Asymptotic Analysis and Spectral Theory (Aspect'22)

Invisibility in acoustic waveguides

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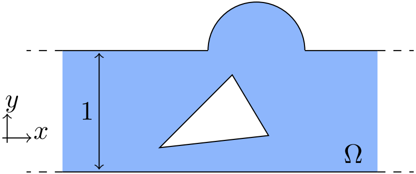
- We consider the propagation of waves in a 2D acoustic waveguide with an obstacle (also relevant in optics, microwaves, water-waves theory,...).



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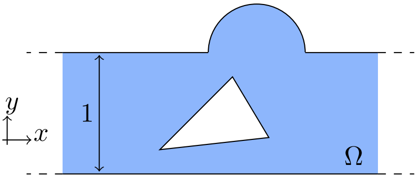

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- We fix $k \in (0;\pi)$ so that only the plane waves $e^{\pm ikx}$ can propagate.

- The scattering of these waves leads us to consider the solutions of $(\mathscr{P})$ with the decomposition

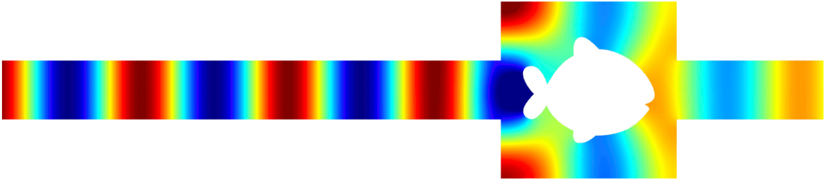
$$u_+ = \left| \begin{array}{l} e^{ikx} + R_+ \, e^{-ikx} + \dots \\ \quad\; T \quad e^{+ikx} + \dots \end{array} \right. \qquad u_- = \left| \begin{array}{ll} \quad\; T \quad e^{-ikx} + \dots & x \to -\infty \\ e^{-ikx} + R_- \, e^{+ikx} + \dots & x \to +\infty \end{array} \right.$$

$R_\pm$, $T \in \mathbb{C}$ are the scattering coefficients, the $\dots$ are expon. decaying terms.

- We have the relations of conservation of energy $|R_\pm|^2 + |T|^2 = 1$.

- Without obstacle, $u_+ = e^{ikx}$ so that $(R_+, T) = (0, 1)$.

- With an obstacle, in general $(R_+, T) \neq (0, 1)$.



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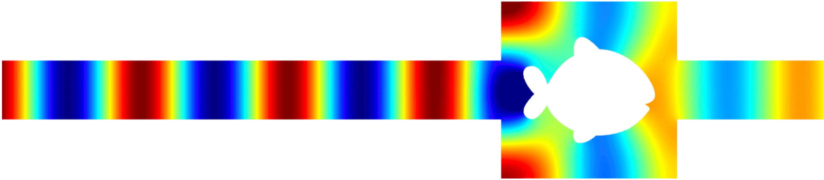
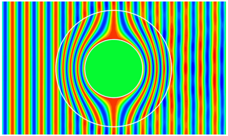


Goal of the talk

We wish to identify situations (geometries, k) where $R_{\pm} = 0$ and/or $T = 1$ (as if there were no obstacle) $\Rightarrow$ cloaking at "infinity".

Introduction 3/3

Difficulty: the scattering coefficients have a non explicit and non linear dependence wrt the geometry and k.



Remark: different from the usual cloaking picture (Pendry *et al.* 06, Leonhardt 06, Greenleaf *et al.* 09) because we wish to control only the scattering coef..

→ Less ambitious but doable without fancy materials (and relevant in practice).

Outline of the talk

We present **two different** points of view on these questions of invisibility:

1. Cloaking of obstacles

   Asymptotic analysis:

   k and Ω are given, we explain how to **perturb the geometry** using **thin resonant ligaments** to get $T \approx 1$.

2. A spectral approach to determine non reflecting wavenumbers

   Spectral theory:

   Ω is given, we explain how to **find non reflecting** k by solving an unusual **spectral problem**.

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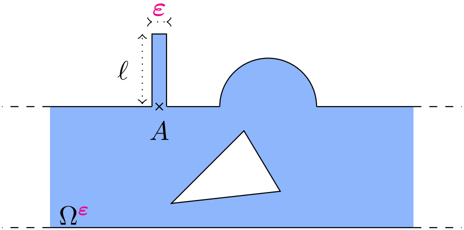
Setting

Main ingredient of our approach: outer resonators of width $\varepsilon \ll 1$.



$$(\mathscr{P}^{\varepsilon})\ \left|\ \begin{array}{rl} \Delta u + k^2 u = 0 & \text{in } \Omega^{\varepsilon}, \\ \partial_n u = 0 & \text{on } \partial\Omega^{\varepsilon} \end{array}\right.$$

- In this geometry, we have the scattering solutions

$$u_+^{\varepsilon} = \left| \begin{array}{r} e^{ikx} + R_+^{\varepsilon}\, e^{-ikx} + \dots \\ T^{\varepsilon}\, e^{+ikx} + \dots \end{array}\right. \qquad u_-^{\varepsilon} = \left| \begin{array}{r} T^{\varepsilon}\, e^{-ikx} + \dots \\ e^{-ikx} + R_-^{\varepsilon}\, e^{+ikx} + \dots \end{array}\right. \qquad \begin{array}{l} x \to -\infty \\ x \to +\infty \end{array}$$

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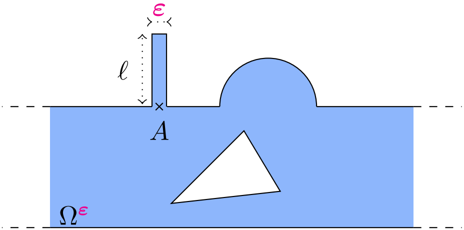

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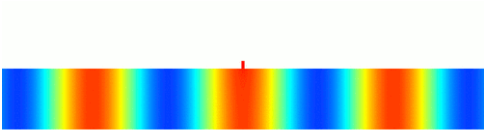
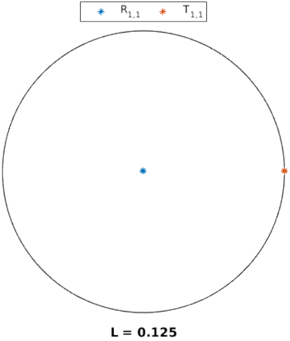
In general, the thin ligament has only a weak influence on the scattering coefficients: $R_{\pm}^{\varepsilon} \approx R_{\pm}$, $T^{\varepsilon} \approx T$. But not always ...

Numerical experiment

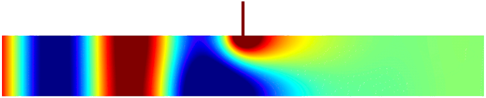
- We vary the length of the ligament:

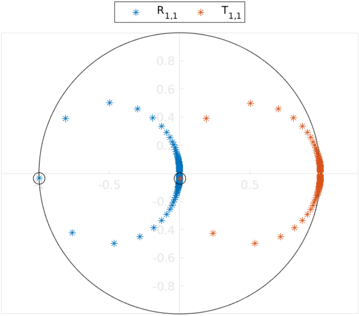




Numerical experiment

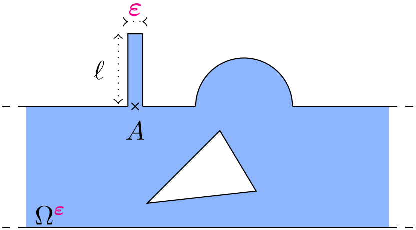
► For one particular length of the ligament, we get a standing mode (zero transmission):





Asymptotic analysis

To understand the phenomenon, we compute an asymptotic expansion of u_+^ε, R_+^ε, T^ε as $\varepsilon \to 0$.



$$(\mathscr{P}^\varepsilon) \left| \begin{array}{rl} \Delta u_+^\varepsilon + k^2 u_+^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ \partial_n u_+^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon \end{array} \right.$$

$$u_+^\varepsilon = \left| \begin{array}{l} e^{ikx} + R_+^\varepsilon\, e^{-ikx} + \dots \\ \qquad\quad T^\varepsilon\, e^{+ikx} + \dots \end{array} \right.$$

► To proceed we use techniques of matched asymptotic expansions (see Beale 73, Gadyl'shin 93, Kozlov et al. 94, Nazarov 96, Maz'ya et al. 00, Joly & Tordeux 06, Lin & Zhang 17, 18, Brandao, Holley, Schnitzer 20, ...).

Asymptotic analysis

- We work with the outer expansions

$$\begin{aligned} u_+^\varepsilon(x,y) &= u^0(x,y) + \dots && \text{in } \Omega, \\ u_+^\varepsilon(x,y) &= \varepsilon^{-1} v^{-1}(y) + v^0(y) + \dots && \text{in the resonator.} \end{aligned}$$

- Considering the restriction of $(\mathscr{P}^\varepsilon)$ to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous 1D problem

$$(\mathscr{P}_{1\mathrm{D}}) \left| \begin{array}{ll} \partial_y^2 v + k^2 v = 0 & \text{in } (1; 1+\ell) \\ v(1) = \partial_y v(1+\ell) = 0. & \end{array} \right.$$

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Asymptotic analysis

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The features of $(\mathscr{P}_{1\mathrm{D}})$ play a key role in the physical phenomena and in the asymptotic analysis.

- We denote by ℓ_{res} (**resonance lengths**) the values of ℓ, given by

$$\ell_{\mathrm{res}} := \pi(m + 1/2)/k, \qquad m \in \mathbb{N},$$

such that $(\mathscr{P}_{1\mathrm{D}})$ admits the non zero solution $v(y) = \sin(k(y-1))$.

Asymptotic analysis – Non resonant case

- Assume that $\ell \neq \ell_{\text{res}}$. Then we find $v^{-1} = 0$ and when $\varepsilon \to 0$, we get

$$
\begin{aligned}
&u^\varepsilon_\pm(x,y) = u_\pm + o(1) && \text{in } \Omega,\\
&u^\varepsilon_\pm(x,y) = u_\pm(A)\, v_0(y) + o(1) && \text{in the resonator},\\
&R^\varepsilon_\pm = R_\pm + o(1), && T^\varepsilon = T + o(1).
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Here $v_0(y) = \cos(k(y-1) + \tan(k(y-\ell)\sin(k(y-1)$.

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The thin resonator has no influence at order ε^0.

$\rightarrow$ **Not interesting for our purpose** because we want $\left| \begin{array}{l} R^{\varepsilon}_{\pm} = 0 + \dots \\ T^{\varepsilon} = 1 + \dots \end{array} \right.$

Asymptotic analysis – Resonant case

- For $\ell = \ell_{\text{res}}$, when $\varepsilon \to 0$, we obtain

$$u_+^\varepsilon(x,y) = u_+(x,y) + ak\gamma(x,y) + o(1) \quad \text{in } \Omega,$$

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$$R_+^\varepsilon = R_+ + iau_+(A)/2 + o(1), \qquad T^\varepsilon = T + iau_-(A)/2 + o(1).$$

Here γ is the outgoing Green function such that $\left| \begin{array}{l} \Delta\gamma + k^2\gamma = 0 \text{ in } \Omega \\ \partial_n\gamma = \delta_A \text{ on } \partial\Omega \end{array} \right.$ and

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This time the thin resonator has an influence at order ε^0

Asymptotic analysis – Resonant case

- For $\ell = \ell_{\text{res}} + \varepsilon\eta$ with $\eta \in \mathbb{R}$ fixed, when $\varepsilon \to 0$, we obtain

$$u_+^\varepsilon(x,y) = u_+(x,y) + a(\eta)k\gamma(x,y) + o(1) \qquad \text{in } \Omega,$$

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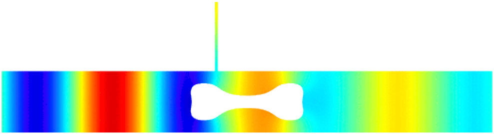
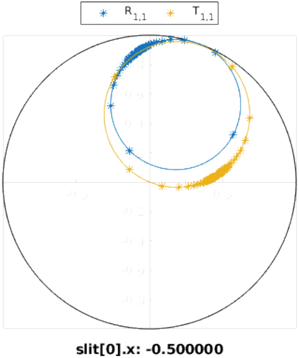
This time the thin resonator has an influence at order ε^0 and it depends on the choice of η!

Almost zero reflection

From this expansion, we find that asymptotically, when the length of the resonator is perturbed **around $\boldsymbol{\ell}_{\mathbf{res}}$**, R_+^ε, T^ε run on **circles** whose **features depend on the choice for A**.





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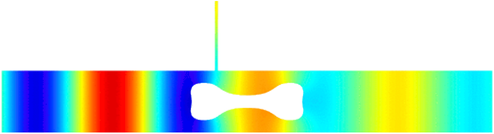

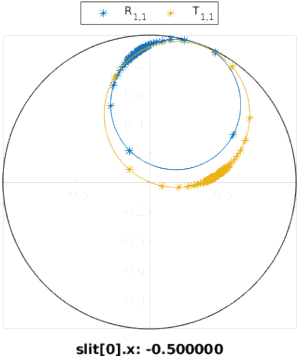

- Using the expansions of $u_\pm(A)$ far from the obstacle, one shows:

PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta)\,|\,\eta\in\mathbb{R}\}$ passes **through zero**.

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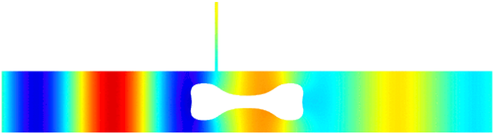

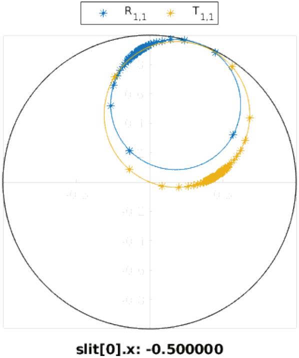

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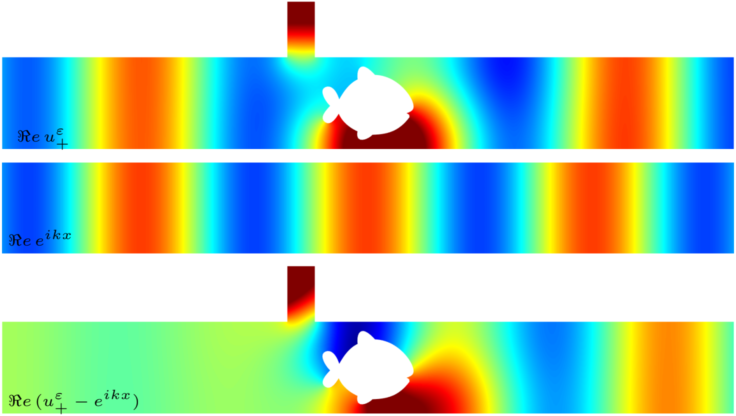
PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta)\,|\,\eta \in \mathbb{R}\}$ passes **through zero**. $\Rightarrow \exists$ situations s.t. $R_+^\varepsilon = 0 + o(1)$.

Almost zero reflection

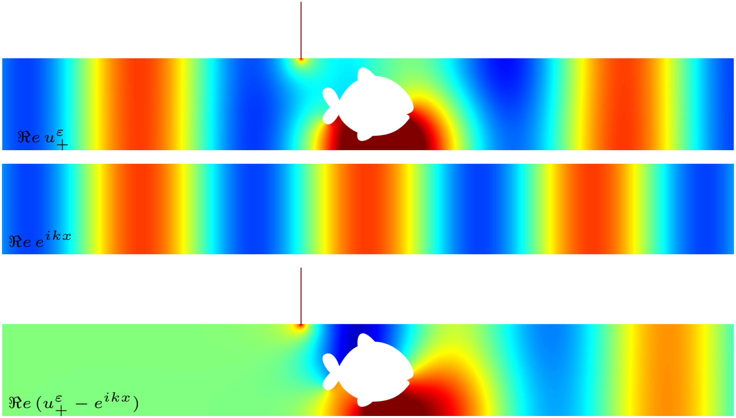
- Example of situation where we have almost zero reflection ($\varepsilon = \mathbf{0.3}$).



Simulations realized with the `Freefem++` *library.*

Almost zero reflection

- Example of situation where we have almost zero reflection ($\varepsilon = \mathbf{0.01}$).



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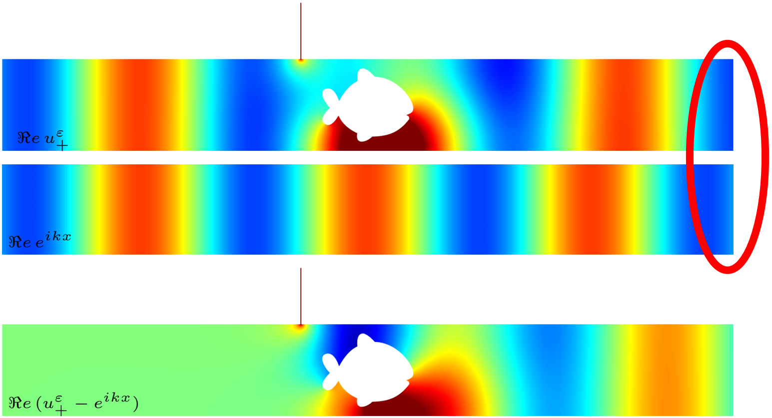

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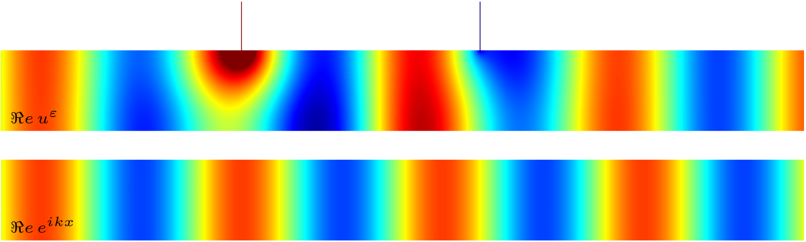
Conservation of energy guarantees that when $R_+^\varepsilon = 0$, $|T^\varepsilon| = 1$.
$\rightarrow$ To cloak the object, it remains to compensate the phase shift!

Phase shifter

- Working with two resonators, we can create phase shifters, that is devices with almost zero reflection and any desired phase.

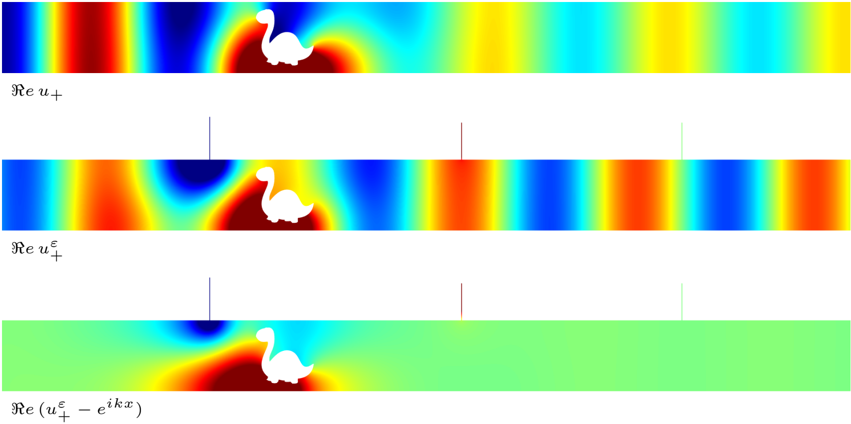


- Here the device is designed to obtain a phase shift approx. equal to $\pi/4$.

Cloaking with three resonators

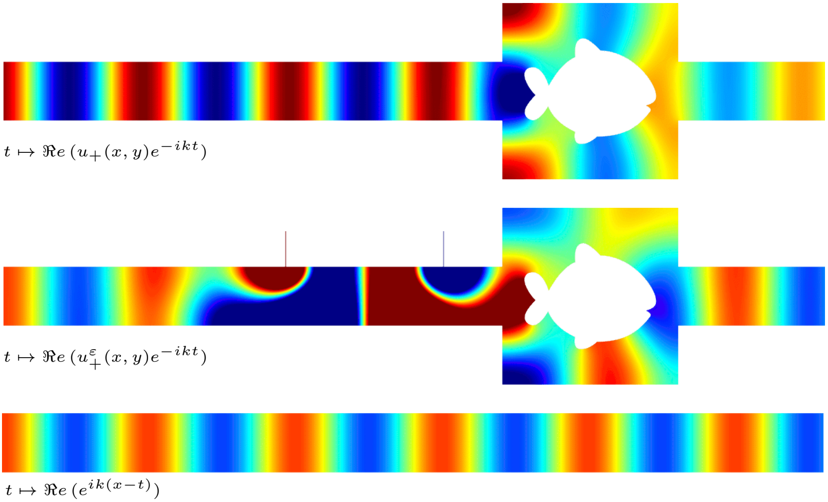
- Now working in two steps, we can approximately cloak any object with three resonators:

1) With one resonant ligament, first we get almost zero reflection;
2) With two additional resonant ligaments, we compensate the phase shift.



Cloaking with two resonators

- Working a bit more, one can show that two resonators are enough to cloak any object.



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Scattering problem

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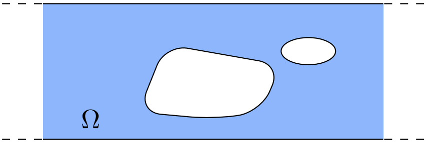

Find $v = v_i + v_s$ s. t.

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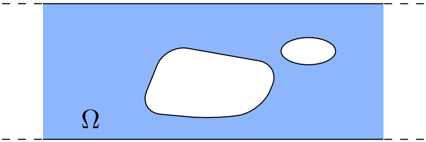

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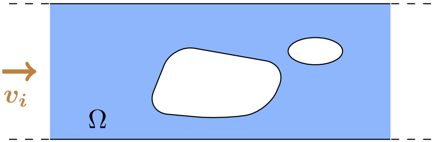
- For this problem, the modes are

Propagating	$w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y)$, $\beta_n = \sqrt{k^2 - n^2\pi^2}$, $n \in [\![0, N-1]\!]$
Evanescent	$w_n^{\pm}(x,y) = e^{\mp \beta_n x}\cos(n\pi y)$, $\beta_n = \sqrt{n^2\pi^2 - k^2}$, $n \geq N$.

Scattering problem

- Consider the scattering problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$



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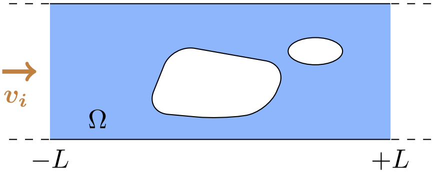

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- v_s is outgoing $\Leftrightarrow$ $v_s = \sum_{n=0}^{+\infty} \gamma_n^{\pm} w_n^{\pm}$ for $\pm x \geq L$, with $(\gamma_n^{\pm}) \in \mathbb{C}^{\mathbb{N}}$.

Goal of the section

DEFINITION: v is a non reflecting mode if v_s is expo. decaying for $x \leq -L$
$\Leftrightarrow \quad \gamma_n^- = 0,\ n \in [\![0, N-1]\!] \quad \Leftrightarrow$ energy is completely transmitted.

GOAL

For a given geometry, we present a method to find values of k such that there is a non reflecting mode v.

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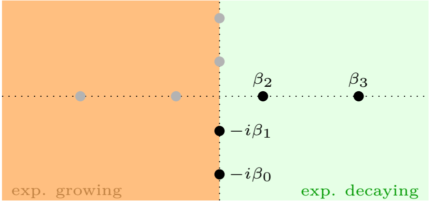
GOAL

For a given geometry, we present a method to find values of k such that there is a non reflecting mode v.

→ Note that non reflection occurs for **particular** $\boldsymbol{v_i}$ to be computed.

Classical complex scaling to compute v_s 1/2

REMINDER: $v_s = \sum_{n=0}^{N-1} \gamma_n^{\pm} e^{\pm i\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \gamma_n^{\pm} e^{\mp \beta_n x} \cos(n\pi y), \ \pm x \geq L.$



Modal exponents for v_s ($x \leq -L$)

Classical complex scaling to compute v_s 1/2

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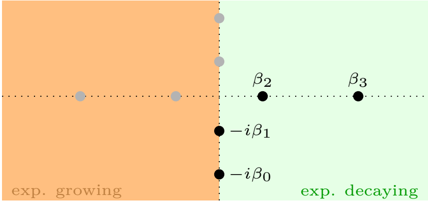

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- For $\theta \in (0; \pi/2)$, consider the complex change of variables

$$\mathcal{I}_\theta(x) = \begin{vmatrix} -L + (x+L)\, e^{i\theta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x-L)\, e^{i\theta} & \text{for } x \geq L. \end{vmatrix}$$

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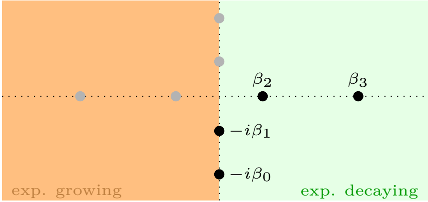

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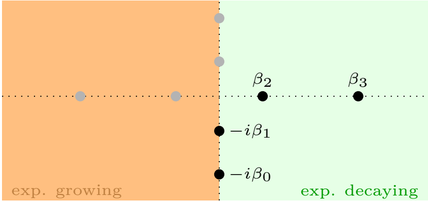

Modal exponents for v_s $(x \leq -L)$

$$v_\theta = \sum_{n=0}^{N-1} \tilde{\gamma}_n^\pm \, e^{\pm i\tilde{\beta}_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \tilde{\gamma}_n^\pm \, e^{\mp \tilde{\beta}_n x} \cos(n\pi y), \quad \pm x \geq L \quad \tilde{\beta}_n = \beta_n e^{i\theta}$$

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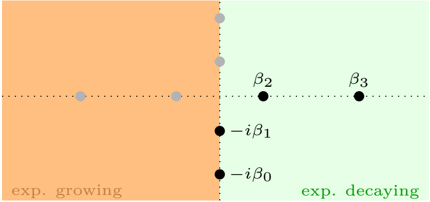

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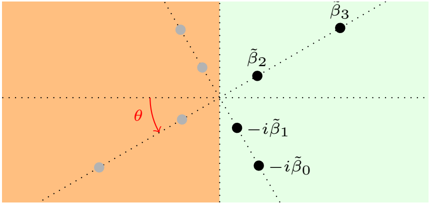

Modal exponents for v_θ $(x \leq -L)$

$$v_\theta = \sum_{n=0}^{N-1} \tilde\gamma_n^{\pm} e^{\pm i\tilde\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \tilde\gamma_n^{\pm} e^{\mp \tilde\beta_n x} \cos(n\pi y), \quad \pm x \geq L \quad \tilde\beta_n = \beta_n e^{i\theta}$$

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1) $v_\theta = v_s$ for $|x| < L$.
2) v_θ is exp. decaying at infinity.

Classical complex scaling to compute v_s 2/2

- v_θ solves

$$(*)\ \left|\ \begin{array}{rcll} \alpha_\theta \dfrac{\partial}{\partial x}\left(\alpha_\theta \dfrac{\partial v_\theta}{\partial x}\right) + \dfrac{\partial^2 v_\theta}{\partial y^2} + k^2 v_\theta & = & 0 & \text{in } \Omega \\ \partial_n v_\theta & = & -\partial_n v_i & \text{on } \partial\Omega. \end{array}\right.$$

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- v_θ solves $(*)\left|\begin{array}{rcll} \alpha_\theta \dfrac{\partial}{\partial x}\left(\alpha_\theta \dfrac{\partial v_\theta}{\partial x}\right) + \dfrac{\partial^2 v_\theta}{\partial y^2} + k^2 v_\theta & = & 0 & \text{in } \Omega \\ \partial_n v_\theta & = & -\partial_n v_i & \text{on } \partial\Omega. \end{array}\right.$

$$\alpha_\theta(x) = 1 \text{ for } |x| < L \qquad \alpha_\theta(x) = e^{-i\theta} \text{ for } |x| \geq L$$

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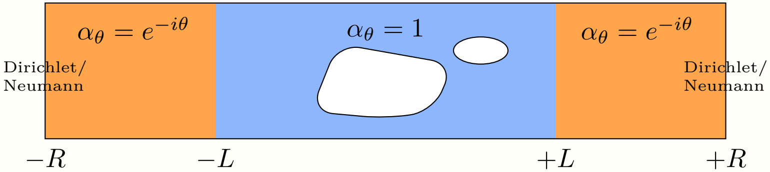
- Numerically we solve $(*)$ in the truncated domain



$\Rightarrow$ We obtain a good approximation of v_s for $|x| < L$.

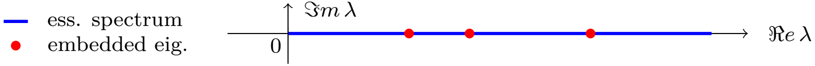
- This is the method of Perfectly Matched Layers (PMLs).

Spectral analysis

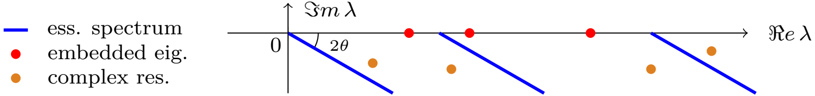
- Define the operators A, A_θ of $\mathrm{L}^2(\Omega)$ such that

$$Av = -\Delta v, \qquad A_\theta v = -\Big(\alpha_\theta \frac{\partial}{\partial x}\Big(\alpha_\theta \frac{\partial v}{\partial x}\Big) + \frac{\partial^2 v}{\partial y^2}\Big) \qquad + \partial_n v = 0 \text{ on } \partial\Omega.$$

- A is selfadjoint and positive.
- $\sigma(A) = \sigma_{\mathrm{ess}}(A) = [0;+\infty)$.
- $\sigma(A)$ may contain embedded eigenvalues in the essential spectrum.

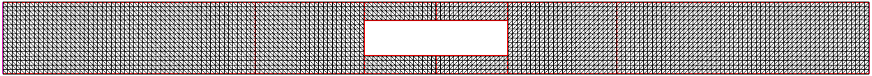


- A_θ is not selfadjoint. $\sigma(A_\theta) \subset \{\rho\, e^{i\gamma},\ \rho \geq 0,\ \gamma \in [-2\theta; 0]\}$.
- $\sigma_{\mathrm{ess}}(A_\theta) = \cup_{n\in\mathbb{N}}\{n^2\pi^2 + t\, e^{-2i\theta},\ t \geq 0\}$.
- real eigenvalues of A_θ = real eigenvalues of A.



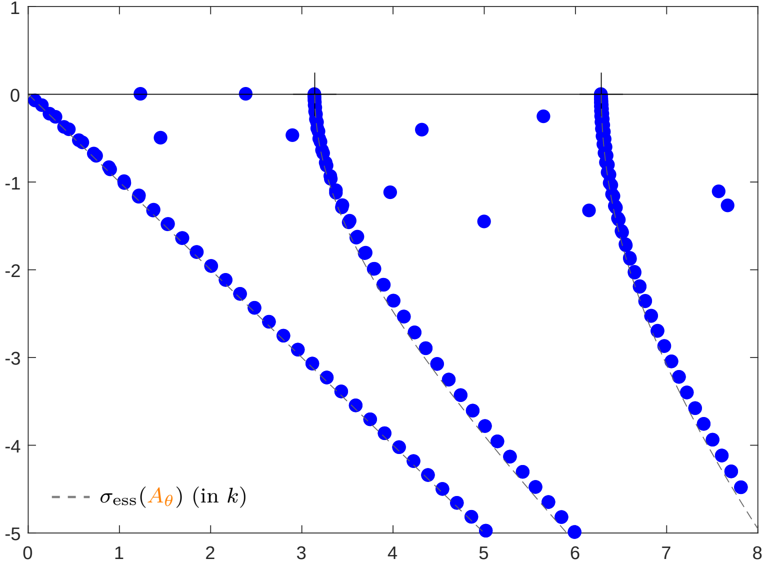
Numerical results

- We work in the geometry



Numerical results

- Discretized spectrum of A_θ in k (not in k^2). We take $\theta = \pi/4$.



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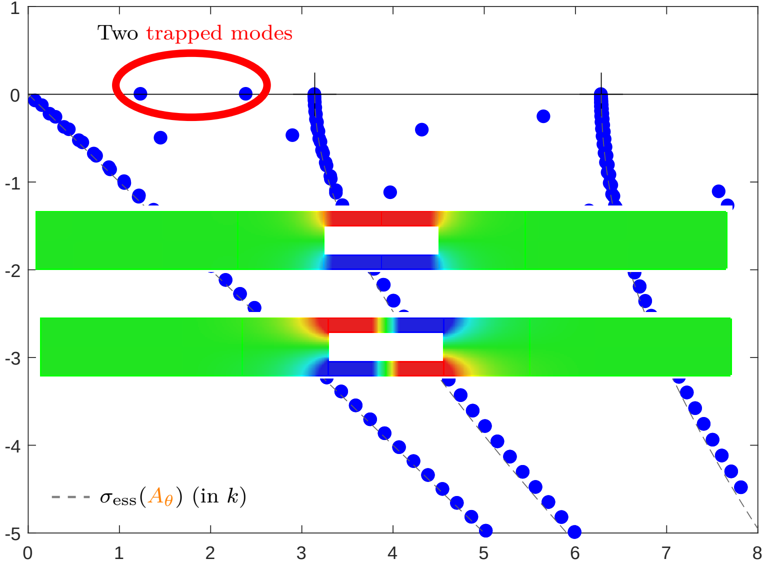

A new complex spectrum for non reflecting v

- Usual complex scaling selects scattered fields which are

outgoing at $-\infty$ and outgoing at $+\infty$.

Important remark: **general** v decompose as

$$v = v_i + \sum_{n=0}^{N-1} \gamma_n^- \, w_n^- + \sum_{n=N}^{+\infty} \gamma_n^- \, w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ \, w_n^+ \quad x \geq L.$$

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IMPORTANT REMARK: **non reflecting** v decompose as

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$$v = \sum_{n=0}^{N-1} \boldsymbol{\alpha_n w_n^+} + \sum_{n=N}^{+\infty} \gamma_n^- \, w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ \, w_n^+ \quad x \geq L.$$

- In other words, **non reflecting** v are

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- In other words, **non reflecting** v are

ingoing at $-\infty$ and outgoing at $+\infty$.

Let us **change the sign** of the complex scaling at $-\infty$!

A new complex spectrum for non reflecting v

- For $\theta \in (0; \pi/2)$, consider the complex change of variables

$$\mathcal{J}_\theta(x) = \left| \begin{array}{ll} -L + (x+L)\ e^{-i\theta} & \text{for } x \leq -L \\ \quad x & \text{for } |x| < L \\ +L + (x-L)\ e^{+i\theta} & \text{for } x \geq L. \end{array} \right.$$

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- Set $u_\theta := v \circ (\mathcal{J}_\theta(x), y)$.

1) $u_\theta = v$ for $|x| < L$.

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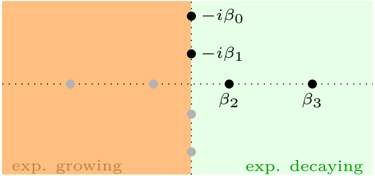
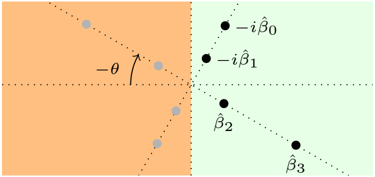


Modal exponents for v $(x \leq -L)$



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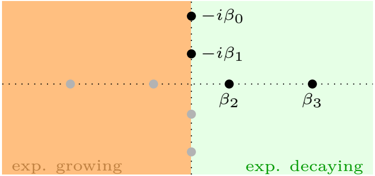

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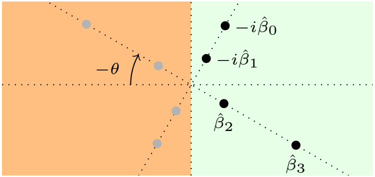

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- u_θ solves

$$(*) \left| \begin{array}{rl} \beta_\theta \dfrac{\partial}{\partial x}\left(\beta_\theta \dfrac{\partial u_\theta}{\partial x}\right) + \dfrac{\partial^2 u_\theta}{\partial y^2} + k^2 u_\theta = 0 & \text{in } \Omega \\ \partial_n u_\theta = 0 & \text{on } \partial\Omega. \end{array} \right.$$

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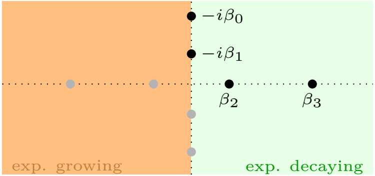

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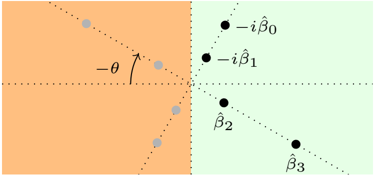

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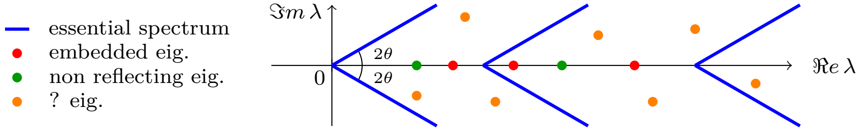
Spectral analysis

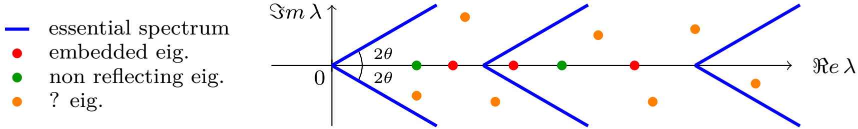
- Define the operator B_θ of $\mathrm{L}^2(\Omega)$ such that

$$B_\theta v = -\Big(\beta_\theta \frac{\partial}{\partial x}\Big(\beta_\theta \frac{\partial v}{\partial x}\Big) + \frac{\partial^2 v}{\partial y^2}\Big) \qquad + \partial_n v = 0 \text{ on } \partial\Omega.$$

- B_θ is not selfadjoint. $\sigma(B_\theta) \subset \{\rho\, e^{i\gamma},\ \rho \geq 0,\ \gamma \in [-2\theta; 2\theta]\}$.
- $\sigma_{\text{ess}}(B_\theta) = \cup_{n\in\mathbb{N}}\{n^2\pi^2 + t\, e^{-2i\theta},\ t \geq 0\} \cup \{n^2\pi^2 + t\, e^{2i\theta},\ t \geq 0\}$.
- real eigenvalues of B_θ = real eigenvalues of A+non reflecting k^2.



Remarks

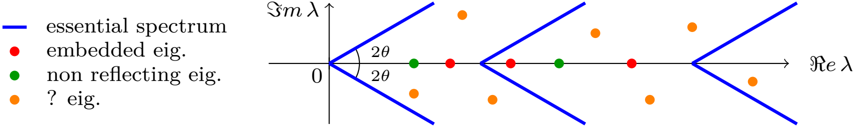


1) **? eig.** correspond to solutions of the Helmholtz equation which are exp. growing at one side of Ω, exp. decaying at the other.

Different from **complex resonances** for which the eigenfunctions are exp. growing both at $\pm\infty$ ...

2) It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete.

Remarks



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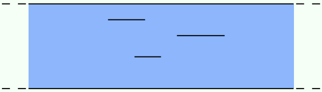
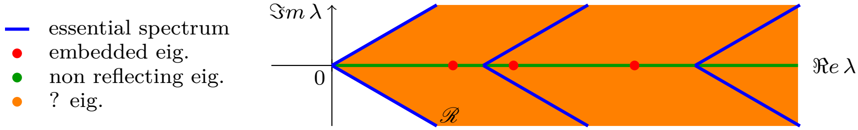


→ **Not true in general!**

$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

Remarks



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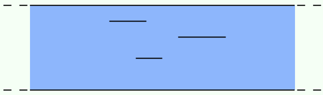
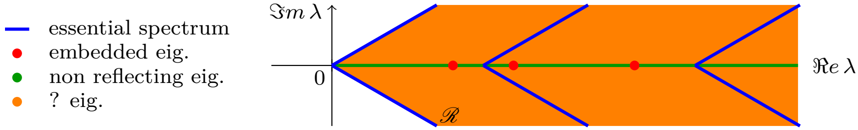


→ **Not true in general!**

$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

Remarks



1) • **? eig.** correspond to solutions of the Helmholtz equation which are exp. growing at one side of Ω, exp. decaying at the other.

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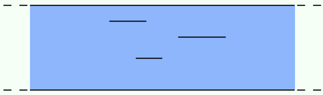

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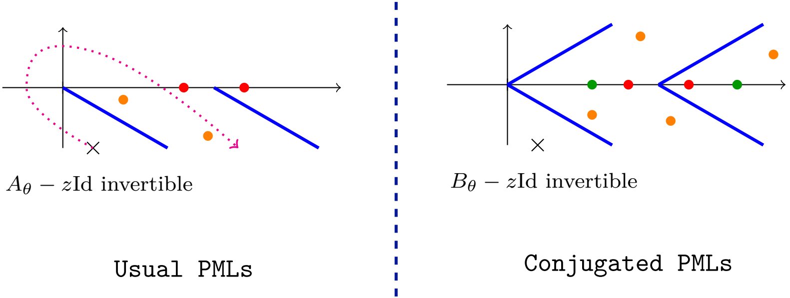

$\rightarrow \mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ is not connected $\Rightarrow$ we cannot apply simply the analytic Fredholm thm.

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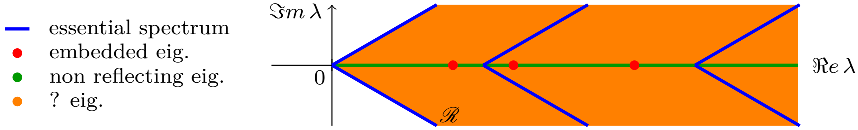

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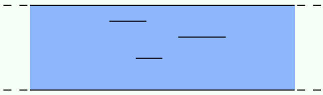

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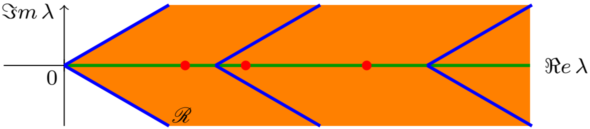
Remarks

- essential spectrum (blue line)
- embedded eig. (red dot)
- non reflecting eig. (green dot)
- ? eig. (orange dot)



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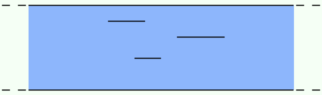

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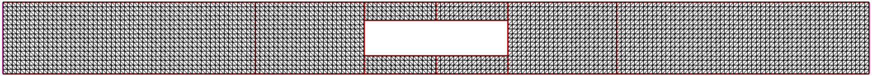
→ $\mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ is not connected ⇒ we cannot apply simply the analytic Fredholm thm.

→ A compact perturbation can change drastically the spectrum (B_θ is not selfadjoint).
Numerical consequences?

Numerical results

- Again we work in the geometry



- Define the operators $\mathcal{P}$ (Parity), $\mathcal{T}$ (Time reversal) such that

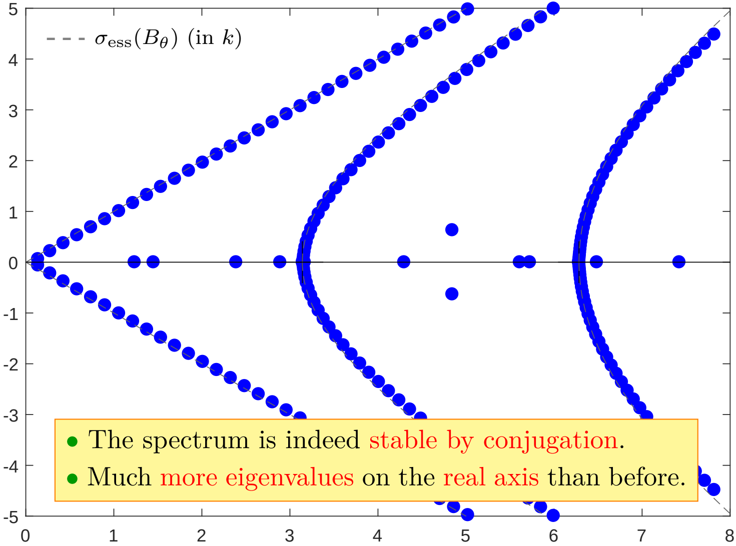
$$\mathcal{P}v(x,y) = v(-x,y) \quad \text{and} \quad \mathcal{T}v(x,y) = \overline{v(x,y)}.$$

> Prop.: For symmetric $\Omega = \{(-x,y)\,|\,(x,y)\in\Omega\}$, B_θ is $\mathcal{PT}$ symmetric:
>
> $$\mathcal{PT}B_\theta\mathcal{PT} = B_\theta.$$
>
> As a consequence, $\sigma(B_\theta) = \overline{\sigma(B_\theta)}$.

$\Rightarrow$ If λ is an "isolated" eigenvalue located close to the real axis, then $\lambda \in \mathbb{R}$!

Numerical results

- Discretized spectrum in k (not in k^2). We take $\theta = \pi/4$.



- The spectrum is indeed stable by conjugation.
- Much more eigenvalues on the real axis than before.

Numerical results

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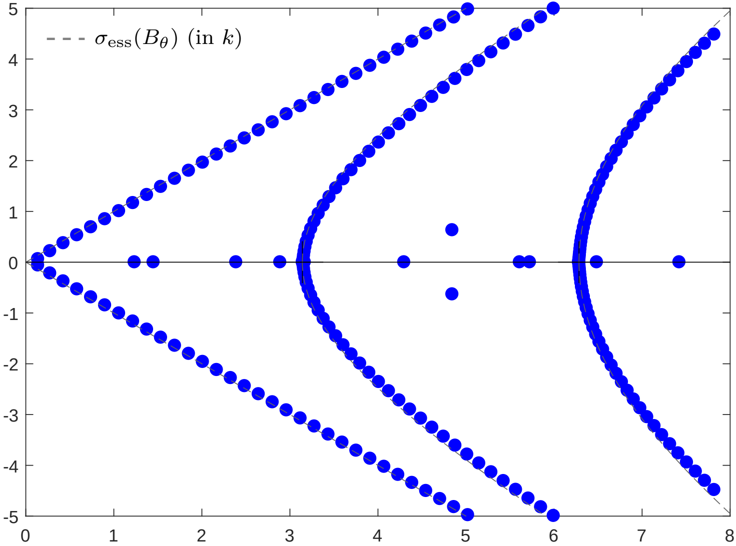

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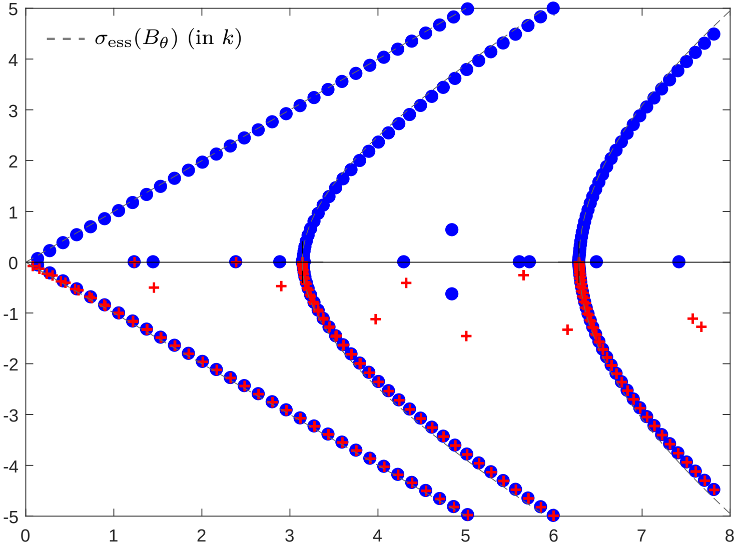
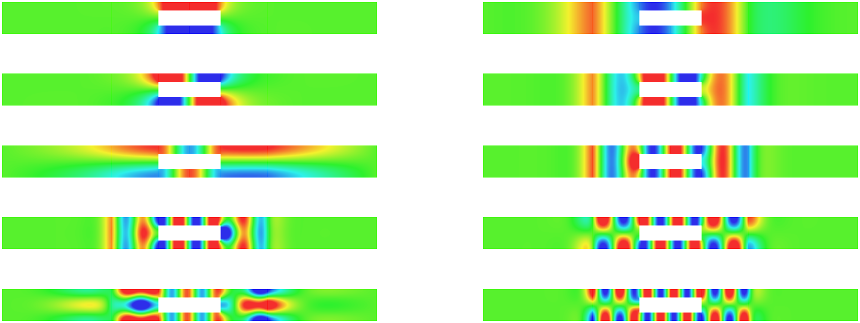


● PMLs with different signs ✚ Classical PMLs

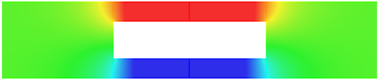
Numerical results

► We display the eigenmodes for the ten first real eigenvalues in the whole computational domain (including PMLs).



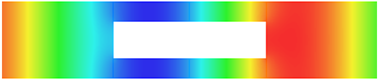
Numerical results

- Let us focus on the eigenmodes such that $0 < k < \pi$.



First trapped mode
$k = 1.2355...$

Second trapped mode
$k = 2.3897...$

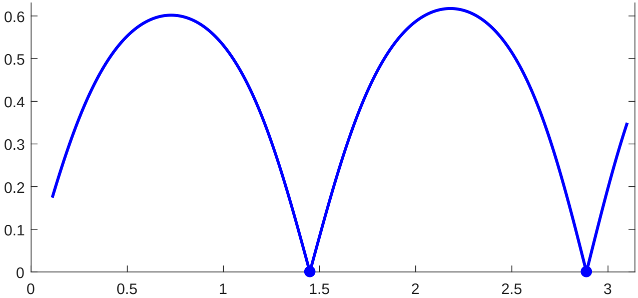


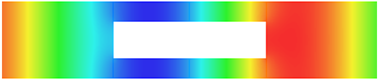
First non reflecting mode
$k = 1.4513...$

Second non reflecting mode
$k = 2.8896...$

Numerical results

- To check our results, we compute $k \mapsto |R(k)|$ for $0 < k < \pi$.





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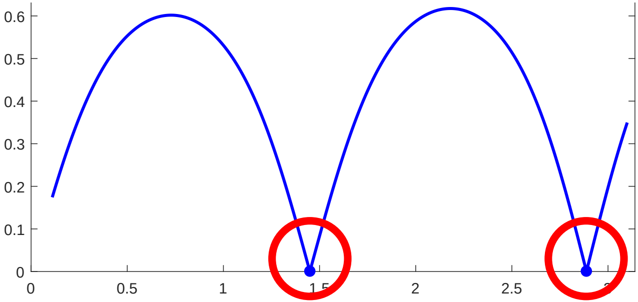

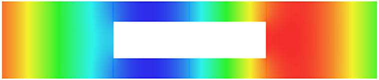

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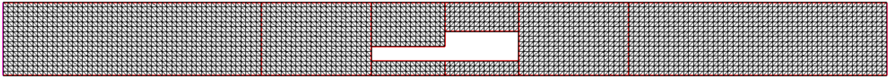
There is perfect agreement!

Numerical results

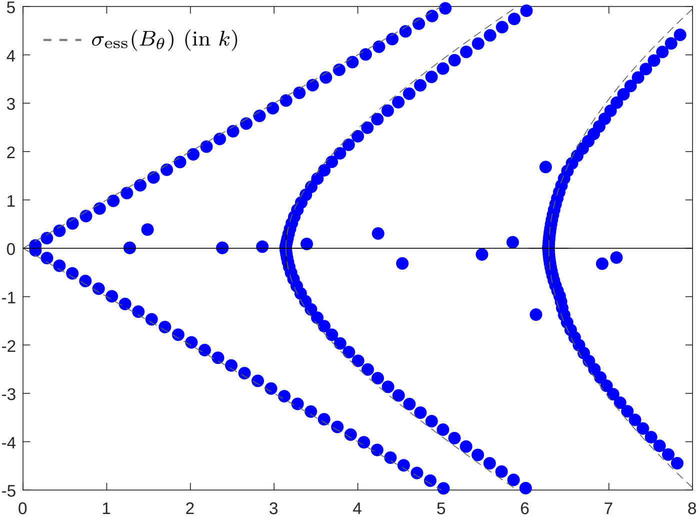
- Now the geometry is not symmetric in x nor in y:



- The operator B_θ is no longer $\mathcal{PT}$-symmetric and we expect:
  - No trapped modes
  - No invariance of the spectrum by complex conjugation.

Numerical results

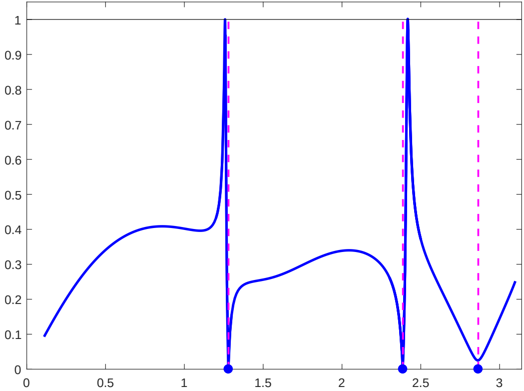
- Discretized spectrum of B_θ in k (not in k^2). We take $\theta = \pi/4$.



- Indeed, the spectrum is not symmetric w.r.t. the real axis.

Numerical results

- We compute $k \mapsto |R(k)|$ for $0 < k < \pi$.



$k = 1.28 + 0.0003i$

$k = 2.3866 + 0.0005i$

$k = 2.8647 + 0.0243i$

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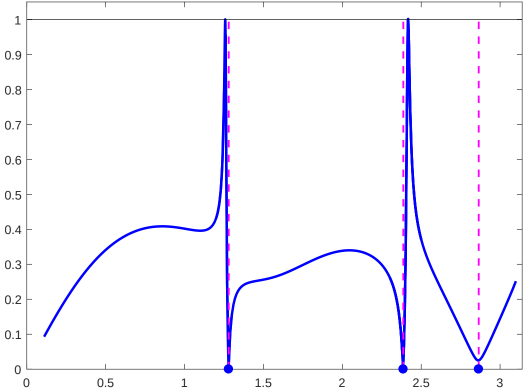

$k = 1.28 + 0.0003i$

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Complex eigenvalues also contain information on almost no reflection.

Outline of the talk

We present **two different** points of view on these questions of invisibility:

1. Cloaking of obstacles

   ASYMPTOTIC ANALYSIS:
   k and Ω are given, we explain how to **perturb the geometry** using **thin resonant ligaments** to get $T \approx 1$.

2. A spectral approach to determine non reflecting wavenumbers

   SPECTRAL THEORY:
   Ω is given, we explain how to **find non reflecting** k by solving an unusual **spectral problem**.

Conclusion

Part I

♠ Method to cloak any object in monomode regime using thin resonators. Two main ingredients:

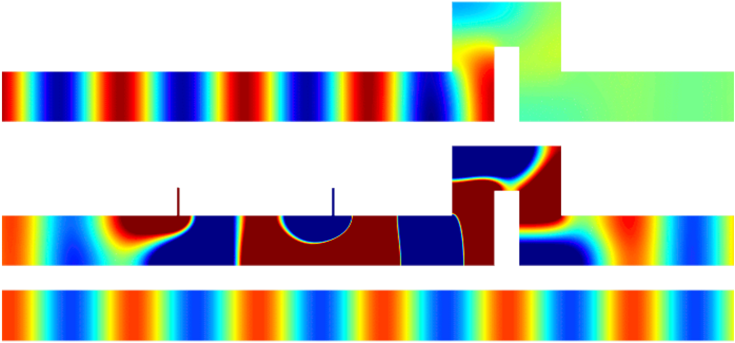
- Around resonant lengths, effects of order ε^0 with perturb. of width ε.
- Explicit dependence wrt to the geometry in the 1D limit resonator.

1) We can similarly hide penetrable obstacles or work in 3D.
2) We can do cloaking at a finite number of wavenumbers (thin structures are resonant at one wavenumber otherwise act at order ε).
3) With Dirichlet BCs, other ideas must be found.

Part II

♠ Spectral approach to compute non reflecting k $(R = 0)$ for a given Ω.

1) Can we find a spectral approach to compute completely reflecting or completely invisible k?
2) Can we prove existence of non reflecting k for the $\mathcal{PT}$-symmetric pb?



Thank you for your attention!

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