A curious spectral behavior for a problem of rounded corner in presence of negative material

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Joint work with X. Claeys[2] and S.A. Nazarov[3]

Collaboration also with A.-S. Bonnet-Ben Dhia[4], C. Carvalho[4] and P. Ciarlet[4]

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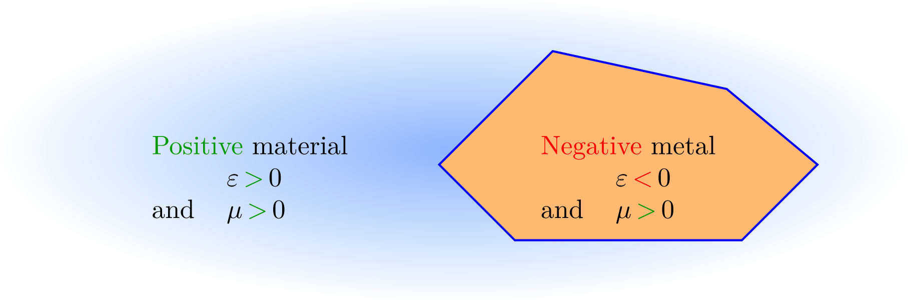
Introduction: general framework

- Scattering by a metal in electromagnetism in time-harmonic regime at optical frequency.
- For metals at optical frequency, $\Re e\,\varepsilon(\omega) < 0$ and $\Im m\,\varepsilon(\omega) << |\Re e\,\varepsilon(\omega)|$.
  $\Rightarrow$ We neglect losses and study the ideal case $\varepsilon(\omega) \in (-\infty; 0)$.



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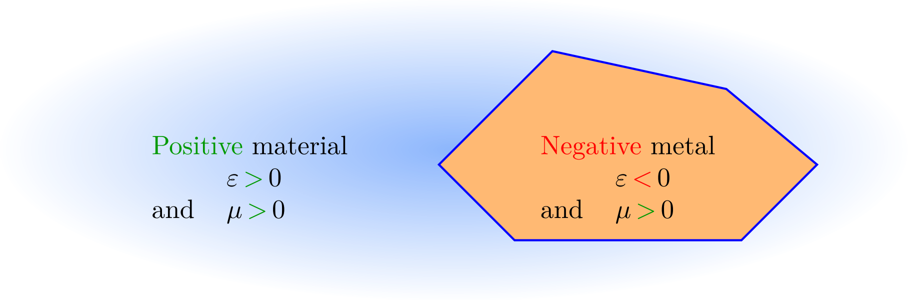
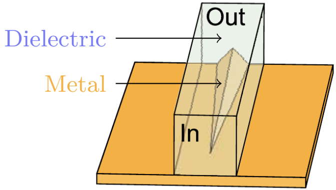
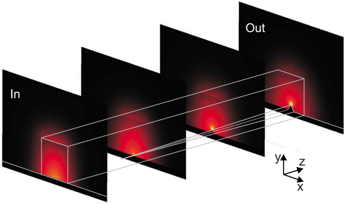


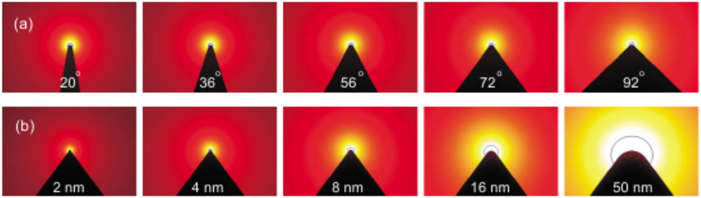
- Waves called Surface Plasmon Polaritons can propagate at the interface between a dielectric and a negative metal.

Introduction: applications

► Surface Plasmons Polaritons can propagate information. Physicists hope to exploit them to reduce the size of computer chips.







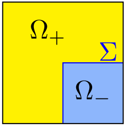
Figures from `O'Connor et al., Appl. Phys. Lett. 95, 171112 (2009)`

► In this context, physicists use singular geometries to focus energy. It allows to stock information.

Original motivation: source term problem

- We study a scalar model problem set in a bounded domain $\Omega \subset \mathbb{R}^2$:

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ -\mathrm{div}(\sigma \nabla u) = f \text{ in } \Omega. \end{array} \right.$$



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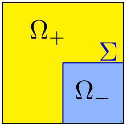

- $\mathrm{H}^1_0(\Omega) = \{v \in \mathrm{L}^2(\Omega) \,|\, \nabla v \in \mathrm{L}^2(\Omega);\ v|_{\partial\Omega} = 0\}$
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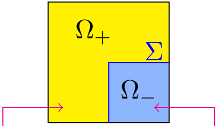

$\sigma|_{\Omega_+} = \sigma_+ > 0$
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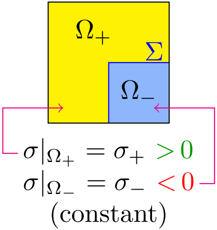

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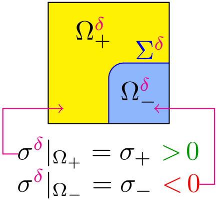

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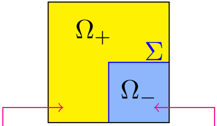

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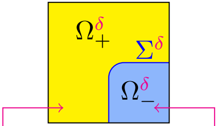

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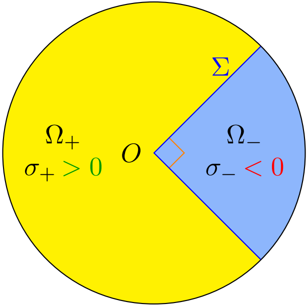
Numerical experiments: setting

- For the numerical experiments, we round the corner in a particular way



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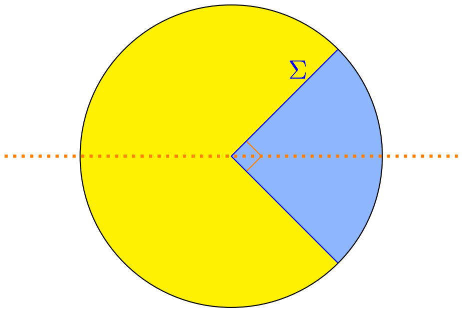

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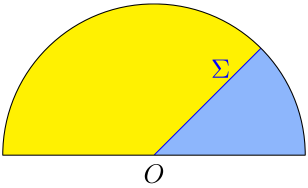

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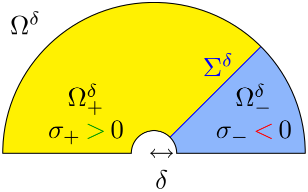

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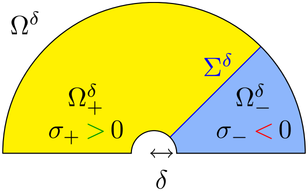

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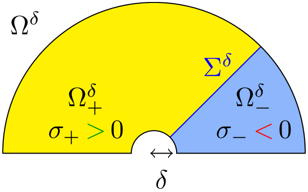

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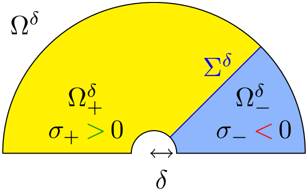

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- We approximate u^δ, *assuming it is well-defined*, by a usual P1 Finite Element Method. We compute the solution u_h^δ of the discretized problem with *FreeFem++*.

We display the behaviour of u_h^δ as $\delta \to 0$.

Numerical experiments: results 1/2

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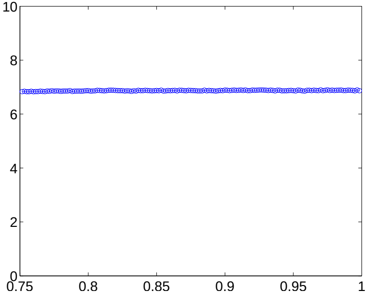

u_h^δ w.r.t. δ | $\|\nabla u_h^\delta\|_{\Omega^\delta}$ w.r.t. $1-\delta$

- For positive materials, it is well-known that $(u^\delta)_\delta$ converges to u, the solution in the limit geometry.
- The rate of convergence depends on the regularity of u.
- To avoid to mesh Ω^δ, we can approximate u^δ by u_h.

Numerical experiments: results 2/2

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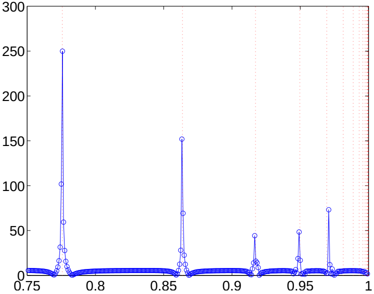

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- For this configuration, u^δ seems to depend critically on δ.

In this talk, our goal is to explain the presence of these peaks.

Outline of the talk

1. Spectral problem in the geometry with a rounded corner
2. Asymptotic analysis
3. Numerical experiments for the spectral problem

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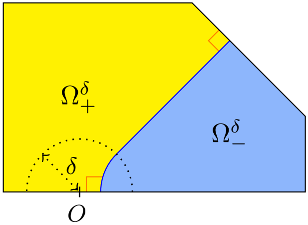
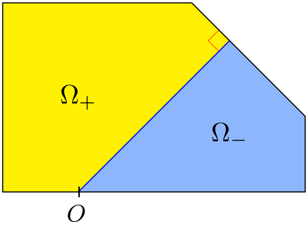
Setting

- For ease of exposition, we consider a half rounded corner





- We are interested in the spectral problem

$$\left| \begin{array}{l} \text{Find } (\lambda^\delta, u^\delta) \in \mathbb{C} \times (\mathrm{H}_0^1(\Omega) \setminus \{0\}) \text{ s.t.:} \\ -\mathrm{div}(\sigma^\delta \nabla u^\delta) = \lambda^\delta u^\delta \quad \text{ in } \Omega. \end{array} \right.$$

- We define the operator $\mathrm{A}^\delta : D(\mathrm{A}^\delta) \to \mathrm{L}^2(\Omega)$ such that

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Known results

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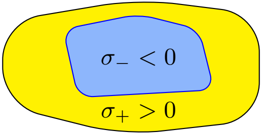
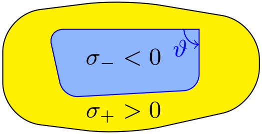
- We have the following properties (see Costabel & Stephan 85, Dauge & Texier 97, Hussein 13, Bonnet-Ben Dhia *et al.* 99,10,12,13):

Smooth interface Σ



✔ A selfadjoint when $\kappa_\sigma = \sigma_-/\sigma_+ \neq -1$.

Interface Σ with a corner



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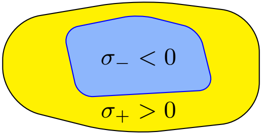

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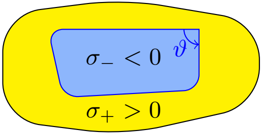

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Results depend on the smoothness of Σ and on σ.

Spectral problem for $\delta > 0$

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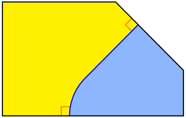

- For $\delta > 0$, the interface is smooth $\Rightarrow \mathrm{A}^\delta$ selfadjoint iff $\kappa_\sigma \neq -1$.

Proposition. If $\kappa_\sigma \neq -1$, $\delta > 0$, A^δ is selfadjoint with compact resolvent. Its spectrum $\mathfrak{S}(\mathrm{A}^\delta)$ consists in two sequences of isolated eigenvalues:

$$-\infty \underset{n\to+\infty}{\leftarrow} \dots \lambda^\delta_{-n} \leq \dots \leq \lambda^\delta_{-1} < 0 \leq \lambda^\delta_1 \leq \lambda^\delta_2 \leq \dots \leq \lambda^\delta_n \dots \underset{n\to+\infty}{\rightarrow} +\infty.$$

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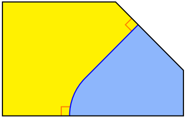

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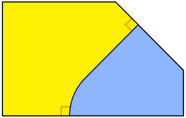

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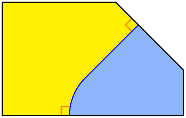

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$\rightarrow$ This depends on the features of the limit operator...

Spectral problem at the limit $\delta = 0$

- Let A denote the limit operator ($\delta = 0$) such that

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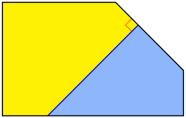

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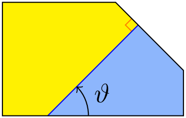

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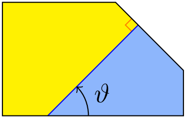

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$$-\infty \underset{n\to+\infty}{\leftarrow} \dots \lambda_{-n} \le \dots \le \lambda_{-1} < 0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_n \dots \underset{n\to+\infty}{\rightarrow} +\infty.$$

In this case, there holds $\mathfrak{S}(\mathrm{A}^\delta) \underset{\delta\to0}{\rightarrow} \mathfrak{S}(\mathrm{A})$.

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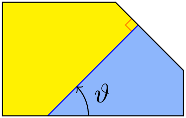

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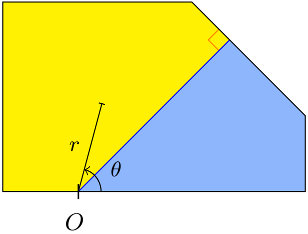
♣ When $\kappa_\sigma \in I_c \setminus \{-1\}$, A is not selfadjoint. *Let us clarify this...*

Spectral problem at the limit $\delta = 0$ inside I_c



- When $\kappa_\sigma \in (-1; -1/\ell)$, there are singularities $r^{\pm i\mu}\phi(\theta)$ with $\mu \in \mathbb{R}^*$, $\phi(0) = \phi(\pi) = 0$, satisfying $\operatorname{div}(\sigma\nabla(r^{\pm i\mu}\phi(\theta)) = 0$ in a neighbour. of O.

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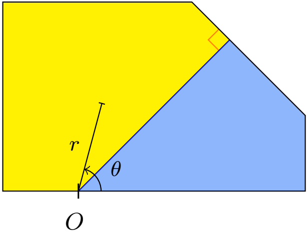

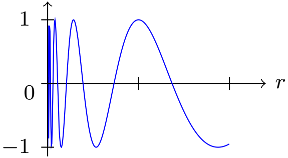

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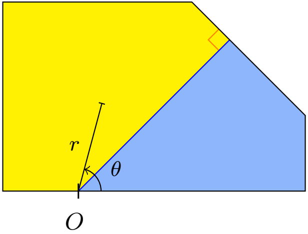

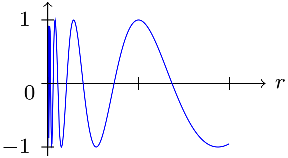

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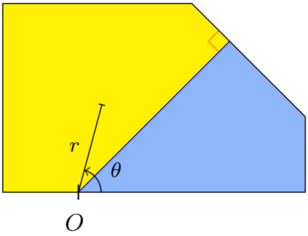

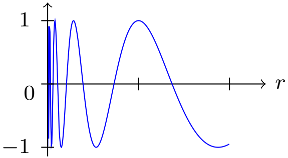

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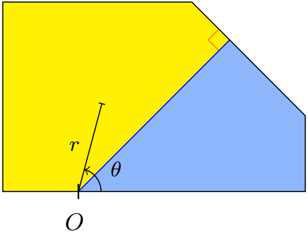

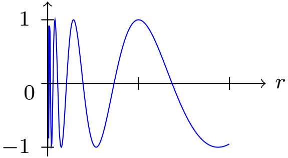

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♣ When $\kappa_\sigma \in (-1; -1/\ell)$, there holds $D(\mathrm{A}^*) = D(\mathrm{A}) \oplus \mathrm{span}(s_+, s_-)$ (in particular A is not selfadjoint). Moreover, $\mathfrak{S}(\mathrm{A}) = \mathbb{C}$.

Selfadjoint extensions of A inside I_c

- The selfadjoint extensions of A are the operators $\mathrm{A}(\tau)$, $\tau \in \mathbb{R}$, such that

$$\left|\begin{array}{l} D(\mathrm{A}(\tau)) = D(\mathrm{A}) \oplus \mathrm{span}(s_+ + e^{i\tau} s_-) \\ \mathrm{A}(\tau)u = \mathrm{div}(\sigma \nabla u). \end{array}\right.$$

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$$(\mathrm{A}^* u_1, u_2)_\Omega - (u_1, \mathrm{A}^* u_2)_\Omega = 2i\mu\lambda_1\overline{\lambda_2}\,(|c_+|^2 - |c_-|^2).$$

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? Maybe $\mathfrak{S}(\mathrm{A}^\delta) \to \mathfrak{S}(\mathrm{A}(\tau))$ for some τ as $\delta \to 0$. But for which τ?

Asymptotic expansion

- From now, we assume that $\kappa_\sigma \in (-1; -1/\ell)$.
- Consider $(\lambda^\delta, u^\delta)$ an eigenpair of the original spectral problem.

$$\left| \begin{array}{l} \text{Find } (\lambda^\delta, u^\delta) \in \mathbb{C} \times (\mathrm{H}^1_0(\Omega) \setminus \{0\}) \text{ s.t.:} \\ -\mathrm{div}(\sigma^\delta \nabla u^\delta) = \lambda^\delta u^\delta \quad \text{in } \Omega. \end{array} \right.$$

- To compute an asymptotic expansion of $(\lambda^\delta, u^\delta)$, we make the ansatz

$$\begin{array}{rcccll} \lambda^\delta & = & \eta^\delta & + & \dots & \\ u^\delta(x) & = & v^\delta(x) & + & \dots & \text{far from } O \\ u^\delta(x) & = & V^\delta(x/\delta) & + & \dots & \text{near } O \end{array}$$

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- Note that η^δ, v^δ, V^δ will be defined as solutions of problems set in geometries independent of δ.

Far field

- The far field is defined in the geometry obtained taking $\delta = 0$.

- We find that the pair (η^δ, v^δ) must verify

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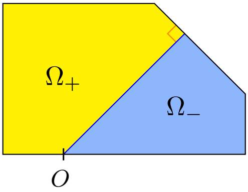

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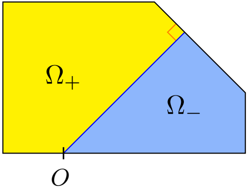


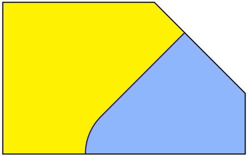
- Since we do not know which behaviour to prescribe at O, we allow decomposition on the two singularities $s_\pm$ and search for v^δ under the form

$$\boxed{\begin{array}{rcccccc} v^\delta & = & c_+^\delta\, s_+ & + & c_-^\delta\, s_- & + & \tilde{v}^\delta \\ & = & c_+^\delta\, r^{i\mu}\phi(\theta) & + & c_-^\delta\, r^{-i\mu}\phi(\theta) & + & \tilde{v}^\delta, \end{array}}$$

where the jauge functions $c_\pm^\delta$ and $\tilde{v}^\delta \in D(\mathrm{A})$ have to be determined.

Near field

- Introduce the rapid coordinate $\xi := x/\delta$ and let $\delta \to 0$.



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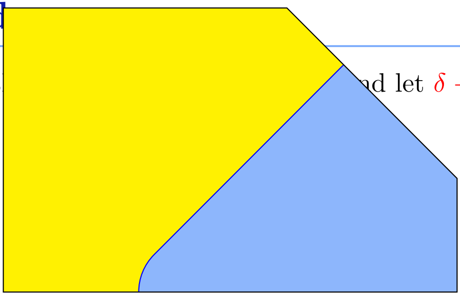
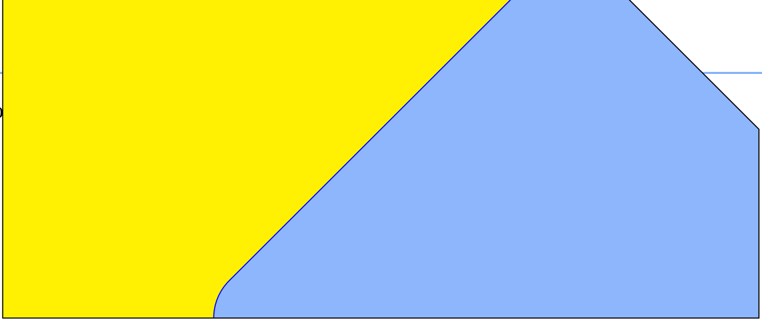
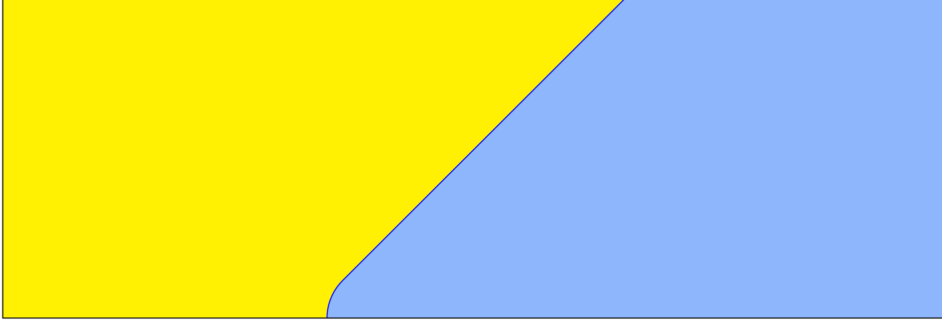
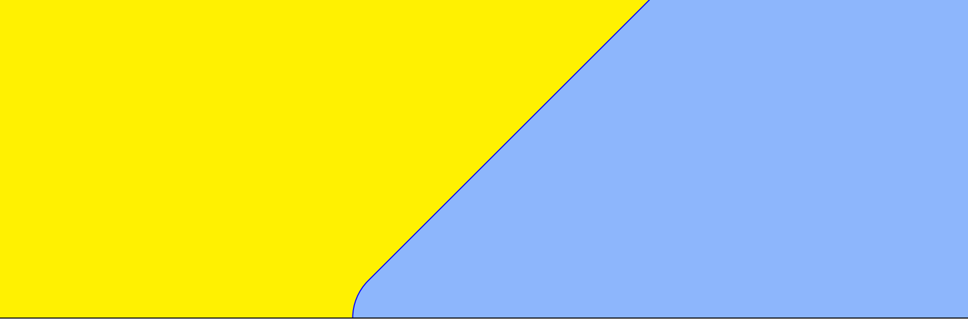
- Introduce t  nd let $\delta \to 0$.

Near

- Intro







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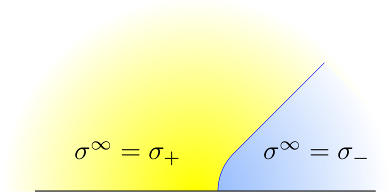

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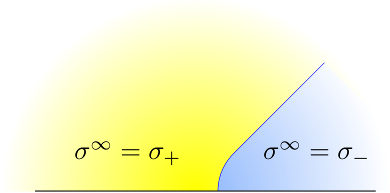

Set $U^\delta(\xi) = u^\delta(\delta\xi)$. We have

$$\begin{aligned} u^\delta(x) &= V^\delta(x/\delta) + \dots \\ \Leftrightarrow \; U^\delta(\xi) &= \; V^\delta(\xi) \; + \dots \; . \end{aligned}$$

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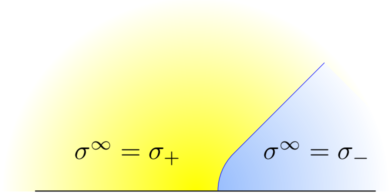

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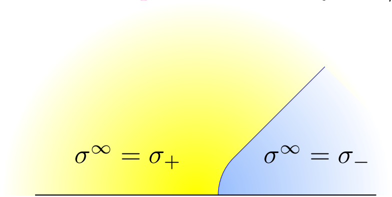

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with $\alpha \in \mathbb{C}$, $\tilde{V}^\delta \in \mathrm{H}^1(\Xi)$.

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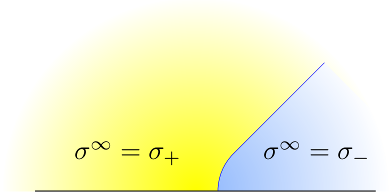

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Important: there holds $|\alpha| = 1$.

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From

$$-\mathrm{div}(\sigma^\infty \nabla V^\delta) = 0 \text{ in } \Xi, \qquad V^\delta = 0 \text{ on } \partial\Xi,$$

multiplying by $\overline{V^\delta}$ and integrating by parts on $\{\xi \in \Xi \,|\, |\xi| < R\}$, we find

$$0 \quad = \quad \Im m \int_{\Xi \cap \{|\xi| = R\}} \sigma^\infty \partial_r V^\delta \overline{V^\delta} \, d\theta$$

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Important: there holds $|\alpha| = 1$.

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- Introduce the rapid coordinate $\xi := x/\delta$ and let $\delta \to 0$.

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- We match the far field and near field expansions in some intermediate region where $r \to 0$ and $r/\delta \to +\infty$ (for example where $r \sim \sqrt{\delta}$).

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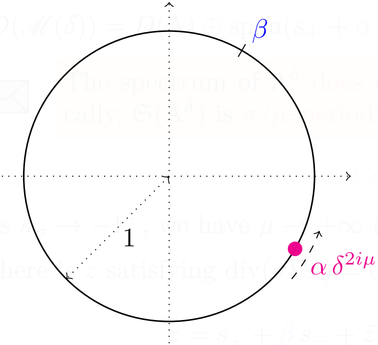

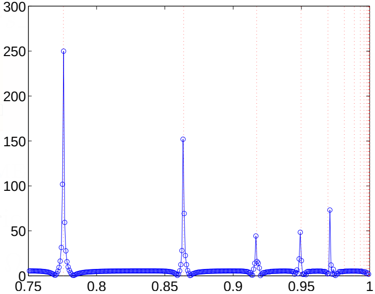

$\|\nabla u_h^\delta\|_{\Omega^\delta}$ w.r.t. $1-\delta$

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Comments concerning the proof

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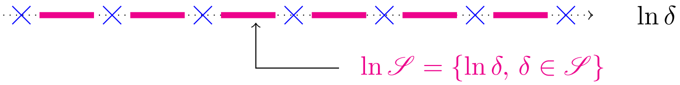

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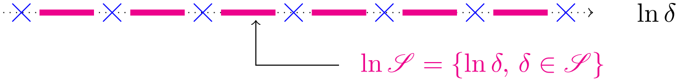

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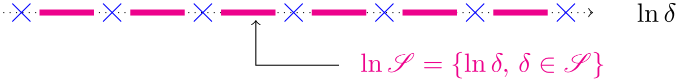

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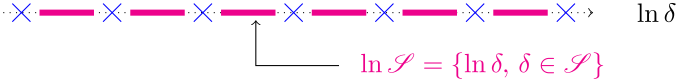

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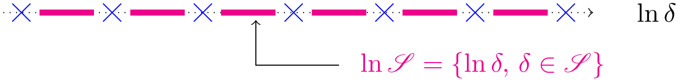

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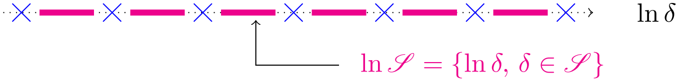

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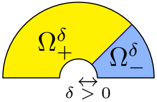
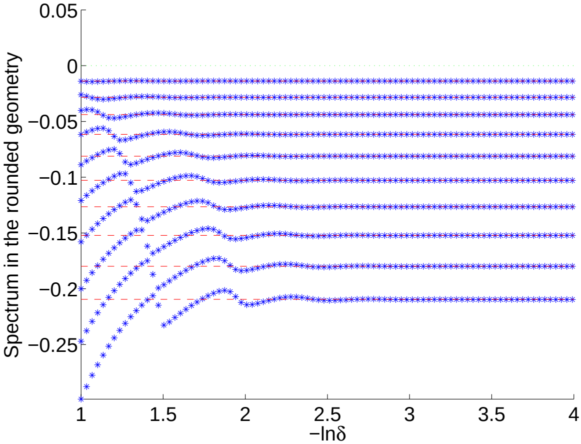
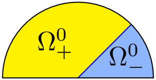
♠ Proving (1), (2) is not straightforward due to the change of sign of σ. This aspect is interesting in itself (`S.A. Nazarov's` technique).

Outside the critical interval

$$\boxed{\kappa_\sigma = -1.0001 \notin [-1; -1/3]}$$



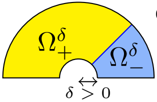


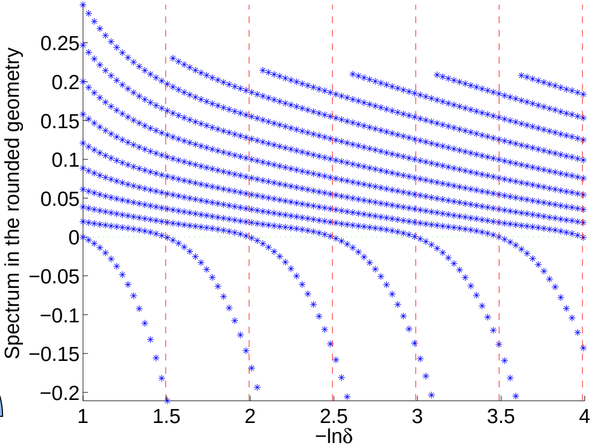


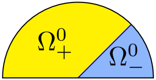
- $\color{blue}{\mathfrak{S}(\mathrm{A}^\delta)}$ converges to $\color{red}{\mathfrak{S}(\mathrm{A})}$ (A is the limit operator) when $\delta \to 0$.

Inside the critical interval

$$\kappa_\sigma = -0.9999 \in [-1; -1/3]$$







- Asymptotically, $\mathfrak{S}(\mathrm{A}^\delta)$ is periodic in $\ln\delta$–scale as $\delta \to 0$.

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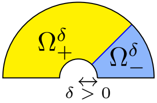

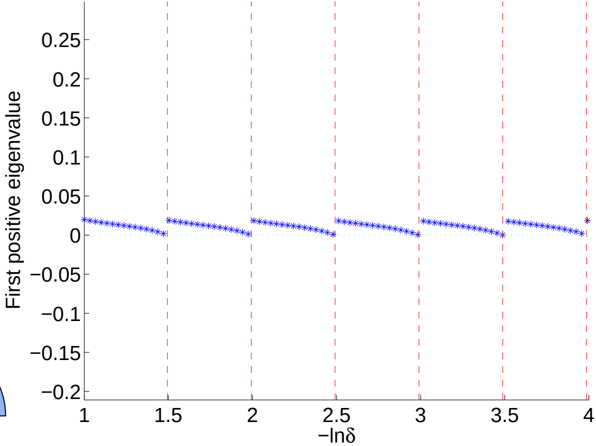

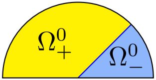

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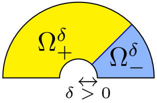

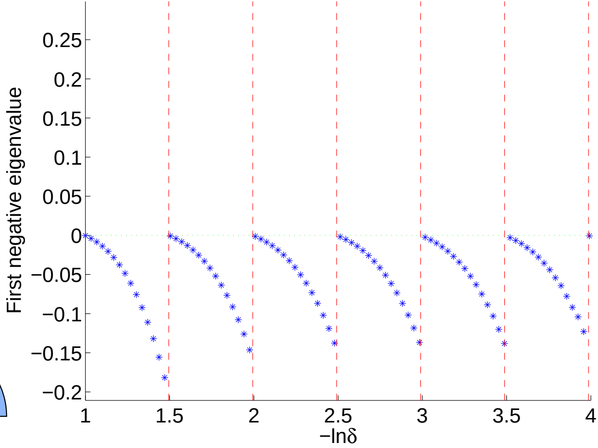

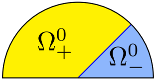

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Conclusion 1/2

Let us remind the initial question:

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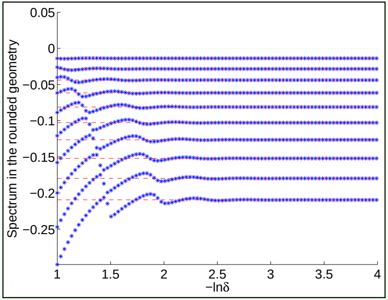
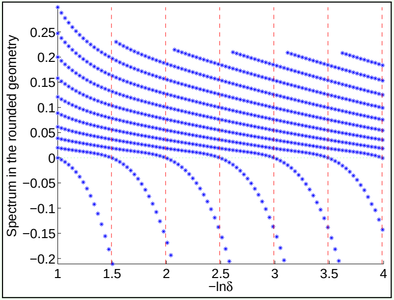
This depends on the features of the limit problem.



$\kappa_\sigma \notin I_c$

⋆ $\mathfrak{S}(\mathrm{A}^{\delta})$ tends to $\mathfrak{S}(\mathrm{A})$ where A is the limit operator for $\delta = 0$.



$\kappa_\sigma \in (-1; -1/\ell)$

⋆ $\mathfrak{S}(\mathrm{A}^{\delta})$ behaves as $\mathfrak{S}(\mathscr{M}(\delta))$, which is periodic in $\ln\delta$-scale.

Conclusion 2/2

And concerning the source term problem?

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$(\dots)$

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$\kappa_\sigma = -0.9999 \in I_c$

When $\kappa_\sigma \in I_c$, $(u^\delta)_\delta$ does not converge, even for the L^2-norm!

In this case, it is impossible to simulate the fields since it is impossible to measure exactly δ. $\Rightarrow$ What happens physically?

Thank you for your attention!

Related works:

- ANR project Metamath coordinated by `S. Fliss`.

L. Chesnel, X. Claeys, S.A. Nazarov, *A curious instability phenomenon for a rounded corner in presence of a negative material*, Asymp. Anal., vol. 88, 1-2:43-74, 2014.

L. Chesnel, X. Claeys, S.A. Nazarov, *Asymptotics of the eigenvalues for a rounded corner in presence of a negative material*, to come, 2015.