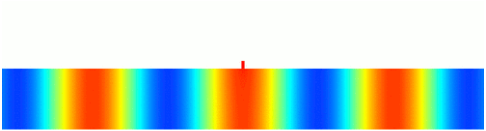
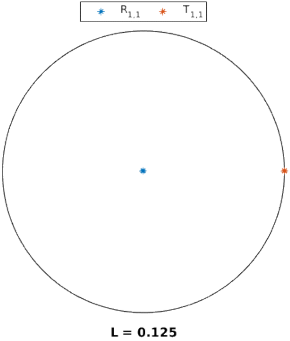
In general, the thin ligament has only a weak influence on the scattering coefficients: $R_\pm^\varepsilon \approx R_\pm$, $T^\varepsilon \approx T$. But not always ...

Numerical experiment

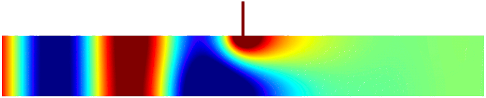
- We vary the length of the ligament:

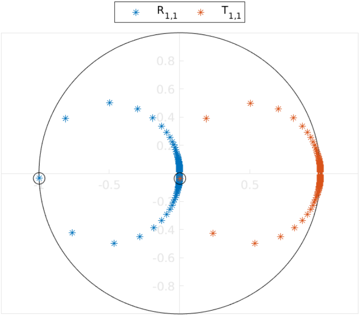




Numerical experiment

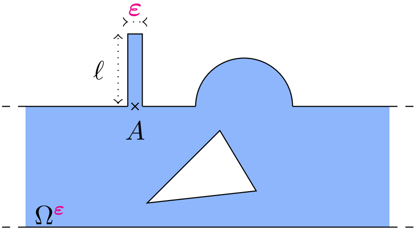
- For one particular length of the ligament, we get a standing mode (zero transmission):





Asymptotic analysis

To understand the phenomenon, we compute an asymptotic expansion of u_+^ε, R_+^ε, T^ε as $\varepsilon \to 0$.



$$(\mathscr{P}^\varepsilon) \left| \begin{array}{rl} \Delta u_+^\varepsilon + k^2 u_+^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ \partial_n u_+^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon \end{array} \right.$$

$$u_+^\varepsilon = \left| \begin{array}{r} e^{ikx} + R_+^\varepsilon\, e^{-ikx} + \dots \\ T^\varepsilon\, e^{+ikx} + \dots \end{array} \right.$$

- To proceed we use techniques of matched asymptotic expansions (see Beale 73, Gadyl'shin 93, Kozlov et al. 94, Nazarov 96, Maz'ya et al. 00, Joly & Tordeux 06, Lin, Shipman & Zhang 17, 18, Brandao, Holley, Schnitzer 20, ...).

Asymptotic analysis

- We work with the outer expansions

$$u_+^\varepsilon(x,y) = u^0(x,y) + \dots \qquad \text{in } \Omega,$$
$$u_+^\varepsilon(x,y) = \varepsilon^{-1} v^{-1}(y) + v^0(y) + \dots \qquad \text{in the resonator.}$$

- Considering the restriction of $(\mathscr{P}^\varepsilon)$ to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous 1D problem

$$(\mathscr{P}_{1\mathrm{D}}) \left| \begin{array}{ll} \partial_y^2 v + k^2 v = 0 & \text{in } (1; 1+\ell) \\ v(1) = \partial_y v(1+\ell) = 0. & \end{array} \right.$$

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The features of $(\mathscr{P}_{1\mathrm{D}})$ play a key role in the physical phenomena and in the asymptotic analysis.

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The features of $(\mathscr{P}_{1\mathrm{D}})$ play a key role in the physical phenomena and in the asymptotic analysis.

- We denote by ℓ_{res} (**resonance lengths**) the values of ℓ, given by

$$\ell_{\mathrm{res}} := \pi(m + 1/2)/k, \qquad m \in \mathbb{N},$$

such that $(\mathscr{P}_{1\mathrm{D}})$ admits the non zero solution $v(y) = \sin(k(y-1))$.

Asymptotic analysis – Non resonant case

- Assume that $\ell \neq \ell_{\text{res}}$. Then we find $v^{-1} = 0$ and when $\varepsilon \to 0$, we get

$$
\begin{aligned}
&u^{\varepsilon}_{\pm}(x,y) = u_{\pm} + o(1) && \text{in } \Omega,\\
&u^{\varepsilon}_{\pm}(x,y) = u_{\pm}(A)\, v_0(y) + o(1) && \text{in the resonator,}\\
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Here $v_0(y) = \cos(k(y-1) + \tan(k(y-\ell)\sin(k(y-1)$.

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The thin resonator has no influence at order ε^0.

$\rightarrow$ **Not interesting for our purpose** because we want $\left|\begin{array}{l} R^\varepsilon_\pm = 0 + \dots \\ T^\varepsilon = 1 + \dots \end{array}\right.$

Asymptotic analysis – Resonant case

- For $\ell = \ell_{\text{res}}$, when $\varepsilon \to 0$, we obtain

$$u_+^\varepsilon(x,y) = u_+(x,y) + ak\gamma(x,y) + o(1) \quad \text{in } \Omega,$$

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$$R_+^\varepsilon = R_+ + iau_+(A)/2 + o(1), \qquad T^\varepsilon = T + iau_-(A)/2 + o(1).$$

Here γ is the outgoing Green function such that $\left| \begin{array}{l} \Delta\gamma + k^2\gamma = 0 \text{ in } \Omega \\ \partial_n \gamma = \delta_A \text{ on } \partial\Omega \end{array} \right.$ and

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This time the thin resonator has an influence at order ε^0

Asymptotic analysis – Resonant case

- For $\ell = \ell_{\text{res}} + \varepsilon\eta$ with $\eta \in \mathbb{R}$ fixed, when $\varepsilon \to 0$, we obtain

$$u_+^\varepsilon(x,y) = u_+(x,y) + a(\eta)k\gamma(x,y) + o(1) \quad \text{in } \Omega,$$

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$$u_+^\varepsilon(x,y) = u_+(x,y) + a(\eta)k\gamma(x,y) + o(1) \quad \text{in } \Omega,$$

$$u_+^\varepsilon(x,y) = \varepsilon^{-1}a(\eta)\sin(k(y-1)) + O(1) \quad \text{in the resonator},$$

$$R_+^\varepsilon = R_+ + ia(\eta)u_+(A)/2 + o(1), \qquad T^\varepsilon = T + ia(\eta)u_-(A)/2 + o(1).$$

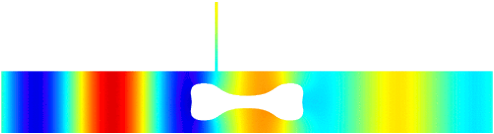
Here γ is the outgoing Green function such that $\left|\begin{array}{l}\Delta\gamma + k^2\gamma = 0 \text{ in } \Omega \\ \partial_n\gamma = \delta_A \text{ on } \partial\Omega\end{array}\right.$ and

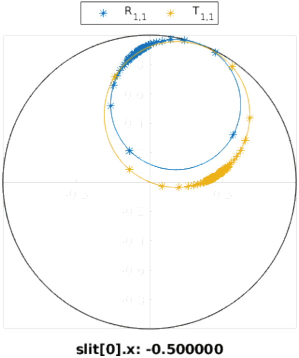
$$a(\eta)k = -\frac{u_+(A)}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi + \eta}.$$

This time the thin resonator has an influence at order ε^0 and it depends on the choice of η!

Almost zero reflection

From this expansion, we find that asymptotically, when the length of the resonator is perturbed **around $\boldsymbol{\ell_{\mathrm{res}}}$**, R_+^ε, T^ε run on **circles** whose **features depend on the choice for** A.





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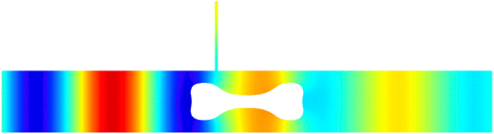

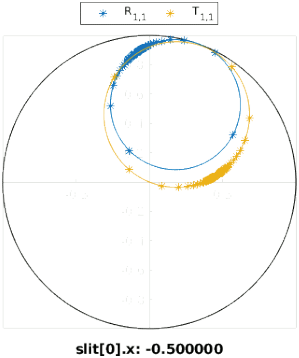

- Using the expansions of $u_\pm(A)$ far from the obstacle, one shows:

PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta)\,|\,\eta\in\mathbb{R}\}$ passes **through zero**.

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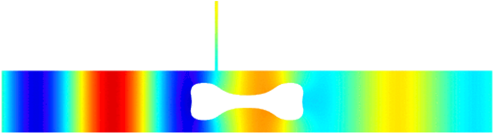
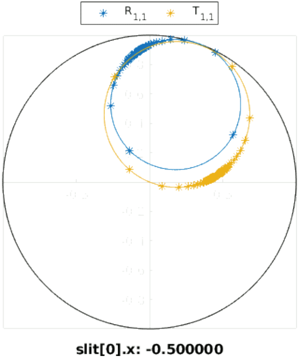


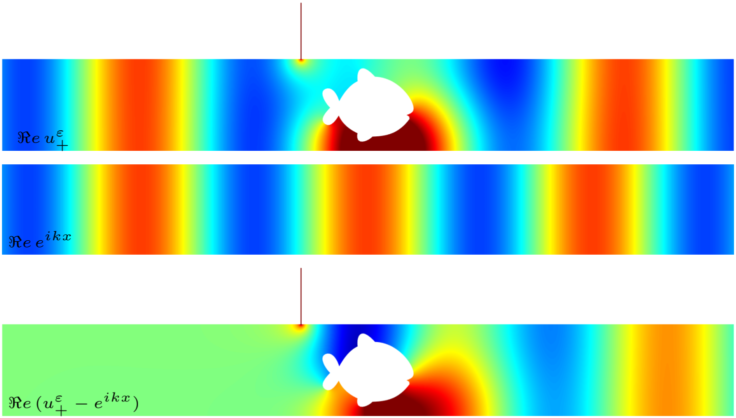


- Using the expansions of $u_\pm(A)$ far from the obstacle, one shows:

PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta)\,|\,\eta \in \mathbb{R}\}$ passes **through zero**. $\Rightarrow \exists$ situations s.t. $R_+^\varepsilon = 0 + o(1)$.

Almost zero reflection

- Example of situation where we have almost zero reflection ($\varepsilon = \mathbf{0.01}$).



Simulations realized with the `Freefem++` *library.*

Almost zero reflection

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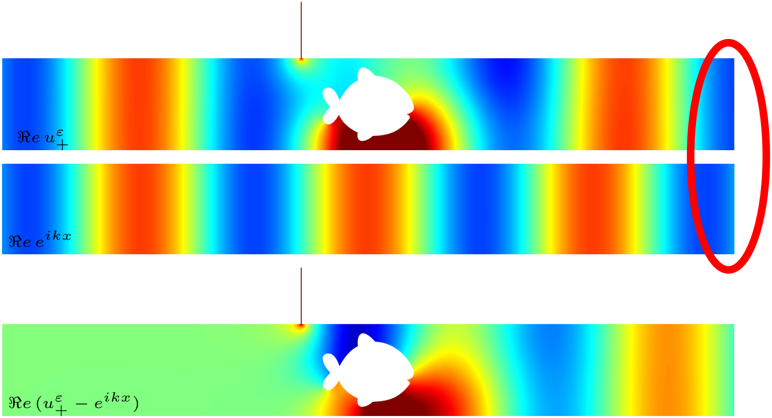
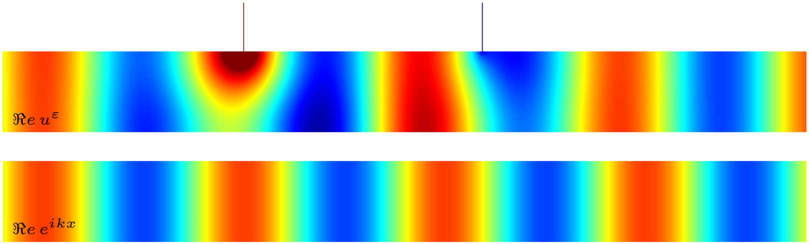


Simulations realized with the `Freefem++` *library.*

Conservation of energy guarantees that when $R_+^\varepsilon = 0$, $|T^\varepsilon| = 1$.
→ To cloak the object, it remains to compensate the phase shift!

Phase shifter

- Working with two resonators, we can create phase shifters, that is devices with almost zero reflection and any desired phase.

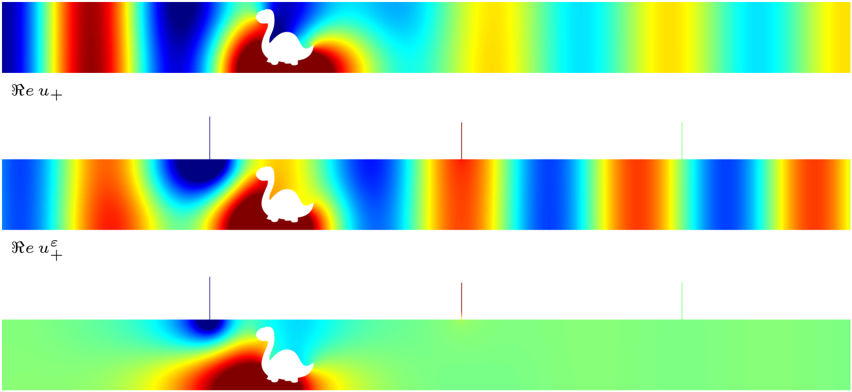


- Here the device is designed to obtain a phase shift approx. equal to $\pi/4$.

Cloaking with three resonators

- Now working in two steps, we can approximately cloak any object with three resonators:

1) With one resonant ligament, first we get almost zero reflection;
2) With two additional resonant ligaments, we compensate the phase shift.



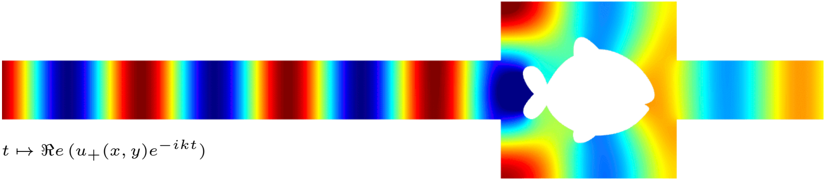
$\Re e\, u_+$

$\Re e\, u_+^{\varepsilon}$

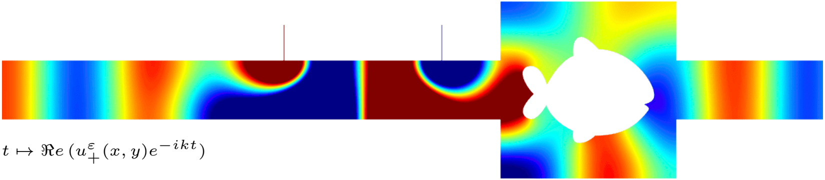
$\Re e\, (u_+^{\varepsilon} - e^{ikx})$

Cloaking with two resonators

- Working a bit more, one can show that two resonators are enough to cloak any object.



$t \mapsto \Re e\,(u_+(x,y)e^{-ikt})$



$t \mapsto \Re e\,(u_+^{\varepsilon}(x,y)e^{-ikt})$

$t \mapsto \Re e\,(e^{ik(x-t)})$

Outline of the talk

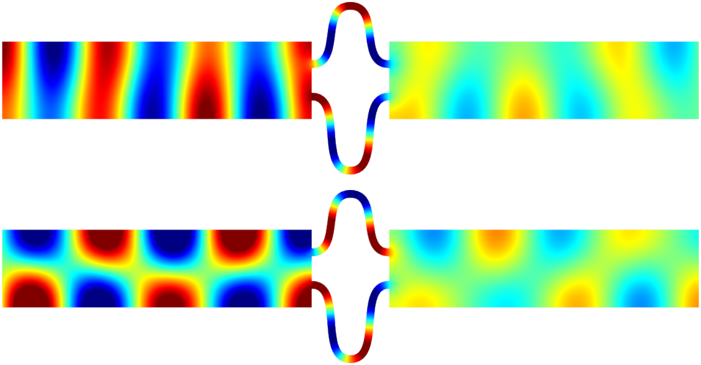
Mode converter

- We work at higher wavenumber so that two modes can propagate.

Goal: find a geometry such that:

1) energy is completely transmitted;
2) mode 1 is transformed into mode 2.

- We decided to work in a geometry with thin ligaments:



$t \mapsto \Re e\,(v_1 e^{-i\omega t})$

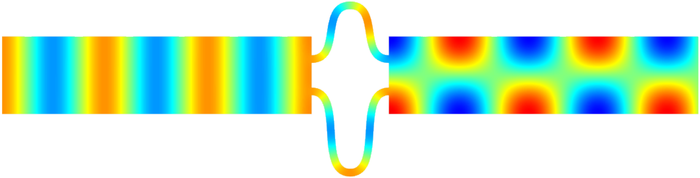
$t \mapsto \Re e\,(v_2 e^{-i\omega t})$

Paradoxical because in general in this Ω, energy is mostly backscattered...

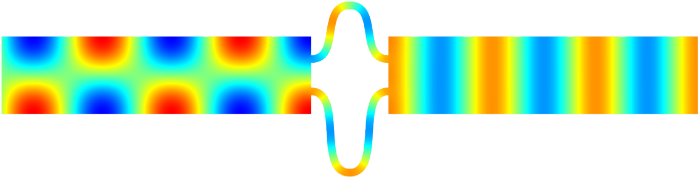
Mode converter 2/2

► Tuning precisely the positions and lengths of the ligaments, we can ensure absence of reflection and mode conversion.

$t \mapsto \Re e\,(v_1 e^{-i\omega t})$

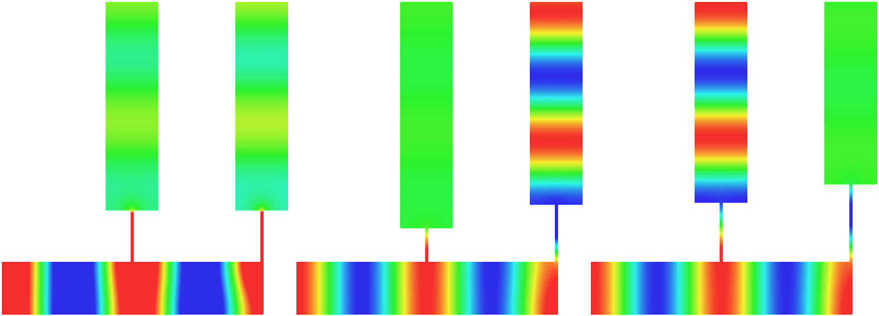


$t \mapsto \Re e\,(v_2 e^{-i\omega t})$



Acoustic energy distributor

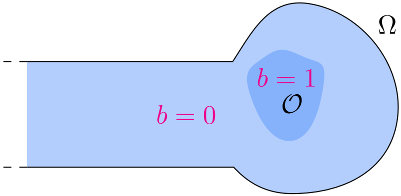
- We display $t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$.



Tuning precisely the length of the two ligaments, we can:

1) ensure absence of reflection;
& 2) control the ratio of energy transmitted in the output channels.

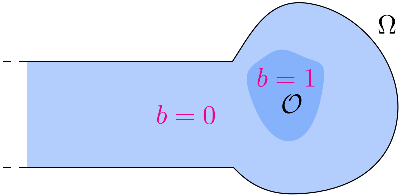
► Consider the scattering of the incident plane wave in a half-waveguide containing a dissipative inclusion $\mathcal{O}$:



$$\left|\begin{array}{rl} \Delta u^\eta + k^2\big(1 + i\eta\, b\big)u^\eta = 0 & \text{in } \Omega \\ \partial_n u^\eta = 0 & \text{on } \partial\Omega \end{array}\right.$$

($\eta > 0$ models the dissipation).

- Consider the scattering of the incident plane wave in a half-waveguide containing a dissipative inclusion $\mathcal{O}$:



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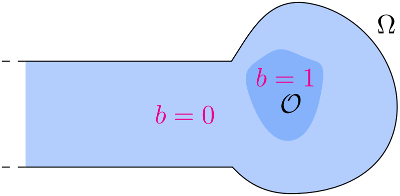
- This problems admits the solution

$$u^\eta = e^{ikx} + R^\eta\, e^{-ikx} + \dots$$

where $R^\eta \in \mathbb{C}$ and the $\dots$ are expon. decaying terms.

Enhancement of dissipation

- Consider the scattering of the incident plane wave in a half-waveguide containing a dissipative inclusion $\mathcal{O}$:



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- This problems admits the solution

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where $R^\eta \in \mathbb{C}$ and the $\dots$ are expon. decaying terms.

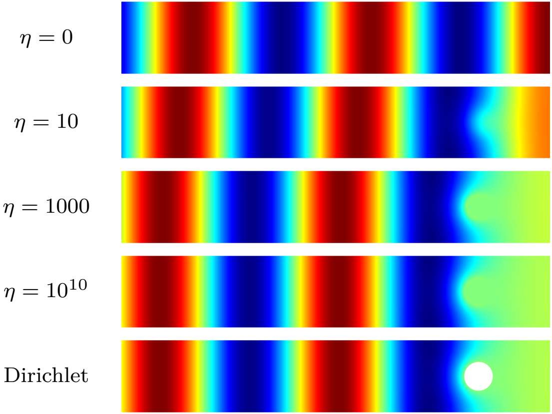
How to obtain complete dissipation in $\mathcal{O}$, i.e. $R^\eta = 0$ for some $\eta > 0$?

Enhancement of dissipation 2/3

- For $\eta = 0$, conservation of energy implies $|R^0| = 1$.

Enhancement of dissipation

- For $\eta = 0$, conservation of energy implies $|R^0| = 1$.
- For $\eta \to +\infty$, the inclusion behaves as a Dirichlet obstacle and $|R^\eta| \to 1$.



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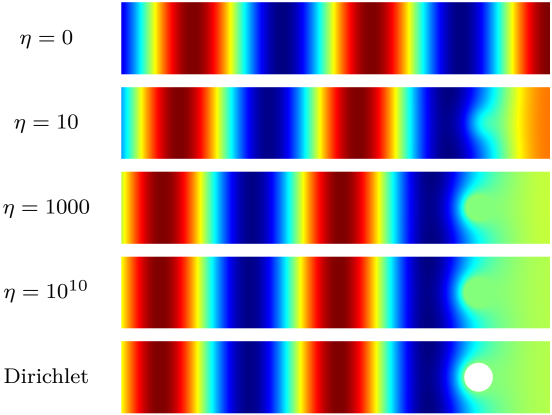
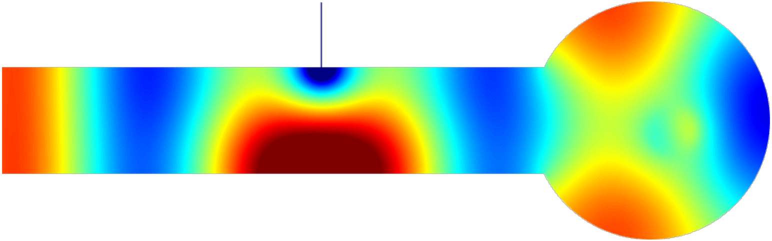


- The curve $\eta \mapsto |R^\eta|$ has a minimum but the latter in general is not zero!

Enhancement of dissipation 3/3

▶ For any Ω, $\mathcal{O}$ and $\eta > 0$, we have shown that we can add a well-designed resonant ligament so that $R^\eta \approx 0$ in the new geometry



$t \mapsto \Re e\,(u^\eta e^{-i\omega t})$

(see also Merkel, Theocharis, Richoux, Romero-Garcia, Pagneux 15).

Outline of the talk

1. Construction of small invisible perturbations
2. Cloaking of given large obstacles with resonant ligaments
3. Playing with resonant ligaments for other applications

Conclusion

What we did

1) We constructed small smooth non reflecting perturbations of the reference strip.

   We explained how clouds of small obstacles can be non reflecting.

2) We showed how to hide approximately ($T \approx 1$) given large obstacles using thin resonant ligaments.

3) We also used thin resonant ligaments to create mode converters, energy distributors and perfect absorbers.

Future work

- Can one hide given large obstacles at higher frequency?
- Can one hide exactly given large obstacles?
- Can we get for example small reflection for an interval of frequencies?
- What can be done for water-waves, electromagnetism,...?

A few references

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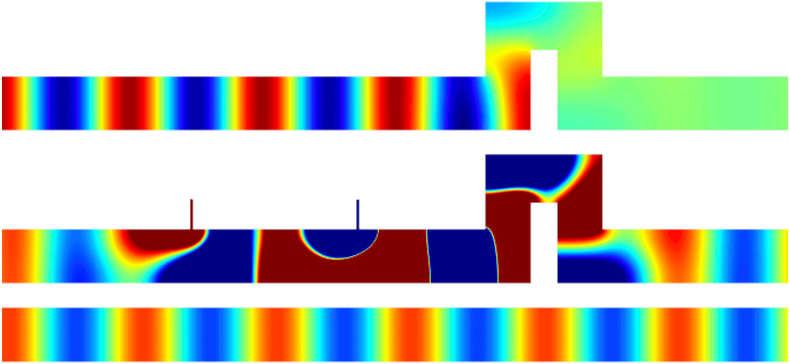

Thank you for your attention!