Séminaire "Analyse numérique et équations aux dérivées partielles"

Invisibilité et camouflage d'obstacles dans des guides d'ondes acoustiques

Lucas Chesnel[1]

Coll. with A.-S. Bonnet-BenDhia[2], J. Heleine[3], S.A. Nazarov[4], V. Pagneux[5]

[1]Idefix team, Inria/Ensta Paris, France
[2]Poems team, Inria/Ensta Paris, France
[3]IMT, Univ. Paul Sabatier, France
[4]FMM, St. Petersburg State University, Russia
[5]LAUM, Univ. du Maine, France

Lille, 21/12/2023

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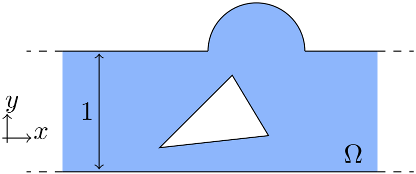

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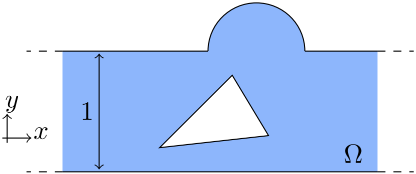

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- For this problem, the modes are

$$\begin{array}{l|l} \text{Propagating} & w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2},\ n \in [\![0, N-1]\!] \\ \text{Evanescent} & w_n^{\pm}(x,y) = e^{\mp\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2},\ n \geq N. \end{array}$$

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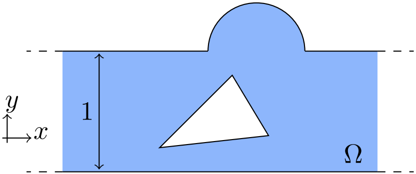

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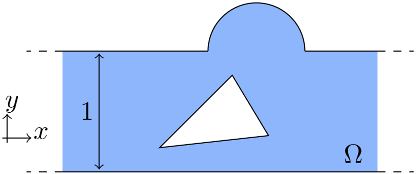

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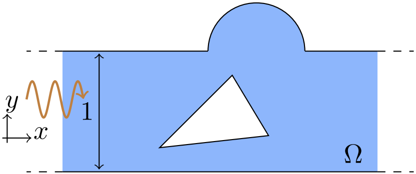

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- The scattering of the wave e^{ikx} leads us to consider the solutions of $(\mathscr{P})$ with the decomposition

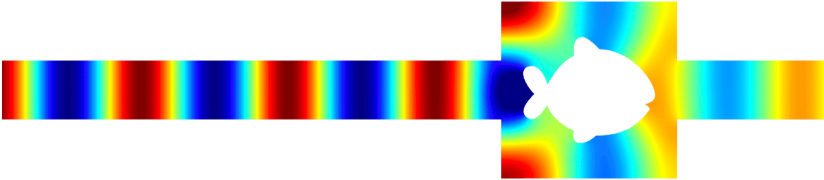
$$u = \left|\begin{array}{ll} e^{ikx} + R\, e^{-ikx} + \dots & x \to -\infty \\ \qquad T\, e^{+ikx} + \dots & x \to +\infty \end{array}\right.$$

R, $T \in \mathbb{C}$ are the scattering coefficients, the $\dots$ are expon. decaying terms.

- We have the relation of conservation of energy $|R|^2 + |T|^2 = 1$.

- Without obstacle, $u = e^{ikx}$ so that $(R, T) = (0, 1)$.

- With an obstacle, in general $(R, T) \neq (0, 1)$.



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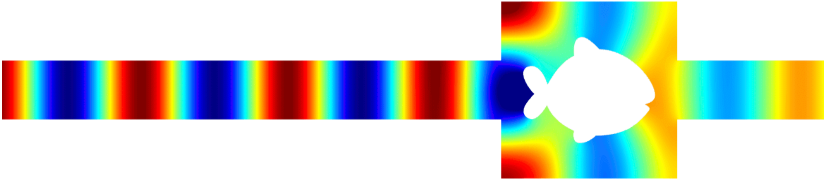
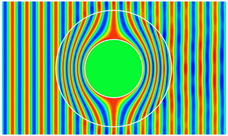


Goal of the talk

We wish to identify situations (geometries, k) where $R = 0$ (zero reflection) or $T = 1$ (perfect invisibility) $\Rightarrow$ cloaking at "infinity".

Difficulty: the scattering coefficients have a non explicit and non linear dependence wrt the geometry and k.
→ Optimization techniques fail due to local minima.



Remark: different from the usual cloaking picture (Pendry *et al.* 06, Leonhardt 06, Greenleaf *et al.* 09) because we wish to control only the scattering coef..

→ Less ambitious but doable without fancy materials (and relevant in practice).

Outline of the talk

1. Smooth non reflecting perturbations of the reference strip

2. Non reflecting clouds of small obstacles

3. Construction of large invisible defects

4. Cloaking of given large obstacles

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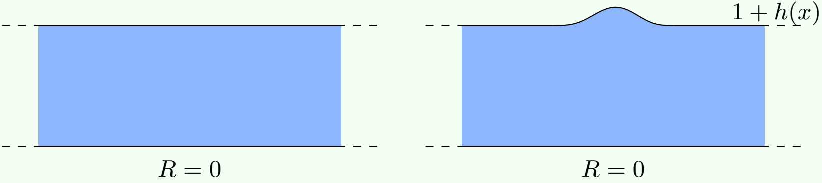
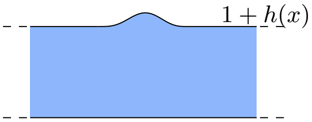
General picture

- *Perturbative* technique: we construct small non reflecting defects using variants of the implicit functions theorem.



Sketch of the method

- For $h \in \mathscr{C}_0^\infty(\mathbb{R})$, denote $R(h) \in \mathbb{C}$ the reflection coef. in the geometry:



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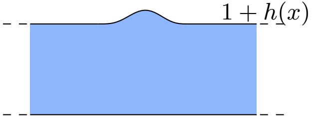

> Note that $R(0) = 0$
> (no obstacle leads to null measurements).

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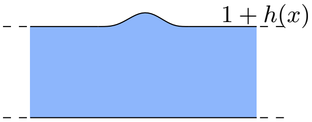

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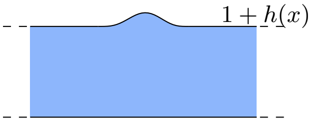

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- We look for h of the form $h = \varepsilon\mu$ with $\varepsilon > 0$ small and μ to determine.

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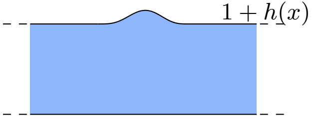

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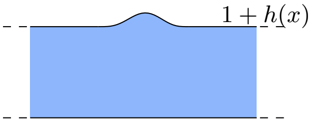

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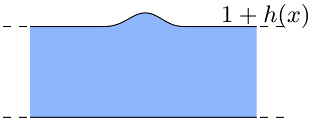

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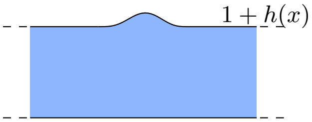

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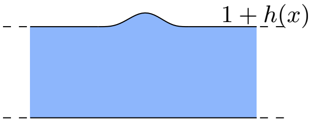

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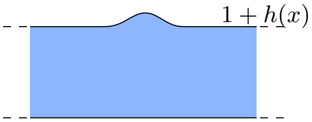

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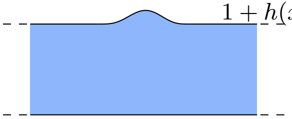

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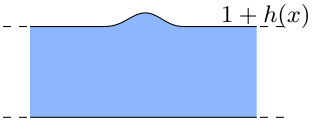

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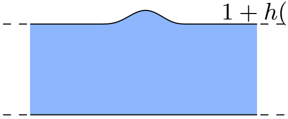

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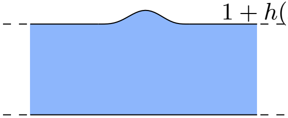

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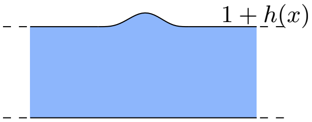

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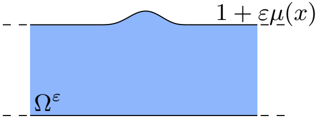
Set $h^{\mathrm{sol}} := \varepsilon\mu^{\mathrm{sol}}$. We have $R(h^{\mathrm{sol}}) = 0$ (non reflecting perturbation).

Calculus of the differential



- Using classical results of asymptotic analysis, we obtain

$$R(\varepsilon\mu) = 0 + \varepsilon \left(-\frac{1}{2} \int_{-\ell}^{\ell} \partial_x \mu(x) e^{2ikx}\, dx \right) + O(\varepsilon^2).$$

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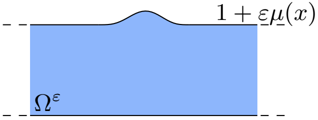

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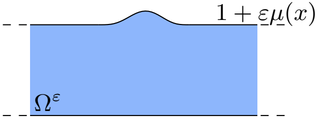

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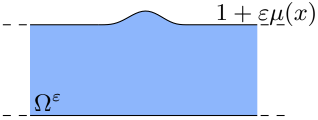

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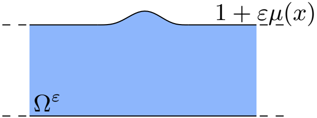
$$T(\varepsilon\mu) - 1 = 0 + \varepsilon\ 0\ + O(\varepsilon^2).$$

Calculus of the differential



- Using classical results of asymptotic analysis, we obtain

$$R(\varepsilon\mu) = 0 + \varepsilon \underbrace{\left(-\frac{1}{2} \int_{-\ell}^{\ell} \partial_x \mu(x) e^{2ikx}\, dx \right)}_{dR(0)(\mu)} + O(\varepsilon^2).$$

$dR(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ is onto $\Rightarrow$ we can get non trivial Ω s.t. $R = 0$.

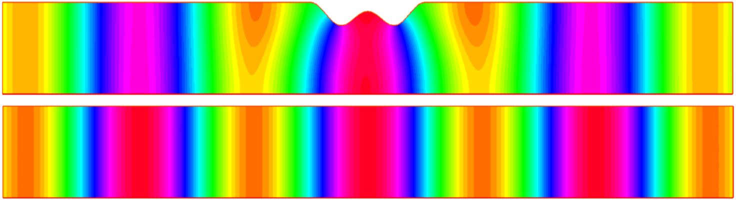
- Can we use the technique to construct Ω such that $T = 1$? We obtain

$$T(\varepsilon\mu) - 1 = 0 + \varepsilon\, \boxed{0} + O(\varepsilon^2).$$

$dT(0)$ is not onto $\Rightarrow$ the approach fails to impose $T = 1$.

Numerical results

- The fixed point problem can be solved iteratively: $\vec{\tau}^{\,n+1} = G^{\varepsilon}(\vec{\tau}^{\,n})$.



Numerics done by a group of students of École Polytechnique with the `Freefem++` *library* $\rightarrow$ P2 FEM + Dirichlet-to-Neumann to truncate Ω.

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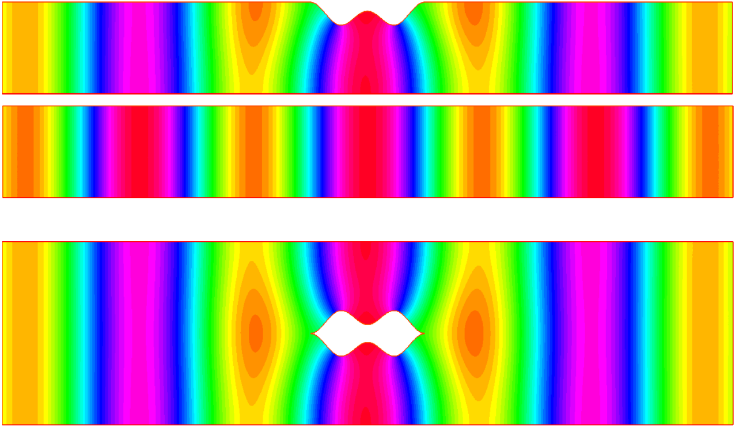

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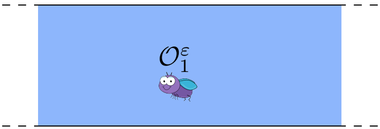
Outline of the talk

Small Dirichlet obstacle

Can one hide a small Dirichlet obstacle centered at M_1 ?



Find $u = u_i + u_s$ s. t.

$$\begin{aligned} \Delta u + k^2 u = 0 & \quad \text{in } \Omega^\varepsilon := \Omega \setminus \overline{\mathcal{O}_1^\varepsilon}, \\ u = 0 & \quad \text{on } \partial\Omega^\varepsilon, \\ u_s & \text{ is outgoing.} \end{aligned}$$

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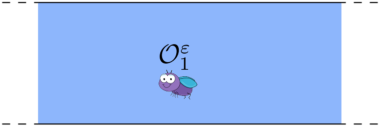

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$$\begin{aligned} R &= 0 + \varepsilon\ (4i\pi\,\mathrm{cap}(\mathcal{O}) w^+(M_1)^2)\ + O(\varepsilon^2) \\ T &= 1 + \varepsilon\ (4i\pi\,\mathrm{cap}(\mathcal{O}) |w^+(M_1)|^2)\ + O(\varepsilon^2). \end{aligned}$$

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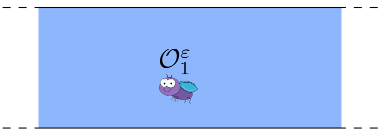

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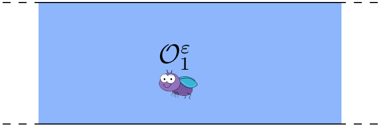

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Non zero terms!
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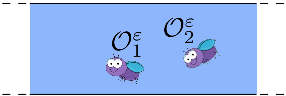
$\Rightarrow$ One single small obstacle cannot even be non reflecting.

Small Dirichlet obstacles



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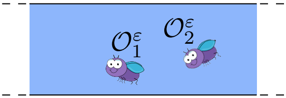

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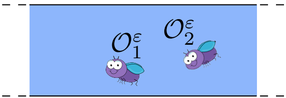

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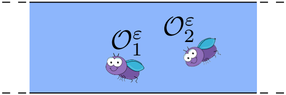

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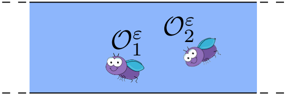

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→ When there are more propagative waves, we need more obstacles.

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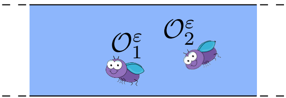

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Acting as a team, flies can become invisible!

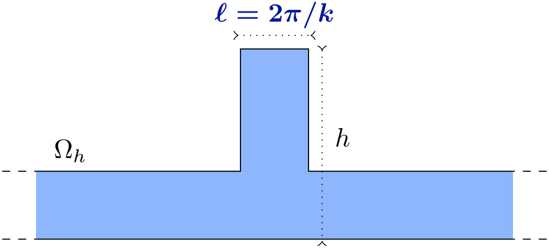
Outline of the talk

1. Smooth non reflecting perturbations of the reference strip
2. Non reflecting clouds of small obstacles
3. Construction of large invisible defects
4. Cloaking of given large obstacles

We constructed small defects such that $R = 0$. How to get large defects with $T = 1$?

Geometrical setting

- Let us work in the geometry



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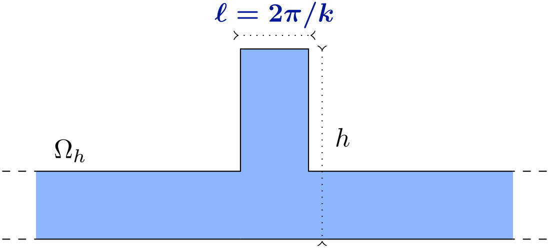
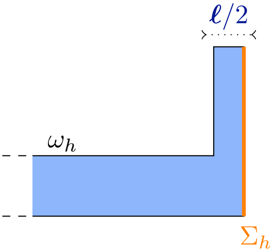


- Introduce the two half-waveguide problems



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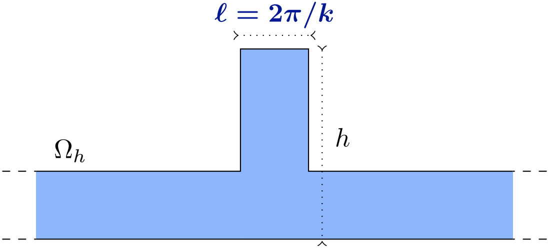

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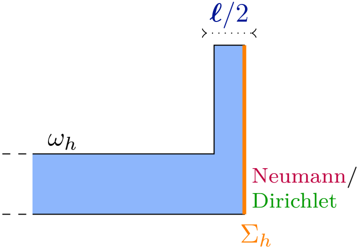

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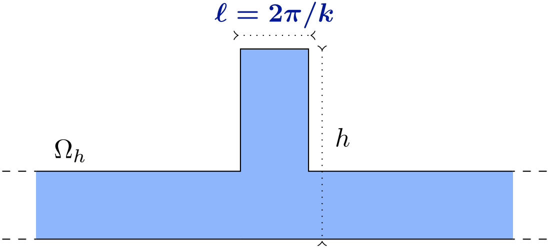

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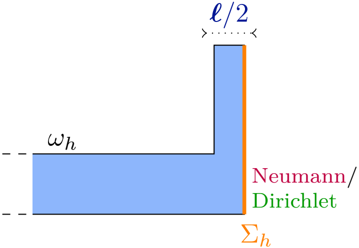

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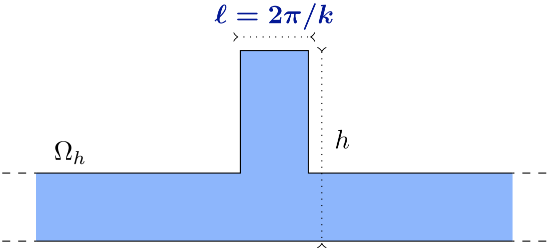

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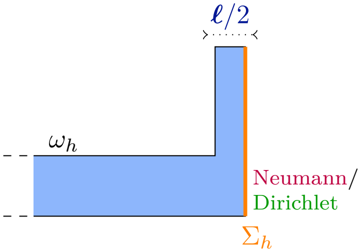

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Mixed B.C.

Relations for the scattering coefficients

- Half-waveguide problems admit the solutions

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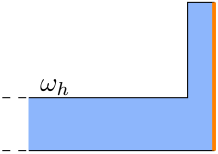

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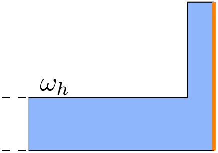

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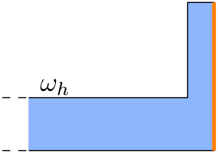

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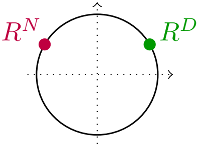

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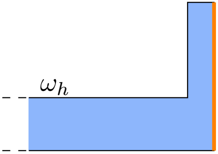

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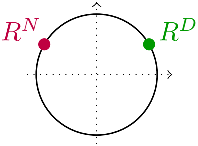

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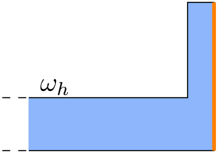

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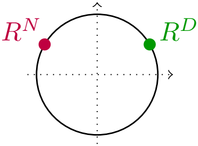

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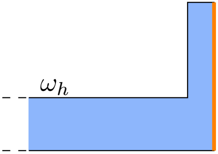

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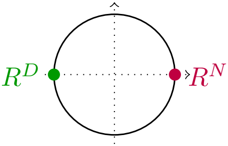

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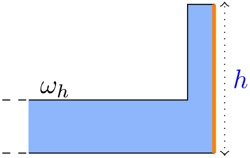

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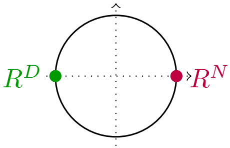

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Crucial point: in this **particular geometry** ω_h,
$u = w^+ + w^- = 2\cos(kx)$ solves the Neum. pb.

$\Rightarrow R^N = 1, \forall h > 1.$

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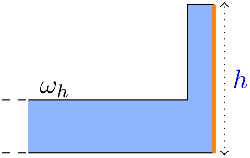

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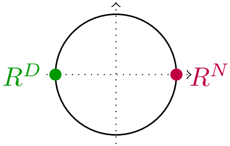


- Using symmetry considerations, one can show that

$$R = \frac{R^N + R^D}{2} \quad \text{and} \quad T = \frac{R^N - R^D}{2}$$

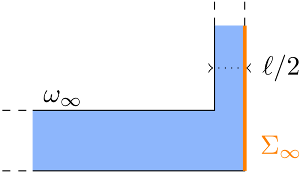
Perfect invisibility
$\Leftrightarrow\ [R^N = 1,\ R^D = -1]$

Crucial point: in this **particular geometry** ω_h, $u = w^+ + w^- = 2\cos(kx)$ solves the Neum. pb. $\Rightarrow R^N = 1, \forall h > 1.$

$\rightarrow$ It remains to study the behaviour of $R^D = R^D(h)$ as $h \rightarrow +\infty$.

Asymptotics of R^D as $h \to +\infty$

Depends on the nb. of propagating modes in the vertical branch of ω_∞



$$
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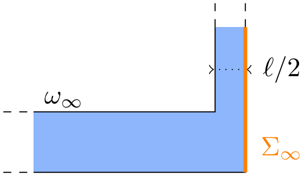

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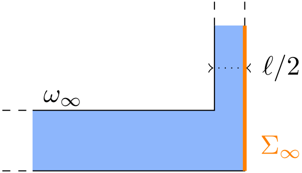

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▶ Using asymptotic analysis, one shows that when $h \to +\infty$,

$$|R^D(h) - R^D_{\mathrm{asy}}(h)| \leq Ce^{-ch}$$

where $R^D_{\mathrm{asy}}(h)$ runs periodically on the unit circle $\mathscr{C}$.

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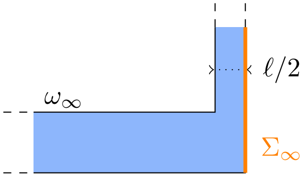

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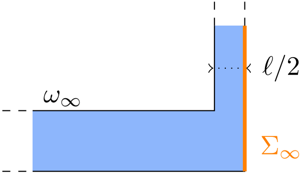

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- Additionally one can prove that $h \mapsto R^D(h)$ runs continuously on $\mathscr{C}$.

$\Rightarrow$ There is a sequence (h_n) with $h_n \to +\infty$ such that $R^D(h_n) = -1$.

Conclusion

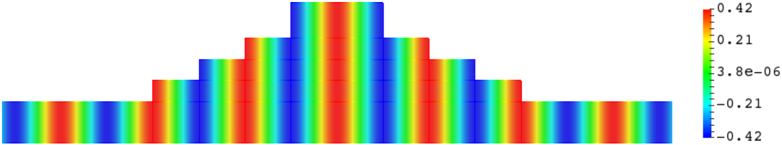
THEOREM: There is an unbounded sequence (h_n) such that for $h = h_n$, we have $T = 1$ (perfect invisibility).

Numerical results

- Works also in the geometry below. When we vary h, the height of the central branch, T runs exactly on the circle $\mathscr{C}(1/2, 1/2)$.

$\rightarrow$ Numerically, we simply sweep in h and extract the h such that $T(h) = 1$.

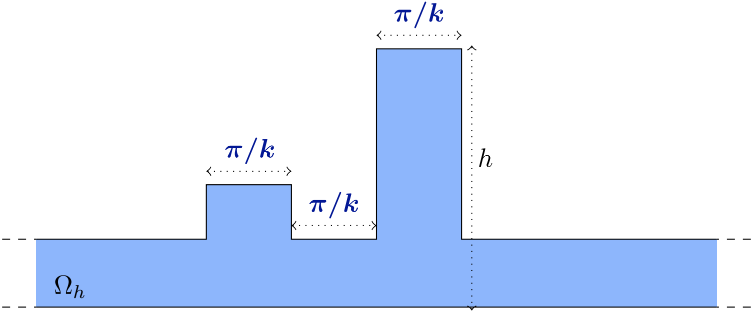
- Perfectly invisible defect ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$)



- Reference waveguide ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$)

Remark

- Actually Ω does not have to be symmetric and we can work in the following geometry:



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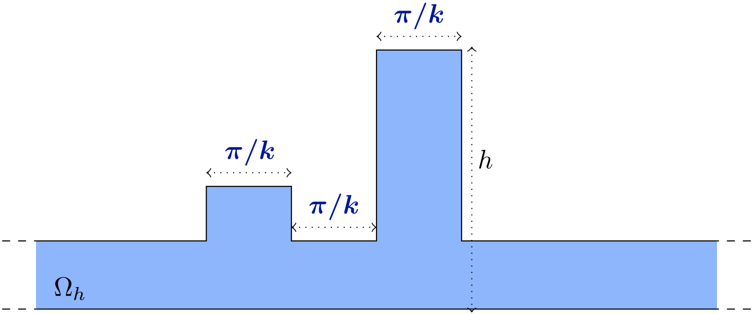


- In this Ω_h, we can show that there holds $R + T = 1$.

- With the identity of energy $|R|^2 + |T|^2 = 1$, this guarantees that T must be on the circle $\mathscr{C}(1/2, 1/2)$.

- Finally, with asy. analysis, we show that T goes through 1 as $h \to +\infty$.

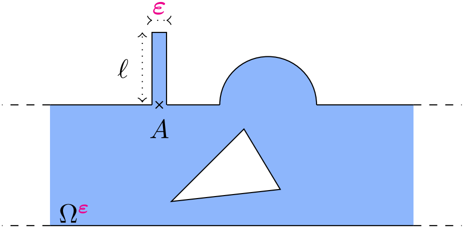
Outline of the talk

1. Smooth non reflecting perturbations of the reference strip
2. Non reflecting clouds of small obstacles
3. Construction of large invisible defects
4. Cloaking of given large obstacles

We constructed invisible defects.
How to hide given large obstacles ?

Setting

Main ingredient of our approach: outer resonators of width $\varepsilon \ll 1$.



$$(\mathscr{P}^{\varepsilon}) \left| \begin{array}{rl} \Delta u + k^2 u = 0 & \text{in } \Omega^{\varepsilon}, \\ \partial_n u = 0 & \text{on } \partial\Omega^{\varepsilon} \end{array} \right.$$

- In this geometry, we have the scattering solutions

$$u_+^{\varepsilon} = \left| \begin{array}{r} e^{ikx} + R_+^{\varepsilon}\, e^{-ikx} + \dots \\ T^{\varepsilon}\, e^{+ikx} + \dots \end{array} \right. \qquad u_-^{\varepsilon} = \left| \begin{array}{r} T^{\varepsilon}\, e^{-ikx} + \dots \\ e^{-ikx} + R_-^{\varepsilon}\, e^{+ikx} + \dots \end{array} \right. \qquad \begin{array}{l} x \to -\infty \\ x \to +\infty \end{array}$$

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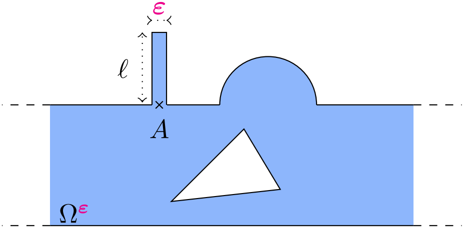

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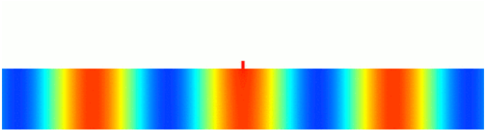
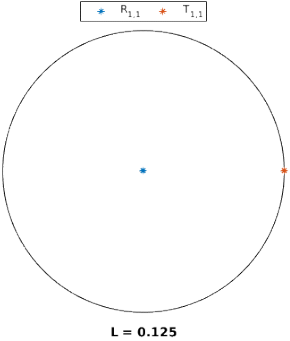
In general, the thin ligament has only a weak influence on the scattering coefficients: $R_{\pm}^{\varepsilon} \approx R_{\pm}$, $T^{\varepsilon} \approx T$. But not always ...

Numerical experiment

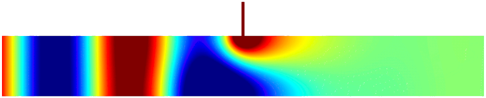
- We vary the length of the ligament:

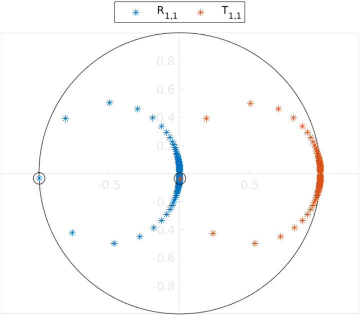




Numerical experiment

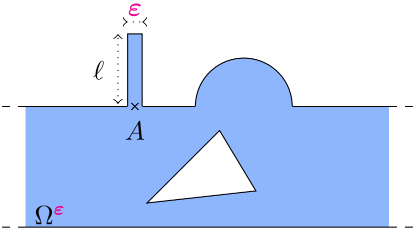
- For one particular length of the ligament, we get a standing mode (zero transmission):





Asymptotic analysis

To understand the phenomenon, we compute an asymptotic expansion of u_+^ε, R_+^ε, T^ε as $\varepsilon \to 0$.



$$(\mathscr{P}^\varepsilon)\left|\begin{array}{rl}\Delta u_+^\varepsilon + k^2 u_+^\varepsilon = 0 & \text{in } \Omega^\varepsilon,\\ \partial_n u_+^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon\end{array}\right.$$

$$u_+^\varepsilon = \left|\begin{array}{r} e^{ikx} + R_+^\varepsilon\, e^{-ikx} + \dots \\ T^\varepsilon\, e^{+ikx} + \dots\end{array}\right.$$

- To proceed we use techniques of matched asymptotic expansions (see Beale 73, Gadyl'shin 93, Kozlov et al. 94, Nazarov 96, Maz'ya et al. 00, Joly & Tordeux 06, Lin, Shipman & Zhang 17, 18, Brandao, Holley, Schnitzer 20, ...).

Asymptotic analysis

- We work with the outer expansions

$$\begin{aligned} u_+^\varepsilon(x,y) &= u^0(x,y) + \dots && \text{in } \Omega, \\ u_+^\varepsilon(x,y) &= \varepsilon^{-1} v^{-1}(y) + v^0(y) + \dots && \text{in the resonator.} \end{aligned}$$

- Considering the restriction of $(\mathscr{P}^\varepsilon)$ to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous 1D problem

$$(\mathscr{P}_{1\mathrm{D}}) \left| \begin{array}{ll} \partial_y^2 v + k^2 v = 0 & \text{in } (1; 1+\ell) \\ v(1) = \partial_y v(1+\ell) = 0. & \end{array} \right.$$

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The features of $(\mathscr{P}_{1\text{D}})$ play a key role in the physical phenomena and in the asymptotic analysis.

- We denote by ℓ_{res} (**resonance lengths**) the values of ℓ, given by

$$\ell_{\text{res}} := \pi(m + 1/2)/k, \qquad m \in \mathbb{N},$$

such that $(\mathscr{P}_{1\text{D}})$ admits the non zero solution $v(y) = \sin(k(y-1))$.

Asymptotic analysis – Non resonant case

- Assume that $\ell \neq \ell_{\text{res}}$. Then we find $v^{-1} = 0$ and when $\varepsilon \to 0$, we get

$$
\begin{aligned}
& u^{\varepsilon}_{\pm}(x,y) = \, u_{\pm} \, + o(1) && \text{in } \Omega, \\
& u^{\varepsilon}_{\pm}(x,y) = u_{\pm}(A)\, v_0(y) + o(1) && \text{in the resonator}, \\
& R^{\varepsilon}_{\pm} = \, R_{\pm} \, + o(1), && T^{\varepsilon} = \, T \, + o(1).
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Here $v_0(y) = \cos(k(y-1) + \tan(k(y-\ell)\sin(k(y-1)$.

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The thin resonator has no influence at order ε^0.

→ **Not interesting for our purpose** because we want $\left| \begin{array}{l} R^{\varepsilon}_{\pm} = 0 + \dots \\ T^{\varepsilon} = 1 + \dots \end{array} \right.$

Asymptotic analysis – Resonant case

- For $\ell = \ell_{\text{res}}$, when $\varepsilon \to 0$, we obtain

$$u_+^\varepsilon(x,y) = u_+(x,y) + ak\gamma(x,y) + o(1) \quad \text{in } \Omega,$$

$$u_+^\varepsilon(x,y) = \varepsilon^{-1} a \sin(k(y-1)) + O(1) \quad \text{in the resonator},$$

$$R_+^\varepsilon = R_+ + iau_+(A)/2 + o(1), \qquad T^\varepsilon = T + iau_-(A)/2 + o(1).$$

Here γ is the outgoing Green function such that $\left| \begin{array}{l} \Delta\gamma + k^2\gamma = 0 \text{ in } \Omega \\ \partial_n \gamma = \delta_A \text{ on } \partial\Omega \end{array} \right.$ and

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This time the thin resonator has an influence at order ε^0

Asymptotic analysis – Resonant case

- For $\ell = \ell_{\text{res}} + \varepsilon\eta$ with $\eta \in \mathbb{R}$ fixed, when $\varepsilon \to 0$, we obtain

$$u_+^\varepsilon(x,y) = u_+(x,y) + a(\eta)k\gamma(x,y) + o(1) \qquad \text{in } \Omega,$$

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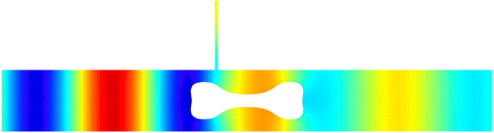
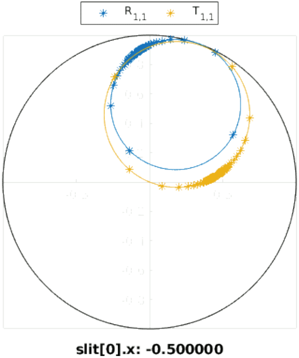
This time the thin resonator has an influence at order ε^0 and it depends on the choice of η!

Almost zero reflection

From this expansion, we find that asymptotically, when the length of the resonator is perturbed **around $\boldsymbol{\ell}_{\mathbf{res}}$**, R_+^ε, T^ε run on **circles** whose **features depend on the choice for A**.





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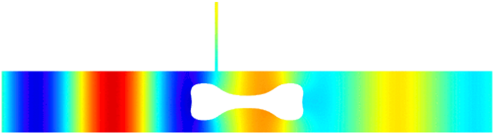

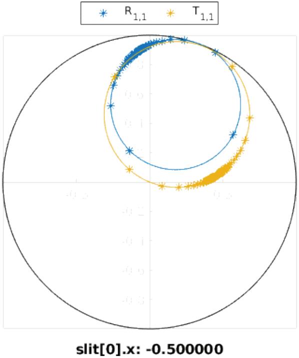

- Using the expansions of $u_\pm(A)$ far from the obstacle, one shows:

Proposition: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta) \,|\, \eta \in \mathbb{R}\}$ passes **through zero**.

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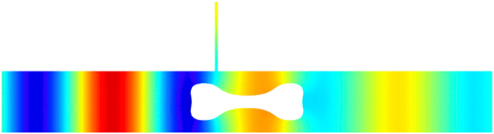

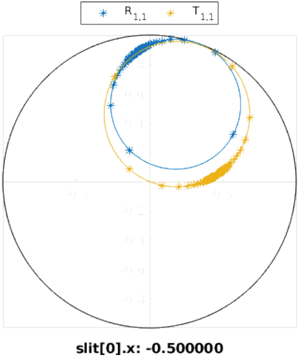

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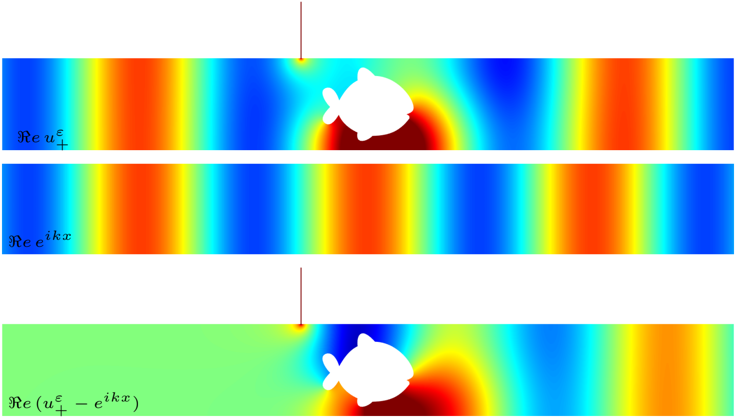
PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta)\,|\,\eta\in\mathbb{R}\}$ passes **through zero**. $\Rightarrow \exists$ situations s.t. $R_+^\varepsilon = 0 + o(1)$.

Almost zero reflection

- Example of situation where we have almost zero reflection ($\varepsilon = \mathbf{0.01}$).



Simulations realized with the `Freefem++` *library.*

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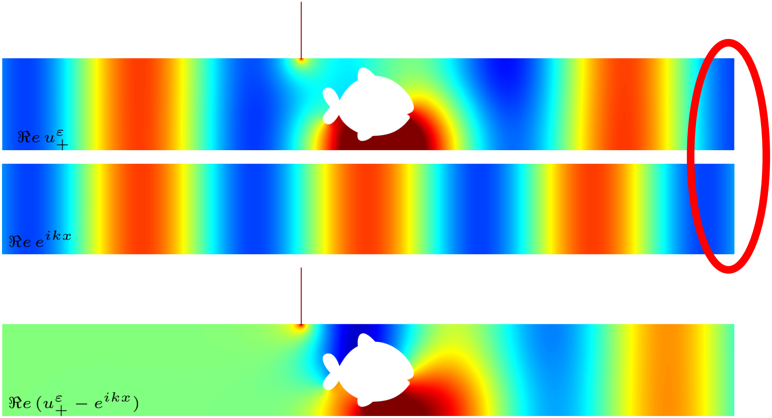
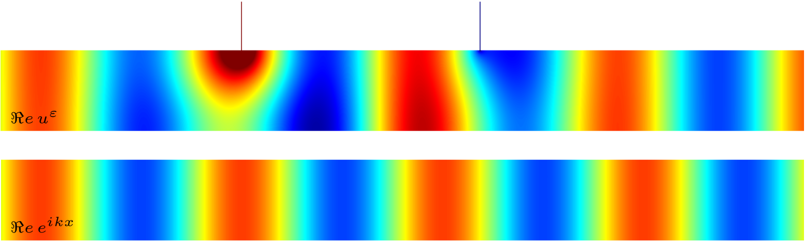


Simulations realized with the `Freefem++` *library.*

Conservation of energy guarantees that when $R_+^\varepsilon = 0$, $|T^\varepsilon| = 1$.
$\rightarrow$ To cloak the object, it remains to compensate the phase shift!

Phase shifter

- Working with two resonators, we can create phase shifters, that is devices with almost zero reflection and any desired phase.

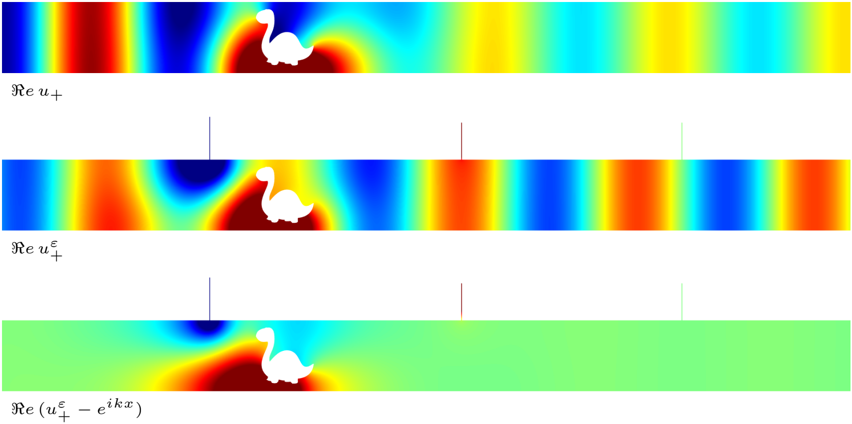


- Here the device is designed to obtain a phase shift approx. equal to $\pi/4$.

Cloaking with three resonators

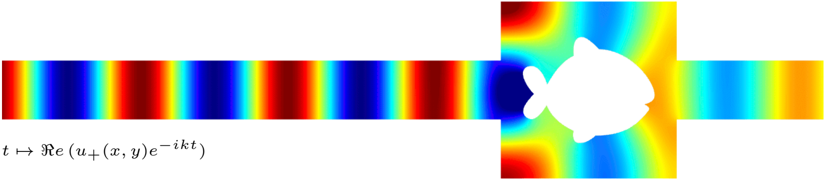
- Now working in two steps, we can approximately cloak any object with three resonators:

1) With one resonant ligament, first we get almost zero reflection;
2) With two additional resonant ligaments, we compensate the phase shift.

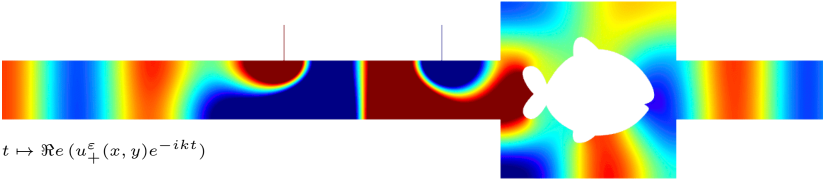


Cloaking with two resonators

- Working a bit more, one can show that two resonators are enough to cloak any object.



$t \mapsto \Re e\,(u_+(x,y)e^{-ikt})$



$t \mapsto \Re e\,(u_+^{\varepsilon}(x,y)e^{-ikt})$

$t \mapsto \Re e\,(e^{ik(x-t)})$

Outline of the talk

1. Smooth non reflecting perturbations of the reference strip

2. Non reflecting clouds of small obstacles

3. Construction of large invisible defects

4. Cloaking of given large obstacles

Conclusion

What we did

1) We constructed small smooth non reflecting perturbations of the reference strip.

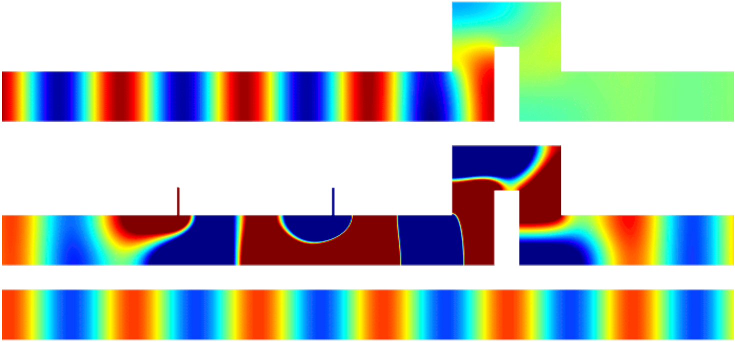
2) We explained how clouds of small obstacles can be non reflecting.

3) We constructed large obstacles which are perfectly invisible.

4) We showed how to hide approximately ($T \approx 1$) given large obstacles.

Future work

- Can one hide given large obstacles at higher frequency?
- Can one hide exactly given large obstacles?
- Can we get for example small reflection for an interval of frequencies?
- What can be done for water-waves, electromagnetism,...?



Thank you for your attention!

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