Thematic day PLASMON 2023

An introduction to transmission problems in presence of negative materials

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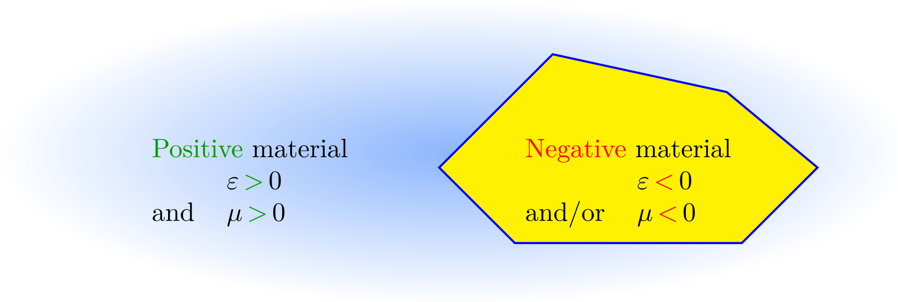
Marseille, 5/12/2023

Introduction: context

Scattering by a negative material in electromagnetism in time-harmonic regime (at a given frequency):



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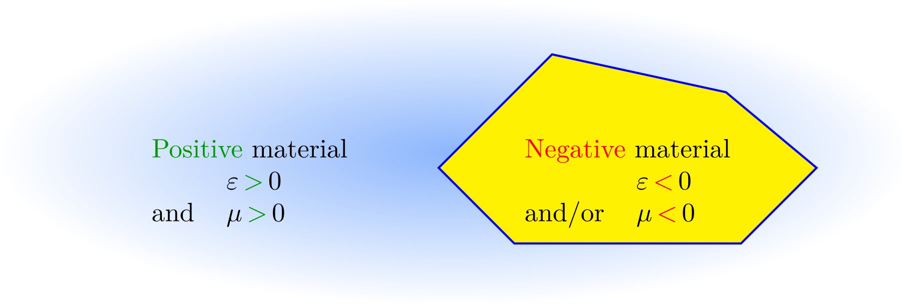

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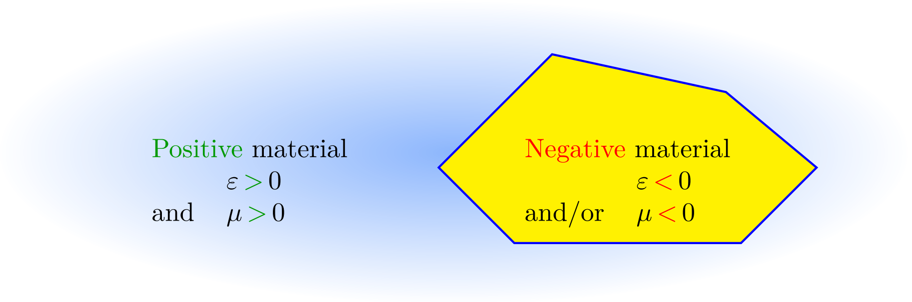

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- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.

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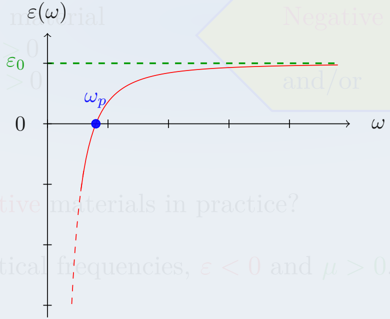
Drude model for a metal (high frequency):

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{{\omega_p}^2}{\omega^2}\right),$$

where ω_p is the plasma frequency.



Positive material $\varepsilon > 0$ and $\mu > 0$

Negative material $\varepsilon < 0$ and/or $\mu < 0$

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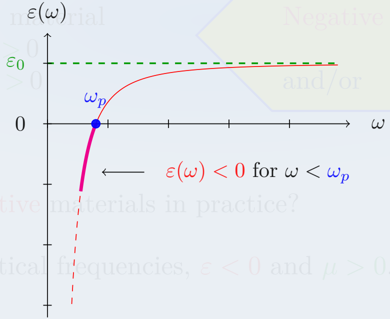

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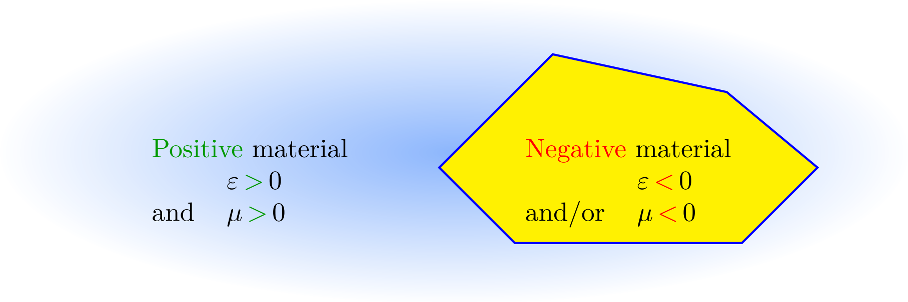

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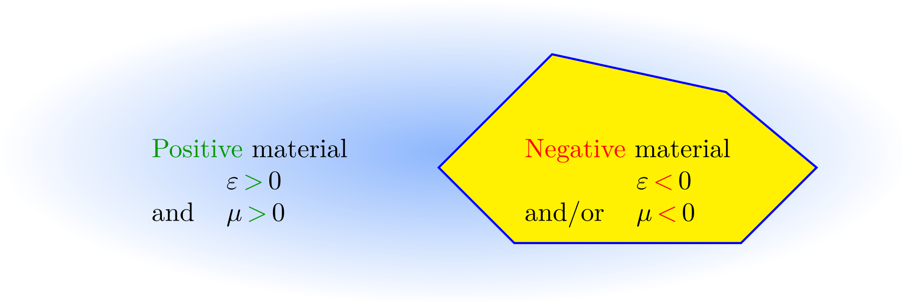
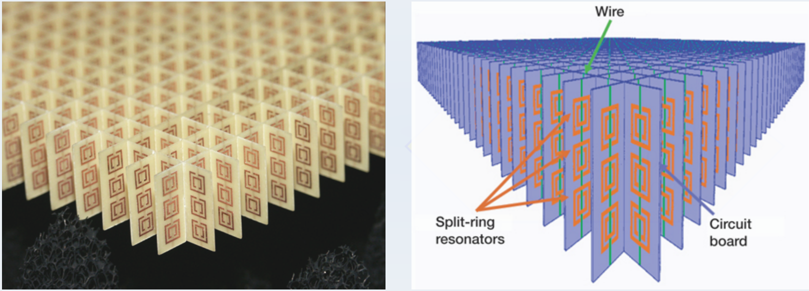


What are these negative materials in practice?

- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.
- Recently, artificial metamaterials have been realized which can be modelled (at some frequency of interest) by $\varepsilon < 0$ and $\mu < 0$.

Introduction: context

Zoom on a metamaterial: practical realizations of metamaterials are achieved by a periodic assembly of small resonators.



Example of metamaterial (NASA)

Mathematical justification of the homogenized model (`Bouchitté, Bourel, Felbacq 09`,...).

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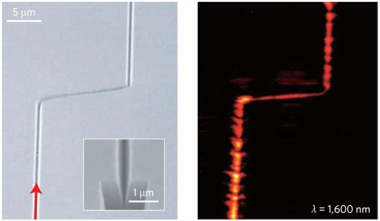
Introduction: applications

► Surface Plasmons Polaritons that propagate at the interface between a metal and a dielectric can help reducing the size of computer chips.



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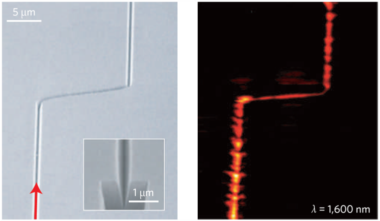

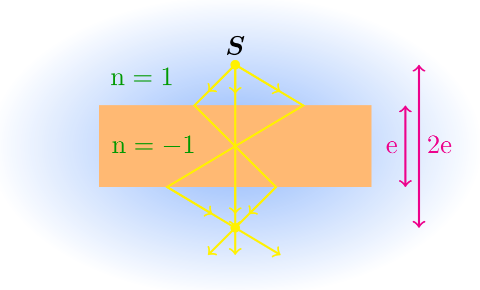

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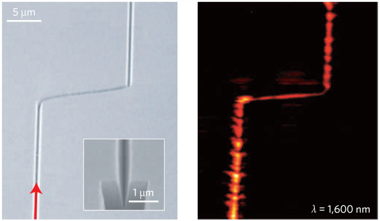

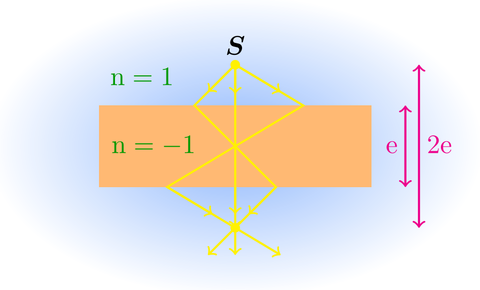

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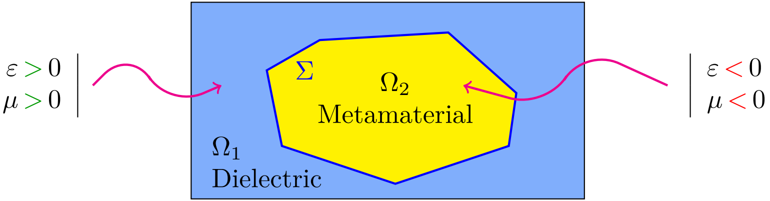
Interfaces between negative materials and dielectrics occur in all (exciting) applications...

Introduction: in this talk

Problem set in a bounded domain Ω:



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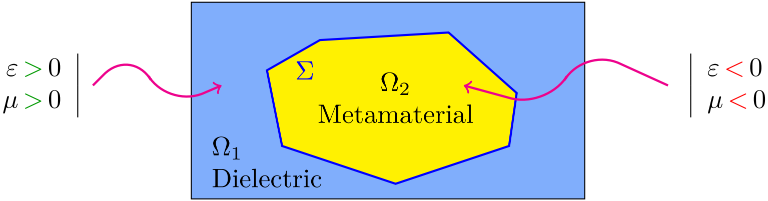

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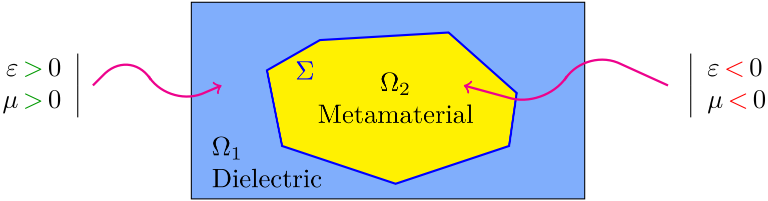

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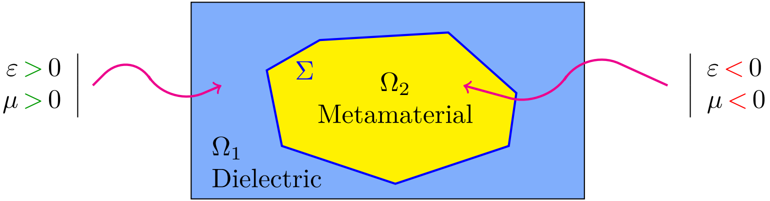

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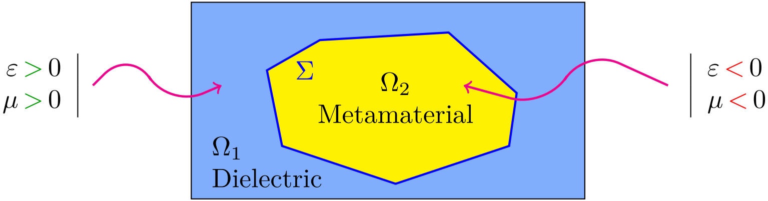

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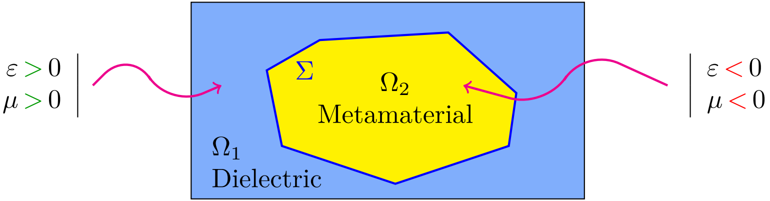

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The relevant question is then: what happens if dissipation is neglected?

- Does well-posedness still hold?
- What is the appropriate functional framework?
- What about the convergence of approximation methods?

Outline of the talk

1. Scalar problem: variational techniques

We develop a $\mathtt{T}$-coercivity approach based on geometrical transformations to study the operator $\mathbf{div}(\mu^{-1}\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$.

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We propose a new functional framework for the scalar problem when $\mathbf{div}(\mu^{-1}\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is not Fredholm.

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   We develop a T-coercivity approach to study the Maxwell's operator $\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\cdot) : \mathbf{X}_N(\varepsilon) \to \mathbf{X}_N(\varepsilon)^*$.

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4. The Interior Transmission Eigenvalue Problem

We study the operator $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$.

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4. The Interior Transmission Eigenvalue Problem

A scalar model problem

Problem for E_z in 2D in case of an invariance with respect to z:

$$\left|\begin{array}{l} \text{Find } E_z \in \mathrm{H}_0^1(\Omega) \text{ such that:} \\ -\mathrm{div}(\mu^{-1}\,\nabla E_z) - \omega^2 \varepsilon E_z = f \quad \text{in } \Omega. \end{array}\right.$$

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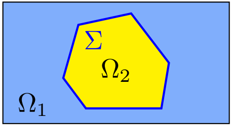

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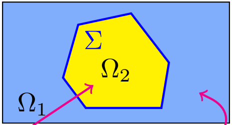

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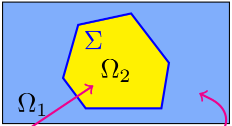

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with $a(u,v) = \displaystyle\int_\Omega \sigma\nabla u \cdot \nabla v$ and $\ell(v) = \langle f, v\rangle_\Omega$.

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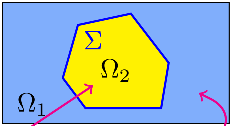

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Definition. We will say that the problem $(\mathscr{P})$ is well-posed if the operator $\mathrm{div}\,(\sigma\nabla\cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$.

Mathematical difficulty

- Classical case $\sigma > 0$ everywhere:

$$a(u,u) = \int_\Omega \sigma |\nabla u|^2 \geq \min(\sigma)\,\|u\|^2_{\mathrm{H}^1_0(\Omega)} \quad \text{coercivity}$$

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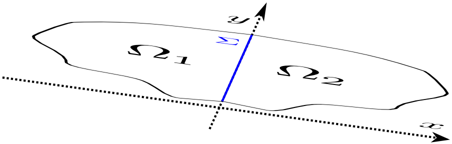
  - When $\sigma_2 = -\sigma_1$, $(\mathscr{P})$ is always ill-posed (`Costabel-Stephan 85`). For a symmetric domain (w.r.t. Σ), we can build a kernel of infinite dimension.

The symmetric case with $\sigma_2 = -\sigma_1$

Consider the case where Ω is symmetric and $\sigma_2 = -\sigma_1$.



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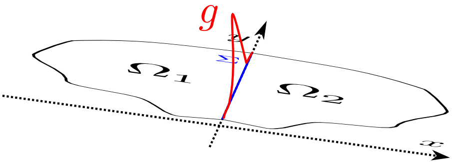

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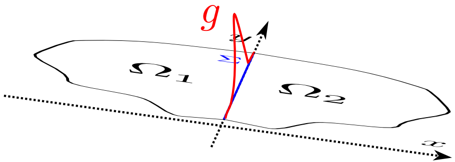

❷ Define u_2 such that $u_2(x, y) = u_1(-x, y)$.

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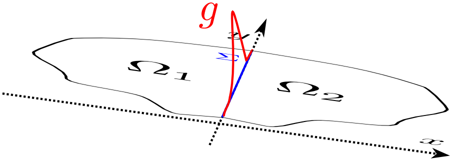

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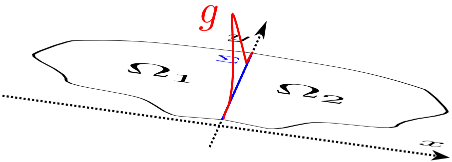
$$\Rightarrow \text{We have} \quad \sigma_1\, \partial_x u_1 = \sigma_2\, \partial_x u_2 \ \text{ on } \Sigma.$$

The symmetric case with $\sigma_2 = -\sigma_1$

Consider the case where Ω is symmetric and $\sigma_2 = -\sigma_1$.

❶ For $g \in \mathscr{C}_0^\infty(\Sigma)$, let $u_1 \in \mathrm{H}^1(\Omega_1)$ be such that

$$\left|\begin{array}{rcll} \Delta u_1 & = & 0 & \text{in } \Omega_1 \\ u_1 & = & 0 & \text{on } \partial\Omega \cap \partial\Omega_1 \\ u_1 & = & g & \text{on } \Sigma. \end{array}\right.$$



❷ Define u_2 such that $u_2(x,y) = u_1(-x,y)$.

$$\Rightarrow \text{We have} \quad \sigma_1\,\partial_x u_1 = \sigma_2\,\partial_x u_2 \ \text{ on } \Sigma.$$

❸ The function $u \in \mathrm{H}_0^1(\Omega)$ s.t. $u|_{\Omega_k} = u_k$ solves $\mathrm{div}(\sigma\nabla u) = 0$ in Ω.

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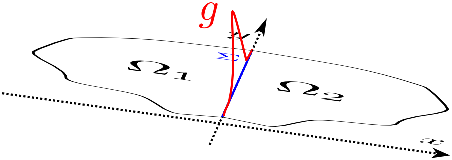

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Proposition. In the symmetric geometry, for $\sigma_2 = -\sigma_1$, $(\mathscr{P})$ has a kernel of infinite dimension.

Idea of the T-coercivity

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$$(\mathscr{P}) \Leftrightarrow (\mathscr{P}_V) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(u,v) = l(v),\ \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Idea of the $\mathtt{T}$-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$$(\mathscr{P}) \Leftrightarrow (\mathscr{P}_V) \Leftrightarrow (\mathscr{P}_V^{\mathtt{T}}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(u, \mathtt{T}v) = l(\mathtt{T}v),\ \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

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Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \sigma\, \nabla u \cdot \nabla(\mathtt{T}u) \geq C\, \|u\|^2_{\mathrm{H}^1_0(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

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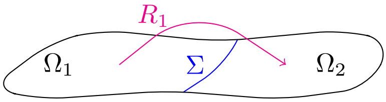

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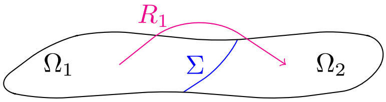

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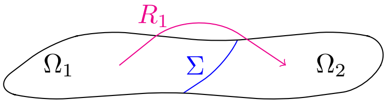

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Idea of the $\mathtt{T}$-coercivity 2/2

3. We find $a(u, \mathtt{T}_1 u) = \displaystyle\int_\Omega |\sigma|\,|\nabla u|^2 - \,2\int_{\Omega_2} \sigma_2\,\nabla u \cdot \nabla(R_1(u|_{\Omega_1}))\,.$

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Conclusion : a is $\mathtt{T}$-coercive when $\sigma_1 > \|R_1\|^2\, |\sigma_2|$

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5. Conclusion:

Theorem. If the contrast $\kappa_\sigma=\sigma_2/\sigma_1\notin[-\|R_2\|^2;-1/\|R_1\|^2]$, then Problem $(\mathscr{P})$ is well-posed.

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5 Conclusion:

The interval depends on the norms of the transfer operators

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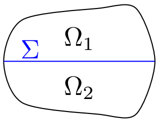
Choice of R_1, R_2?

- A simple case: the symmetric domain



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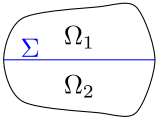

$$R_1 = R_2 = S_\Sigma$$

One checks that $\|R_1\| = \|R_2\| = 1$

$(\mathscr{P})$ well-posed $\Leftrightarrow \kappa_\sigma \neq -1$

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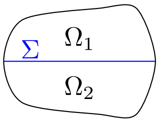

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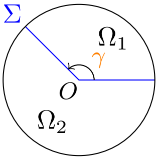
- Interface with a 2D corner



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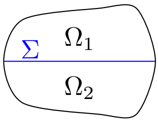

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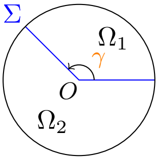

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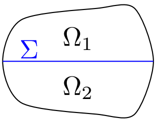

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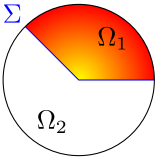

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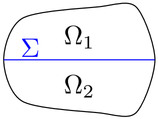

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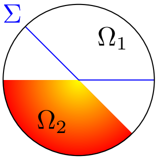

Action of R_1: symmetry

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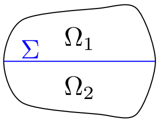

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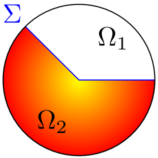

Action of R_1: symmetry + dilatation in θ

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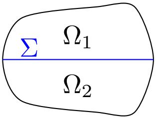

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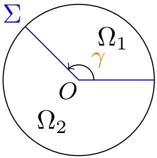

Action of R_1: symmetry + dilatation in θ

$$\|R_1\|^2 \qquad = \mathcal{R}_\gamma := (2\pi - \gamma)/\gamma$$

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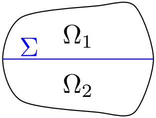

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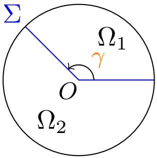

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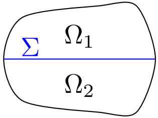

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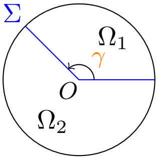

Action of R_1: symmetry + dilatation in θ
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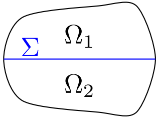

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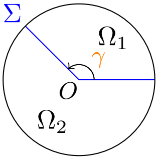

Action of R_1: symmetry + dilatation in θ

Action of R_2: symmetry + contraction in θ

$$\|R_1\|^2 = \|R_2\|^2 = \mathcal{R}_\gamma := (2\pi - \gamma)/\gamma$$

$(\mathscr{P})$ well-posedness $\Leftrightarrow$ $\kappa_\sigma \notin [-\mathcal{R}_\gamma; -1/\mathcal{R}_\gamma]$

General geometry

Idea: work by localisation

- With Riesz, define the operator $A : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ such that

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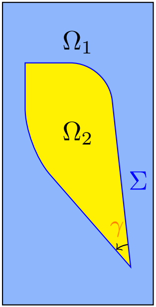

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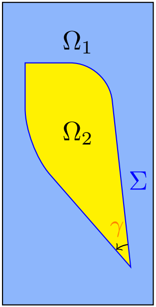

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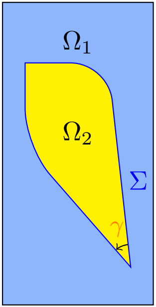

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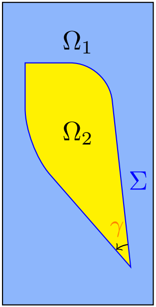

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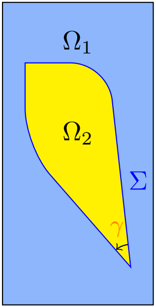

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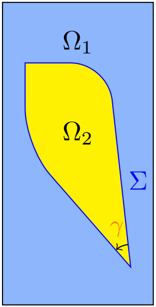

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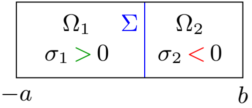
$\Rightarrow$ If Σ is smooth, $(\mathscr{P})$ is well-posed in the Fredholm sense when $\kappa_\sigma \neq -1$.

Summary of the results for the 2D cavity

Problem

$$(\mathscr{P}) \quad \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ -\mathrm{div}\,(\sigma \nabla u) = f \qquad \text{in } \Omega. \end{array} \right.$$



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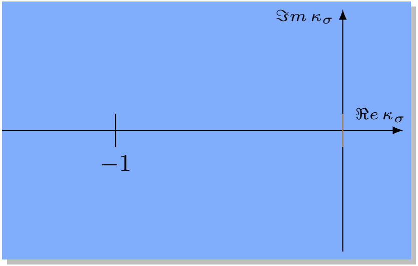
For $\kappa_\sigma \in \mathbb{C}\backslash\mathbb{R}_-$, $(\mathscr{P})$ well-posed (Lax-Milgram)



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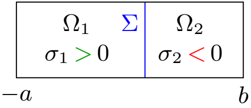

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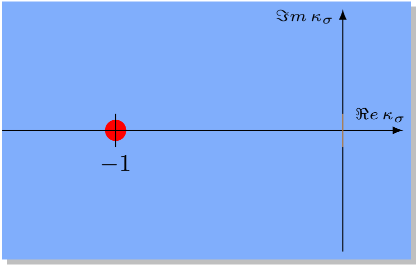
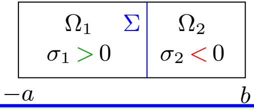


Summary of the results for the 2D cavity

Problem

$(\mathscr{P})$ | Find $u \in \mathrm{H}^1_0(\Omega)$ s.t.:
$-\mathrm{div}\,(\sigma\nabla u) = f \quad$ in Ω.



PROPOSITION. The operator $A = \mathrm{div}\,(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is an isomorphism if and only $\kappa_\sigma \in \mathbb{C}^* \setminus \mathscr{S}$ with $\mathscr{S} = \{-\tanh(n\pi b)/\tanh(n\pi a),\ n \in \mathbb{N}^*\} \cup \{-1\}$. For $\kappa_\sigma = -\tanh(n\pi b)/\tanh(n\pi a)$, we have $\ker A = \mathrm{span}\,\varphi_n$ with

$$\varphi_n(x,y) = \begin{cases} \sinh(n\pi(x+a))\sin(n\pi y) & \text{on } \Omega_1 \\ -\dfrac{\sinh(n\pi a)}{\sinh(n\pi b)}\sinh(n\pi(x-b))\sin(n\pi y) & \text{on } \Omega_2 \end{cases}.$$

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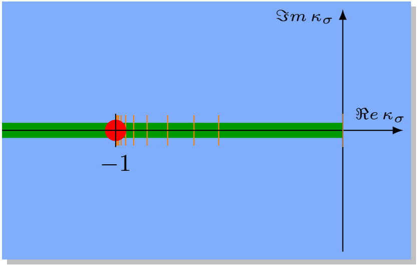

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Let u_g be the potential for an electrostatic charge g distributed on Σ. If we normalize the total energy in Ω, what is the minimum of energy in Ω_2?

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Solving the Poincaré problem gives the contrasts for which our problem has a non zero kernel.

Extensions for the scalar case

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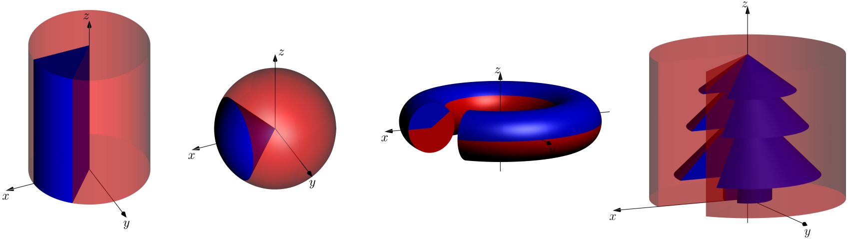

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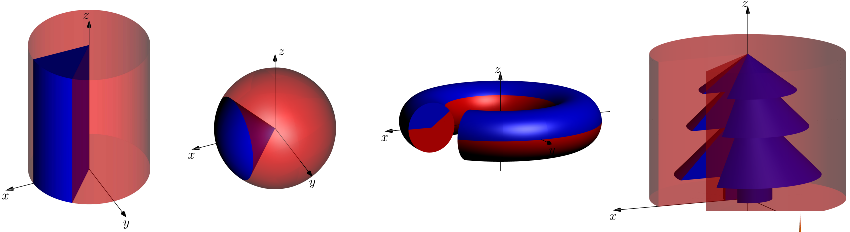
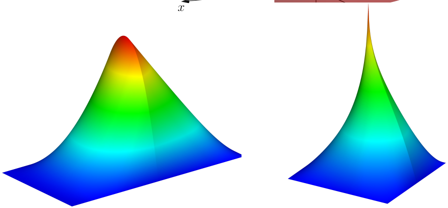


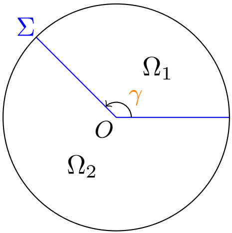
- T-coercivity can be used to justify the convergence of standard FEM with assumptions on meshes (Nicaise, Venel 11, Bonnet-Ben Dhia, Carvalho, Ciarlet 18).



$\rightarrow$ for other methods without mesh assumption based on optimization techniques, see Abdulle, Lemaire 23, Ciarlet, Lassounon, Rihani 22.

Transition: from variational methods to Fourier/Mellin techniques

For the corner case, what happens when the contrast lies inside the criticial interval, *i.e.* when $\kappa_\sigma \in [-\mathcal{R}_\gamma; -1/\mathcal{R}_\gamma]$?



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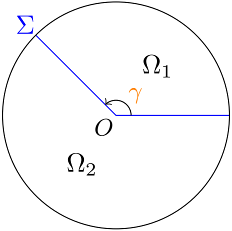

Idea: let us study the regularity of the "solutions" using the Kondratiev's tools, *i.e.* the Fourier/Mellin transform (Dauge, Texier 97, Nazarov, Plamenevsky 94).

Problem considered in this section

- We recall the problem under consideration

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ -\mathrm{div}(\sigma\nabla u) = f \qquad \text{in } \Omega. \end{array} \right.$$

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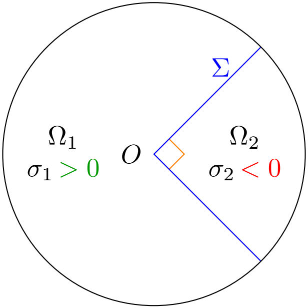

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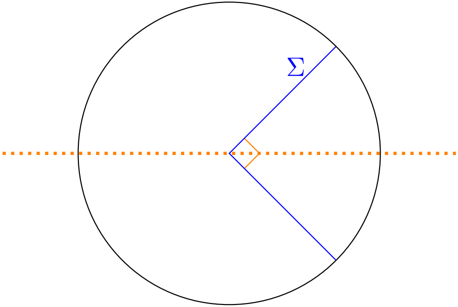

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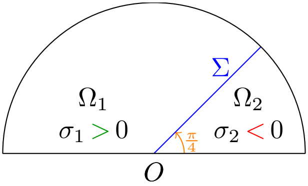

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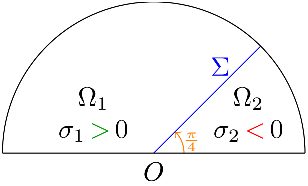

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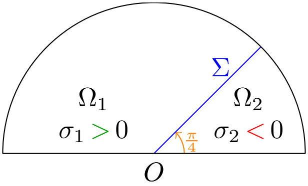

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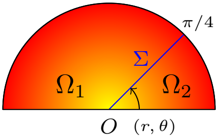
Analogy with a waveguide problem

- Bounded sector Ω



- Equation:

$$\underbrace{-\mathrm{div}(\sigma\nabla u)}_{-r^{-2}(\sigma(r\partial_r)^2+\partial_\theta\sigma\partial_\theta)u} \quad = f$$

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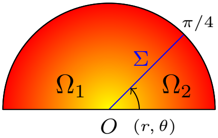

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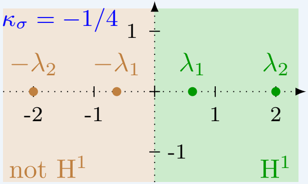
- Singularities in the sector

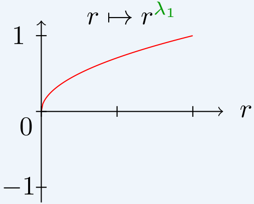
$s(r,\theta) = r^{\lambda}\varphi(\theta)$

Analogy with a waveguide problem

We compute the singularities $s(r, \theta) = r^{\lambda}\varphi(\theta)$ and we observe two cases:

- Outside the critical interval





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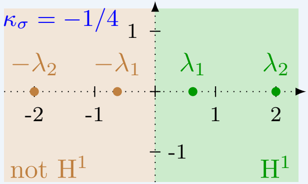

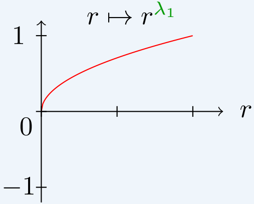

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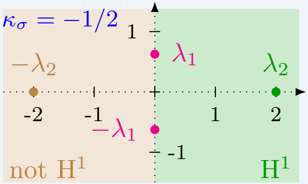

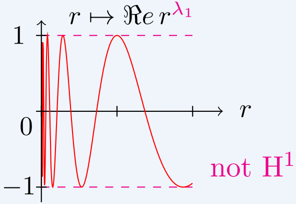

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For κ_σ inside the critical interval, there are singularities of the form $s(r,\theta) = r^{\pm i\eta}\varphi(\theta)$ with $\eta \in \mathbb{R} \setminus \{0\}$.

► By using these singularities, one breaks the *a priori* estimate

$$\xcancel{\|u\|_{\mathrm{H}^1_0(\Omega)} \leq C\left(\|Au\|_{\mathrm{H}^1_0(\Omega)} + \|u\|_{\mathrm{L}^2(\Omega)}\right)} \qquad \forall u \in \mathrm{H}^1_0(\Omega).$$

► This shows that one cannot have $A = I + K$ where I is an isomorphism of $\mathrm{H}^1_0(\Omega)$ and K is a compact operator of $\mathrm{H}^1_0(\Omega)$.

Proposition. For $\kappa_\sigma \in (-1;-1/3)$, $\mathrm{div}(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is not of Fredholm type.

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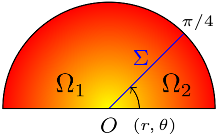
Let us see how to modify the functional framework to recover Fredholmness.

Analogy with a waveguide problem

- Bounded sector Ω



- Equation:

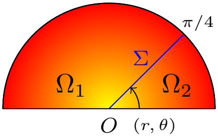
$$\underbrace{-\mathrm{div}(\sigma\nabla u)}_{-r^{-2}(\sigma(r\partial_r)^2+\partial_\theta\sigma\partial_\theta)u} = f$$

- Singularities in the sector

$s(r,\theta) = r^\lambda \varphi(\theta)$

Analogy with a waveguide problem

- Bounded sector Ω



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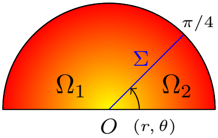
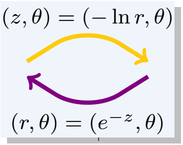
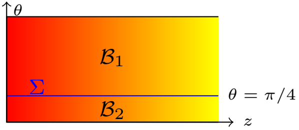




- Half-strip $\mathcal{B}$



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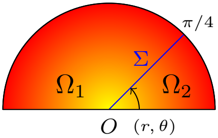
Analogy with a waveguide problem

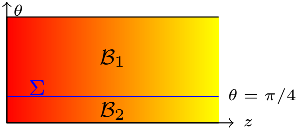
- Bounded sector Ω



$(z,\theta) = (-\ln r, \theta)$

$(r,\theta) = (e^{-z}, \theta)$

- Half-strip $\mathcal{B}$



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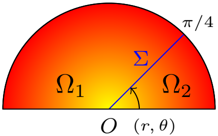

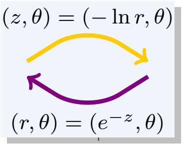

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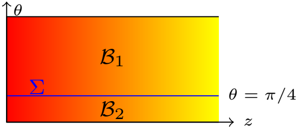

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- Modes in the strip

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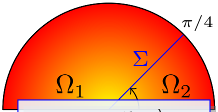
Analogy with a waveguide problem

- Bounded sector Ω



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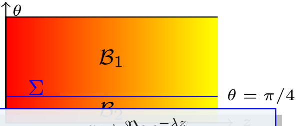

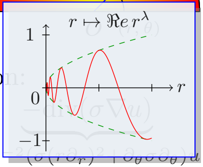

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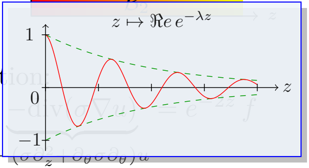

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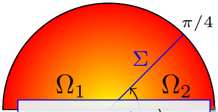
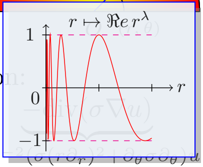
$s \in \mathrm{H}^1(\Omega)$ $\quad\Re e\,\lambda > 0\quad$ m is evanescent

Analogy with a waveguide problem

- Bounded sector Ω





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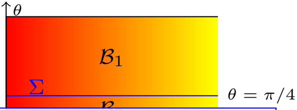

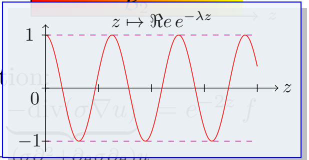

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$$= \cancel{r^a}\,(\cos b\ln r + i\sin b\ln r)\varphi(\theta)$$

- Modes in the strip

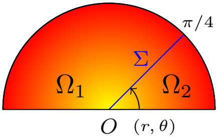
$$m(z,\theta) = e^{-\lambda z}\varphi(\theta)$$
$$= \cancel{e^{-az}}\,(\cos bz - i\sin bz)\varphi(\theta)$$

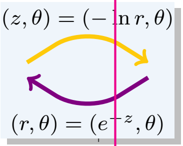
$(\Re e\,\lambda = a,\ \Im m\,\lambda = b)$

$s \in \mathrm{H}^1(\Omega)$	$\Re e\,\lambda > 0$	m is evanescent
$s \notin \mathrm{H}^1(\Omega)$	$\Re e\,\lambda = 0$	m is propagative

Analogy with a waveguide problem

- Bounded sector Ω





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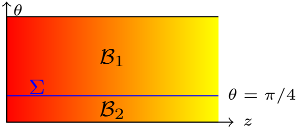

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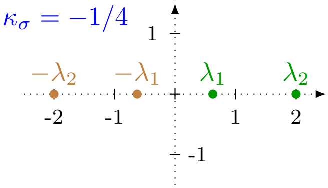
▶ This encourages us to use modal decomposition in the half-strip.

Modal analysis in the waveguide



► Outside the critical interval. All the modes are exponentially growing or decaying.

→ We look for an exponentially decaying solution. $\boxed{\mathrm{H}^1 \text{ framework}}$

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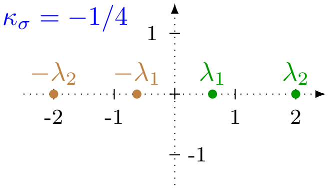

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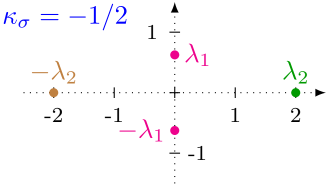
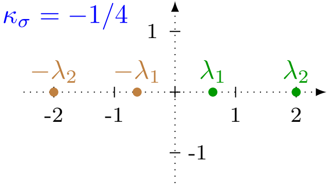


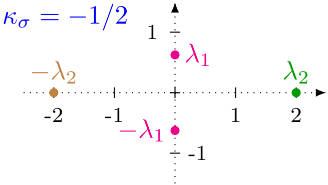
► Inside the critical interval. There are exactly two propagative modes.

Modal analysis in the waveguide



- Outside the critical interval. All the modes are exponentially growing or decaying.

$\rightarrow$ We look for an exponentially decaying solution. $\boxed{\text{H}^1 \text{ framework}}$



- Inside the critical interval. There are exactly two propagative modes.

$\rightarrow$ The decomposition on the outgoing modes leads to look for a solution of the form

$$u = \underbrace{c_1\, \varphi_1\, e^{\lambda_1 z}}_{\text{propagative part}} + \underbrace{u_e}_{\text{evanescent part}}.$$

$\boxed{\text{non H}^1 \text{ framework}}$

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$\mathrm{W}_{-\beta} = \{v \,|\, e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\}$ space of exponentially decaying functions

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$$
\begin{aligned}
\mathrm{W}_{-\beta} &= \{v \,|\, e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} && \text{space of exponentially decaying functions}\\
\mathrm{W}^{+} &= \mathrm{span}(\zeta\varphi_1\, e^{\lambda_1 z}) \oplus \mathrm{W}_{-\beta} && \text{propagative part + evanescent part}\\
\mathrm{W}_{\beta} &= \{v \,|\, e^{-\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} && \text{space of exponentially growing functions}
\end{aligned}
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THEOREM. Let $\kappa_\sigma \in (-1; -1/3)$ and $0 < \beta < 2$. The operator A^{+} : $\mathrm{div}(\sigma \nabla \cdot)$ from W^{+} to ${\mathrm{W}_\beta}^*$ is an isomorphism.

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THEOREM. Let $\kappa_\sigma \in (-1; -1/3)$ and $0 < \beta < 2$. The operator $A^+ : \mathrm{div}(\sigma\nabla\cdot)$ from W^+ to ${\mathrm{W}_\beta}^*$ is an isomorphism.

IDEAS OF THE PROOF:

1. $A_{-\beta} : \mathrm{div}(\sigma\nabla\cdot)$ from $\mathrm{W}_{-\beta}$ to ${\mathrm{W}_\beta}^*$ is injective but not surjective.

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

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Theorem. Let $\kappa_\sigma \in (-1; -1/3)$ and $0 < \beta < 2$. The operator $A^+ : \mathrm{div}(\sigma\nabla\cdot)$ from W^+ to ${\mathrm{W}_\beta}^*$ is an isomorphism.

Ideas of the proof:

1. $A_{-\beta} : \mathrm{div}(\sigma\nabla\cdot)$ from $\mathrm{W}_{-\beta}$ to ${\mathrm{W}_\beta}^*$ is injective but not surjective.
2. $A_{\beta} : \mathrm{div}(\sigma\nabla\cdot)$ from W_{β} to ${\mathrm{W}_{-\beta}}^*$ is surjective but not injective.
3. The intermediate operator $A^+ : \mathrm{W}^+ \to {\mathrm{W}_\beta}^*$ is injective (energy integral) and surjective (residue theorem).

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3. The intermediate operator $A^+ : \mathrm{W}^+ \to {\mathrm{W}_\beta}^*$ is injective (energy integral) and surjective (residue theorem).
4. Limiting absorption principle to select the outgoing mode.

Naive approximation

► Let us try a usual Finite Element Method (P1 Lagrange Finite Element). We solve the problem

$$\boxed{\left|\begin{array}{l}\text{Find } u_h \in \mathrm{V}_h \text{ s.t.:}\\ \displaystyle\int_\Omega \sigma \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h, \quad \forall v \in \mathrm{V}_h,\end{array}\right.}$$

where V_h approximates $\mathrm{H}^1_0(\Omega)$ as $h \to 0$ (h is the mesh size).

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where V_h approximates $\mathrm{H}_0^1(\Omega)$ as $h \to 0$ (h is the mesh size).

- We display u_h as $h \to 0$.

Naive approximation

- Let us try a usual Finite Element Method (P1 Lagra
Element). We solve the problem

The sequence (u_h) does not converge as $h \to 0$!!!

$$\left|\begin{array}{l}\text{Find } u_h \\ \displaystyle\int_{\Omega} \quad u_h \cdot \nabla v_h = \int_{\Omega} f v_h, \quad \forall v \in \mathrm{V}_h,\end{array}\right.$$

where V_h approximates $\mathrm{H}^1_0(\Omega)$ as $h \to 0$ (h is the mesh size).

- We display u_h as $h \to 0$.

$(\dots)$

Contrast $\kappa_\sigma = -0.999 \in (-1; -1/3)$.

Remark

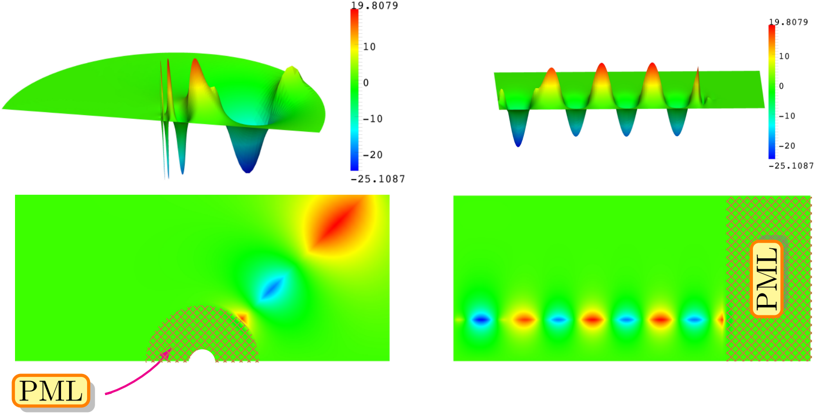
- Outside the critical interval, the sequence (u_h) converges with the naive approximation.

$(\dots)$

Contrast $\kappa_\sigma = -1.001 \notin (-1; -1/3)$.

How to approximate the solution?

- We use a PML (*Perfectly Matched Layer*) to bound the domain $\mathcal{B}$ + finite elements in the truncated strip ($\kappa_\sigma = -0.999 \in (-1; -1/3)$) (`Bonnet-Ben Dhia, Carvalho, Chesnel, Ciarlet 16`).



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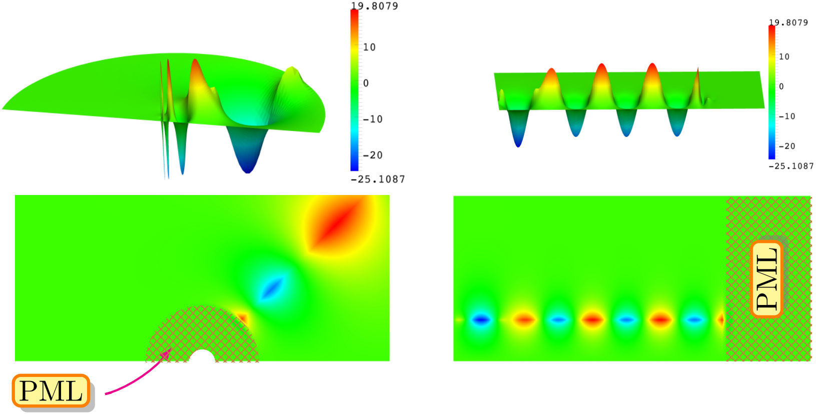
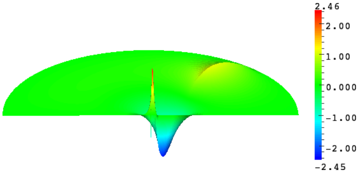


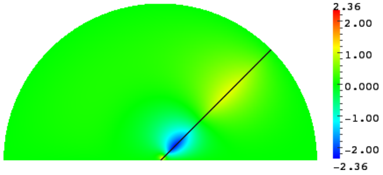
Without the PML, the solution in the truncated strip of length L does not converge when $L \to \infty$.

A black hole phenomenon

- The same phenomenon occurs for the problem with a non zero ω.

$$(\boldsymbol{x}, t) \mapsto \Re e\left(u(\boldsymbol{x})e^{-i\omega t}\right) \quad \text{for } \kappa_\sigma = -1/1.3$$





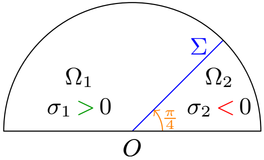
The corner point is like infinite: it is necessary to impose a radiation condition to select the outgoing behaviour.

- Analogous phenomena occur in cuspidal domains in the theory of water-waves and in elasticity (Cardone, Nazarov, Taskinen 11).

Summary of the results for the scalar problem

Problem

$$(\mathscr{P}) \left|\begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ -\mathrm{div}\,(\sigma \nabla u) = f \quad \text{ in } \Omega. \end{array}\right.$$



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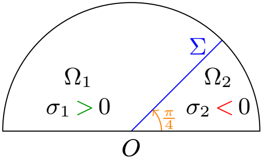

Results

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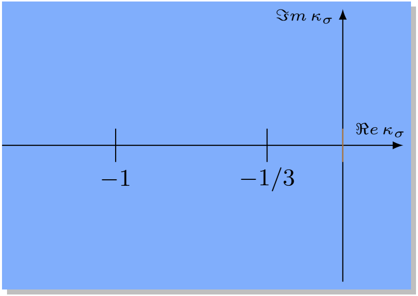

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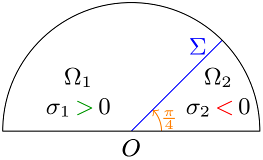

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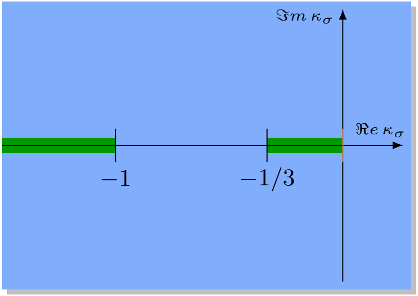
- For $\kappa_\sigma \in \mathbb{R}^*_- \setminus [-1; -1/3]$, $(\mathscr{P})$ well-posed in $\mathrm{H}^1_0(\Omega)$ ($\mathtt{T}$-coercivity)



Summary of the results for the scalar problem

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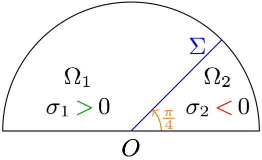

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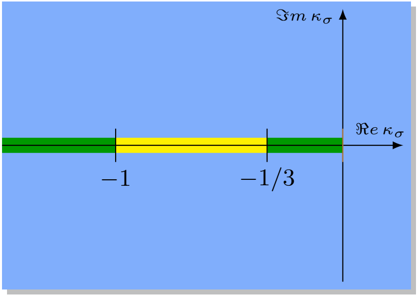

Summary of the results for the scalar problem

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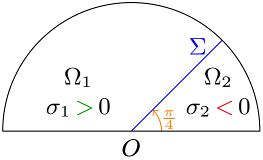

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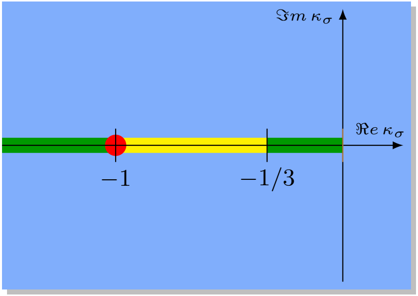
$\kappa_\sigma = -1$, $(\mathscr{P})$ ill-posed in $\mathrm{H}^1_0(\Omega)$



Problem formulation

For $\boldsymbol{F} \in \mathbf{L}^2(\Omega)$ s.t. $\operatorname{div} \boldsymbol{F} = 0$, consider the problem for the electric field $\boldsymbol{E}$

$$\left| \begin{array}{l} \text{Find } \boldsymbol{E} \in \mathbf{X}_N(\varepsilon) \text{ such that for all } \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon) : \\ \underbrace{\int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \overline{\boldsymbol{E}'}}_{a(\boldsymbol{E},\boldsymbol{E}')} - \omega^2 \underbrace{\int_\Omega \varepsilon \boldsymbol{E} \cdot \overline{\boldsymbol{E}'}}_{c(\boldsymbol{E},\boldsymbol{E}')} = \underbrace{\int_\Omega \boldsymbol{F} \cdot \overline{\boldsymbol{E}'}}_{\ell(\boldsymbol{E}')}, \end{array} \right.$$

with $\mathbf{X}_N(\varepsilon) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,) \,|\, \operatorname{div}(\varepsilon \boldsymbol{u}) = 0 \text{ in } \Omega,\ \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

Difficulties:

When μ changes sign, $a(\cdot,\cdot)$ is not coercive.

When ε changes sign, is the embedding $\mathbf{X}_N(\varepsilon) \subset \mathbf{L}^2(\Omega)$ compact?

$\mathbb{T}$-coercivity for Maxwell 1/2

If $\mathbb{T}$ is an isomorphism of $\mathbf{X}_N(\varepsilon)$, we have

$$\begin{array}{rrcll} & a(\boldsymbol{E},\boldsymbol{E}') - \omega^2 c(\boldsymbol{E},\boldsymbol{E}') & = & \ell(\boldsymbol{E}'), & \forall \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon) \\ \Leftrightarrow & a(\boldsymbol{E},\mathbb{T}\boldsymbol{E}') - \omega^2 c(\boldsymbol{E},\mathbb{T}\boldsymbol{E}') & = & \ell(\mathbb{T}\boldsymbol{E}'), & \forall \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon). \end{array}$$

Goal: to construct $\mathbb{T}$ such that

$$a(\boldsymbol{E},\mathbb{T}\boldsymbol{E}') = \int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, (\overline{\mathbb{T}\boldsymbol{E}'})$$

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Let us try $\mathbb{T}\boldsymbol{E} = \left| \begin{array}{ll} \boldsymbol{E}_1 & \text{in } \Omega_1 \\ -\boldsymbol{E}_2 + 2R_1\boldsymbol{E}_1 & \text{in } \Omega_2 \end{array}\right.$, If $\mathbb{T}$ is an isomorphism of $\mathbf{X}_N(\varepsilon)$, we have

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Not possible!

$\mathbb{T}$-coercivity for Maxwell 2/2

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$. We would like to have

$$\mathbf{curl}\,(\mathbb{T}\boldsymbol{E}) = \mu\mathbf{curl}\,\boldsymbol{E}$$

to get $a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \displaystyle\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,(\overline{\mathbb{T}\boldsymbol{E}})\,dx = \int_\Omega |\mathbf{curl}\,\boldsymbol{E}|^2\,dx.$

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To present the construction, define the **scalar** operators $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$, $A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ such that

$$(A_\varepsilon\varphi, \varphi')_{\mathrm{H}^1_0(\Omega)} = \int_\Omega \varepsilon\nabla\varphi \cdot \nabla\overline{\varphi'}\,dx, \qquad \forall \varphi, \varphi' \in \mathrm{H}^1_0(\Omega).$$

$$(A_\mu\varphi, \varphi')_{\mathrm{H}^1_\#(\Omega)} = \int_\Omega \mu\nabla\varphi \cdot \nabla\overline{\varphi'}\,dx, \qquad \forall \varphi, \varphi' \in \mathrm{H}^1_\#(\Omega).$$

where $\mathrm{H}^1_\#(\Omega) := \{\varphi \in \mathrm{H}^1(\Omega) \mid \int_\Omega \varphi\,dx = 0\}$.

$\mathbb{T}$-coercivity for Maxwell

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

$\mathbb{T}$-coercivity for Maxwell

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❶ Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\, \boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

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☛ Ok when A_μ is an isom.

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2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

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4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$.

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3. Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. To proceed, solve

$$\int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'\,dx = \int_\Omega \varepsilon\boldsymbol{u}\cdot\nabla\varphi'\,dx, \quad \forall\varphi' \in \mathrm{H}^1_0(\Omega).$$

Ok when A_ε is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. There holds:

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\overline{\boldsymbol{u}}\,dx$$

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

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$\mathbb{T}$-coercivity for Maxwell 2/2

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Lemma. Suppose that

$$A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega) \text{ is an isomorphism}$$

$$A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega) \text{ is an isomorphism.}$$

Then, there exists $\mathbb{T} : \mathbf{X}_N(\varepsilon) \to \mathbf{X}_N(\varepsilon)$ such that, for all $\boldsymbol{E}$, $\boldsymbol{E}'$

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}') = a(\mathbb{T}\boldsymbol{E}, \boldsymbol{E}') = \int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\overline{\boldsymbol{E}'}\,dx$$

(this implies in particular that $\mathbb{T}$ is an isomorphism of $\mathbf{X}_N(\varepsilon)$).

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Compact embedding and final result

Using a similar construction, we prove the

THEOREM. If $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isomorphism, then $\mathbf{X}_N(\varepsilon)$ is compactly embedded in $\mathbf{L}^2(\Omega)$ and $(\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)$ is a inner product in $\mathbf{X}_N(\varepsilon)$.

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- This yields the final result (Bonnet-BenDhia, Chesnel, Ciarlet 14'):

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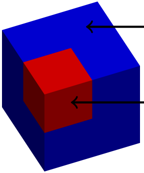
Then, the problem for the electric field is well-posed for all $\omega \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

Comments and example

- We have a similar result for the magnetic problem.
- These results extend to:

- situations where A_ε, A_μ are Fredholm of index zero with a non zero kernel;
- situations where Ω is not simply connected/$\partial\Omega$ is not connected.

Example of the Fichera's cube:



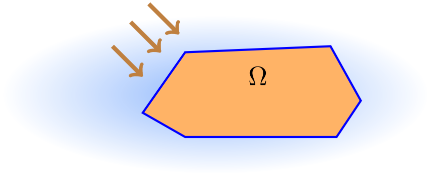
Proposition. Assume that

$$\frac{\varepsilon_-}{\varepsilon_+} \notin [-7; -\frac{1}{7}] \qquad \text{and} \qquad \frac{\mu_-}{\mu_+} \notin [-7; -\frac{1}{7}]. \quad *$$

Then, the problems for the electric and magnetic fields are well-posed for all $\omega \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

* *Note that 7 is the ratio of the blue volume over the red volume. This interval is not optimal.*

The ITEP in nutshell



- We want to determine the support of an inclusion Ω embedded in a reference medium ($\mathbb{R}^2$) using the Linear Sampling Method.

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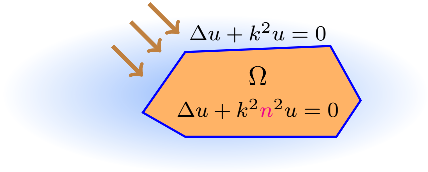

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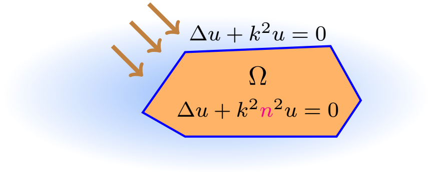

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- We can use the method when k is not an eigenvalue of the Interior Transmission Eigenvalue Problem:

$$\left| \begin{array}{l} \text{Find } (k, v) \in \mathbb{C} \times \mathrm{H}_0^2(\Omega) \setminus \{0\} \text{ such that:} \\ \displaystyle\int_\Omega \frac{1}{1 - n^2} (\Delta v + k^2 n^2 v)(\Delta v' + k^2 v') = 0, \quad \forall v' \in \mathrm{H}_0^2(\Omega). \end{array} \right.$$

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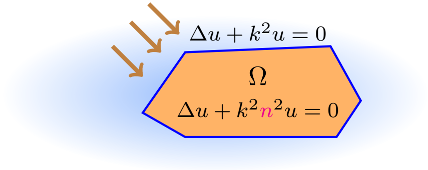

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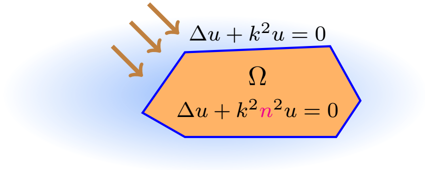

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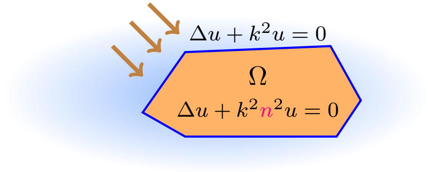
The ITEP in nutshell



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TRANSMISSION PROBLEM WITH A SIGN-CHANGING COEFFICIENT

$$\left|\begin{array}{l}\text{Find } (k, v) \in \mathbb{C} \times \mathrm{H}^2_0(\Omega) \setminus \{0\} \text{ such that:} \\ \displaystyle\int_\Omega \frac{1}{1-n^2}(\Delta v + k^2 n^2 v)(\Delta v' + k^2 v') = 0, \quad \forall v' \in \mathrm{H}^2_0(\Omega).\end{array}\right.$$

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What happens when $1 - n^2$ changes sign?

A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

$$(\mathscr{F}_V) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^2_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \sigma \Delta v \Delta v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{\ell(v')}, \quad \forall v' \in \mathrm{H}^2_0(\Omega). \end{array} \right.$$

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Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\mathrm{div}\,(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

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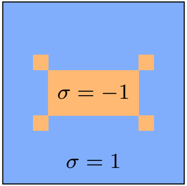
Theorem. The problem $(\mathscr{F}_V)$ is well-posed in the Fredholm sense as soon as σ does not change sign in a neighbourhood of $\partial\Omega$.

Fredholm



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Ideas of the proof: We have

$$a(v, v) = (\sigma\Delta v, \Delta v)_\Omega.$$

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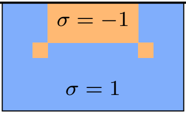

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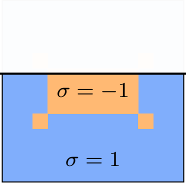

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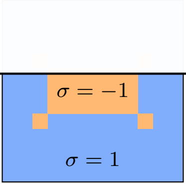

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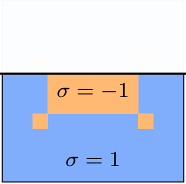

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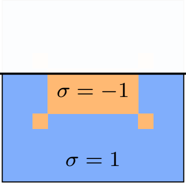

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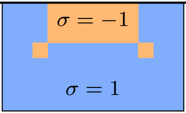

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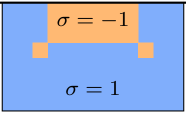

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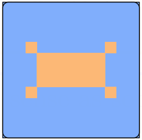

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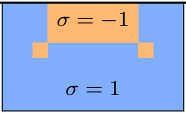

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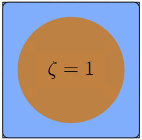

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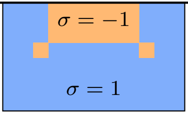

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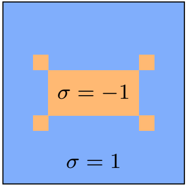
Theorem. The problem $(\mathscr{F}_V)$ is well-posed in the Fredholm sense as soon as σ does not change sign in a neighbourhood of $\partial\Omega$.

Fredholm



A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

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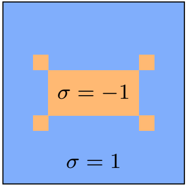
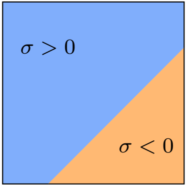
... but $(\mathscr{F}_V)$ can be ill-posed (not Fredholm) when σ changes sign "on $\partial\Omega$".

Fredholm





Not always Fredholm

Conclusions

Scalar problem outside the critical interval $\mathrm{div}\,(\mu^{-1}\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$

- Concerning the approximation of the solution by FEM, in practice, usual methods converge. Only partial proofs are available.
- In 3D, are the interval obtained optimal?

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Scalar problem inside the critical interval $\quad \mathrm{div}\,(\mu^{-1}\nabla\cdot) : \mathrm{V}^+(\Omega) \to \mathrm{V}_\beta(\Omega)^*$

- What happens in 3D (edge, intersection of edges,...)?
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Interior Transmission Eigenvalue Problem $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$

- How to compute the transmission eigenvalues when there are oscillating singularities? (coll. with F. Monteghetti).

Open questions

♠ The new model in the critical interval raises many questions related to the physics of plasmonics and metamaterials.

Can we observe this black-hole effect in practice? For rounded corners, we showed that the solution is unstable with respect to the rounding parameter...

♠ The case $\kappa_\sigma = -1$ (the graal for applications) has still to be studied. New frameworks have been proposed (Joly-Vinoles, Nguyen, Benhellal-Pankrashkin,...): $\Rightarrow$ how to approximate the solutions?

♠ For metamaterials, can we reconsider the homogenization process to take into account interfacial phenomena?

$\Rightarrow$ See the work of Claeys-Fliss-Vinoles.

♠ In practice ε and μ depend on ω.

What happens for the spectral problems? in time-domain regime? Is the limiting amplitude principle still valid?

$\Rightarrow$ See the works of Hazard-Paolantoni, Cassier-Joly-Kachanovska.

Thank you for your attention!!!