Construction of invisible defects in acoustic waveguides

Lucas Chesnel[1]

Coll. with A. Bera[2], A.-S. Bonnet-Ben Dhia[2] and S.A. Nazarov[3].

[1]Defi team, CMAP, École Polytechnique, France
[2]Poems team, Ensta ParisTech, France
[3]FMM, St. Petersburg State University, Russia

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General setting

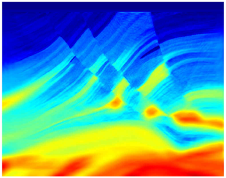
- We are interested in methods based on the propagation of waves to determine the shape, the physical properties of objects, in an exact or qualitative manner, from given measurements.

- General principle of the methods:

  i) send waves in the medium;

  ii) measure the scattered field;

  iii) deduce information on the structure.



- Many techniques: Xray, ultrasound imaging, seismic tomography, ...
- Many applications: biomedical imaging, non destructive testing of materials, geophysics, ...

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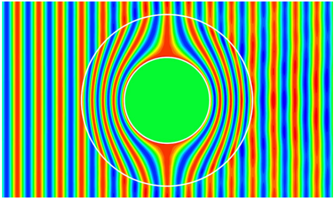

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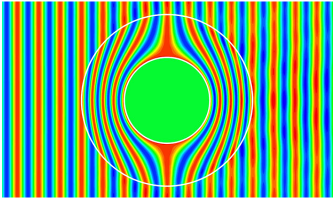

  - Also relevant for applications, in particular in waveguides.

- At least two reasons to study invisibility questions:

  - We can wish to hide objects.
  - It allows to understand limits of imaging techniques.

Outline of the talk

1. Construction of invisible penetrable defects

2. Can one hide a small Dirichlet obstacle?

3. Can one hide a perturbation of the wall?

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Waveguide problem

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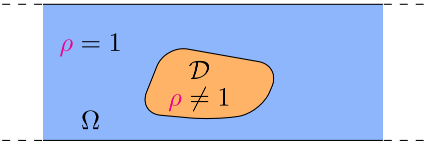

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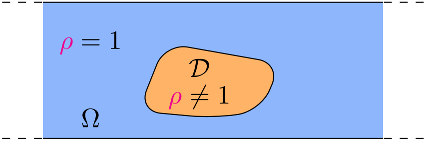

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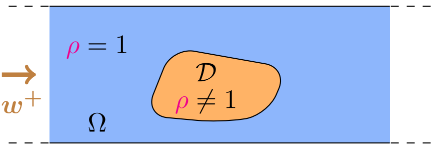

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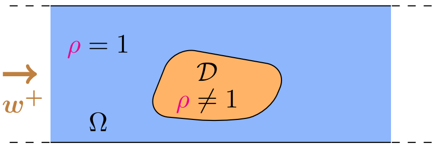

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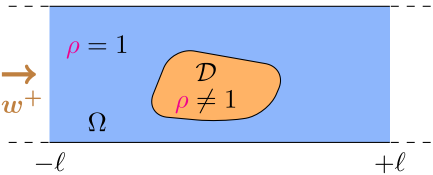

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($\chi^{\pm}$ *are smooth cut-off functions s.t.* $\chi^{\pm} = 1$ *for* $\pm x \geq 2\ell$, $\chi^{\pm} = 0$ *for* $\pm x \leq \ell$)

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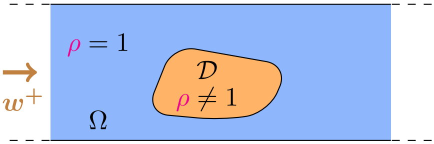

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Definition:
u_{i} = incident field (data)
u = total field (defined by $(\mathscr{P})$)
u_{s} = scattered field (defined by $(\mathscr{P})$).

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We explain how to construct inclusions such that

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$$s^- = 0 \quad \text{or} \quad s^+ = 0.$$

- These inclusions cannot be detected from far field measurements.
- We assume that k and the support of the inclusion $\overline{\mathcal{D}}$ are given.

Goal

Find a real valued function $\rho \not\equiv 1$, with $\rho - 1$ supported in $\overline{\mathcal{D}}$, such that the solution to the problem

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satisfies $s^- = 0$ or $s^+ = 0$.

Sketch of the method

- We will work as in the proof of the implicit functions theorem.

  The idea was used in Nazarov 11 to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.

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- Define $\sigma = \rho - 1$ and gather the measurements in the vector
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- No obstacle leads to null measurements $\Rightarrow$ $F(0) = 0$.

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- We look for small perturbations of the reference medium: $\sigma = \varepsilon\mu$ where $\varepsilon > 0$ is a small parameter and where μ has be to determined.

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Assume that $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.

$$\exists \mu_0, \mu_1, \dots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t. } \left| \begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \dots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array} \right.$$

- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the τ_n are real parameters to set:

$$0 = F(\varepsilon\mu) \qquad \Leftrightarrow \qquad \boxed{\vec{\tau} = G^\varepsilon(\vec{\tau})}$$

where $\vec{\tau} = (\tau_1, \dots, \tau_{2N})^\top$ and $G^\varepsilon(\vec{\tau}) = -\varepsilon \tilde{F}^\varepsilon(\mu)$.

If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \dots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

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If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.
Set $\sigma^{\mathrm{sol}} := \varepsilon \mu^{\mathrm{sol}}$. We have $F(\sigma^{\mathrm{sol}}) = 0$ (invisible inclusion).

Calculus of $dF(0)$

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re e\, s^-, \Im m\, s^-, \Re e\, s^+, \Im m\, s^+,).$$

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- $is^\pm = -k^2 \displaystyle\int_{\mathcal{D}} (\rho^\varepsilon - 1)\,(u_{\mathrm{i}} + u_{\mathrm{s}}^\varepsilon)\,\overline{w^\pm}\, d\boldsymbol{x}.$

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Conclusion: $dF(0)(\mu) = \Big(\int_{\mathcal{D}} \mu \cos(2kx)\, d\boldsymbol{x}, \int_{\mathcal{D}} \mu \sin(2kx)\, d\boldsymbol{x}, \int_{\mathcal{D}} \mu\, d\boldsymbol{x}, 0 \Big).$

Penetrable inclusion

- For $F(\sigma) = (\Re e \, \frac{s^-}{ik^2}, \Im m \, \frac{s^-}{ik^2}, \Re e \, \frac{s^+}{ik^2}, \Im m \, \frac{s^+}{ik^2})$, we obtain

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Is $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^4$ onto ?

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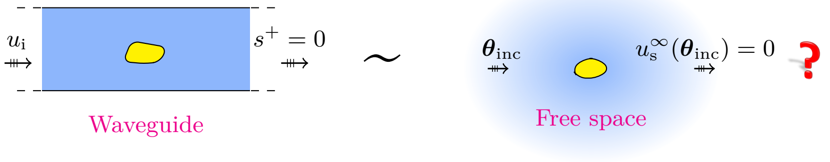

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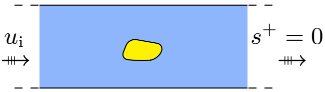
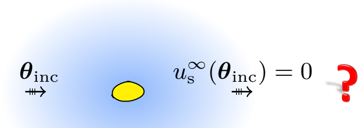


Waveguide

$\sim$



Free space

$u_{\rm s}^\infty(\boldsymbol{\theta}_{\rm inc}) = 0 \ \Rightarrow\ u_{\rm s} \equiv 0 \in \mathbb{R}^d \setminus \overline{D}$

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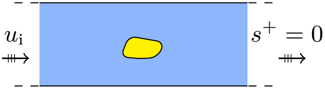

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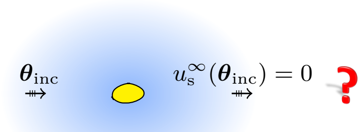

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Impose $\left|\begin{array}{l} s^- = 0 \\ \Im m\, s^+ = 0 \end{array}\right.$. *Then,* $\left|\begin{array}{l} |s^-|^2 + |1 + s^+|^2 = 1 \\ s^+ = O(\varepsilon) \end{array}\right.$ $\Rightarrow$ $s^+ = 0$.

Main result

PROPOSITION: For $\boldsymbol{\varepsilon}$ **small enough**, define $\rho^{\,\mathrm{sol}} = 1 + \varepsilon\mu^{\,\mathrm{sol}}$ with

$$\mu^{\,\mathrm{sol}} = \mu_0 + \sum_{n=1}^{3} \tau_n^{\,\mathrm{sol}}\,\mu_n.$$

Then the solution of the scattering problem

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u_{\mathrm{s}}^\varepsilon + w^+ \\ -\Delta u \quad = \quad k^2 \rho^{\,\mathrm{sol}} u \quad \text{in } \Omega, \\ \qquad\qquad u_{\mathrm{s}} \text{ is outgoing} \end{array}\right.$$

satisfies $s^- = s^+ = 0$.

COMMENTS:

→ We need $\boldsymbol{\varepsilon}$ to be **small enough** to prove that G^ε is a contraction.

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$\rightarrow$ We have $\tau^{\,\mathrm{sol}} = O(\varepsilon) \Rightarrow \mu^{\,\mathrm{sol}} \approx \mu_0$. We control the main form of the defect.

Numerical experiments: algorithm and data

- We can solve the fixed point problem using an iterative procedure: we set $\vec{\tau}^{\,0} = (0,0,0)^\top$ then define

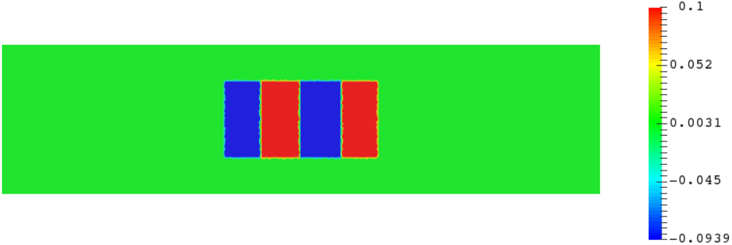
$$\boxed{\vec{\tau}^{\,n+1} = G^\varepsilon(\vec{\tau}^{\,n}).}$$

- At each step, we solve a scattering problem. We use a P2 finite element method in $\Omega_4 := (-4;4) \times (0;1)$. On $x = \pm 4$, a truncated Dirichlet-to-Neumann map with 10 harmonics serves as a transparent boundary condition.

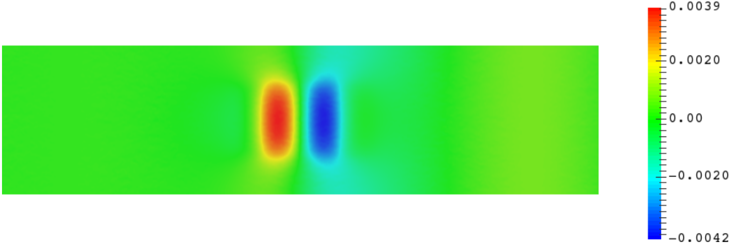
- We set $k = 3$ and $\mathcal{D} = (-\pi/(2k);\pi/(2k)) \times (1/4;3/4)$

Results at the end of the fixed point loop

$\sigma = \rho - 1$



$\Re e\, u_{\mathrm{s}}$



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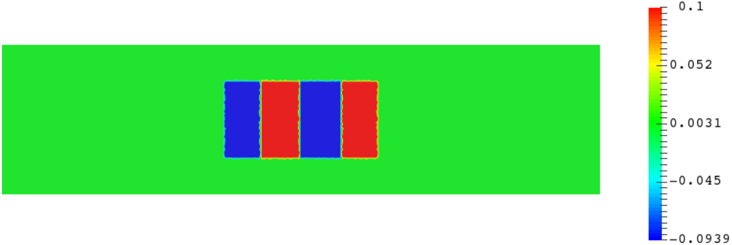

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Can we increase the perturbation to obtain larger defects ?

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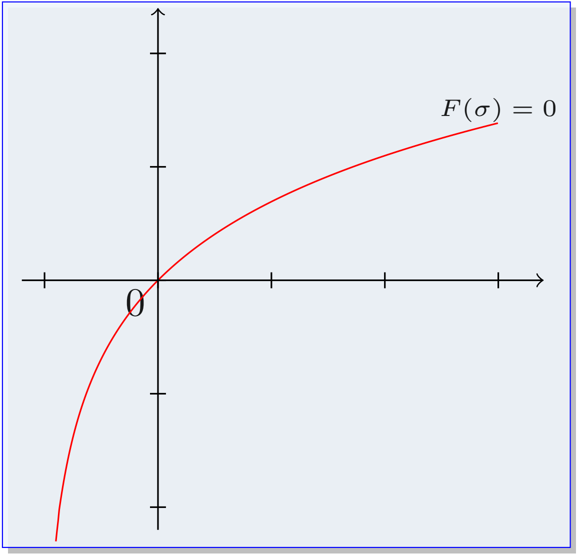
- Schematic view of what we did ($F : \mathbb{R}^2 \to \mathbb{R}$ is the measurements map):



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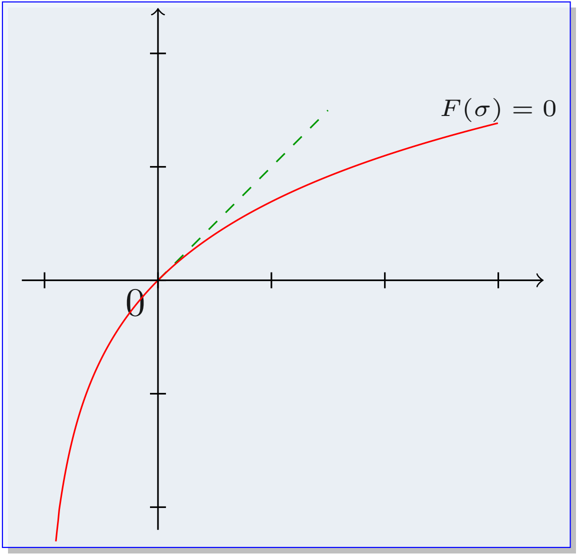

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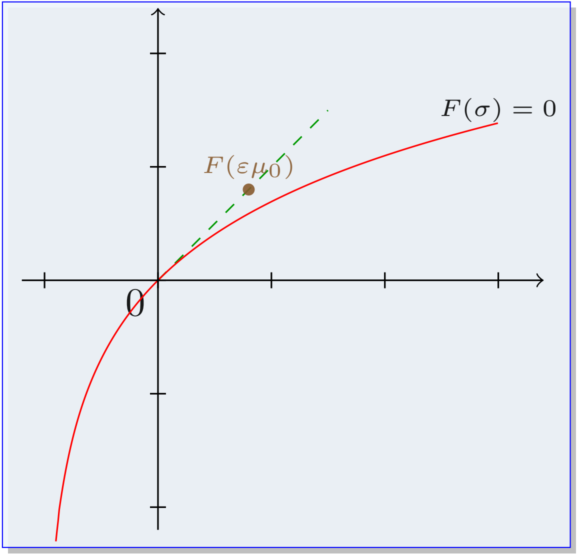

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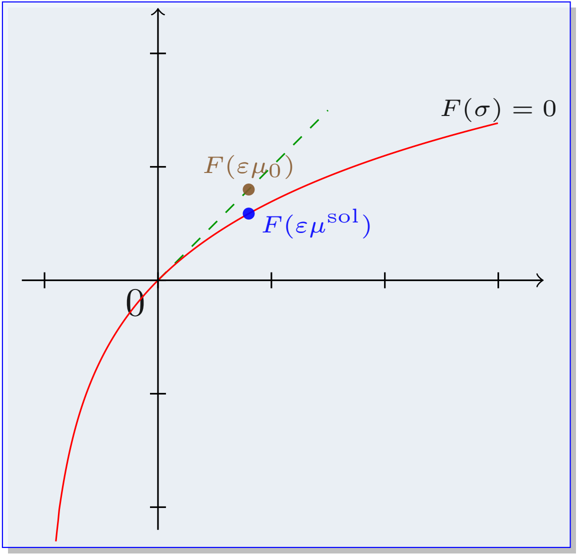

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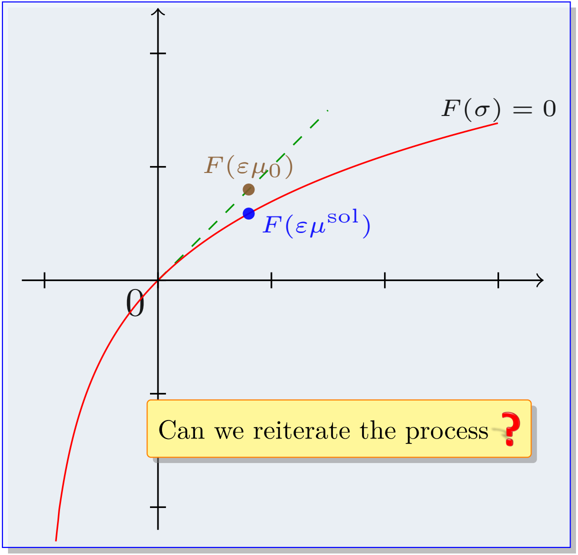

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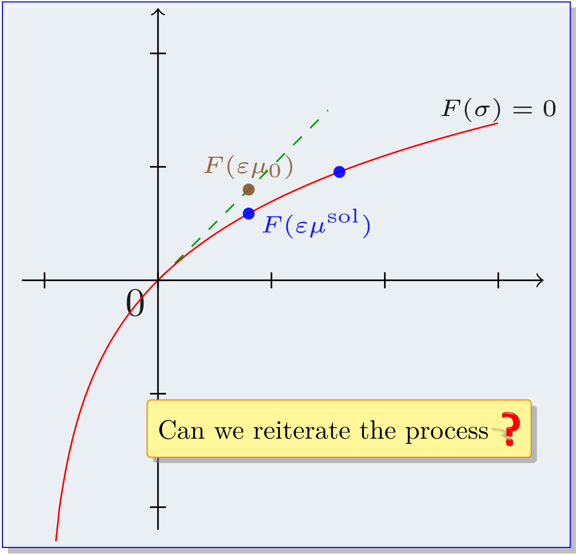

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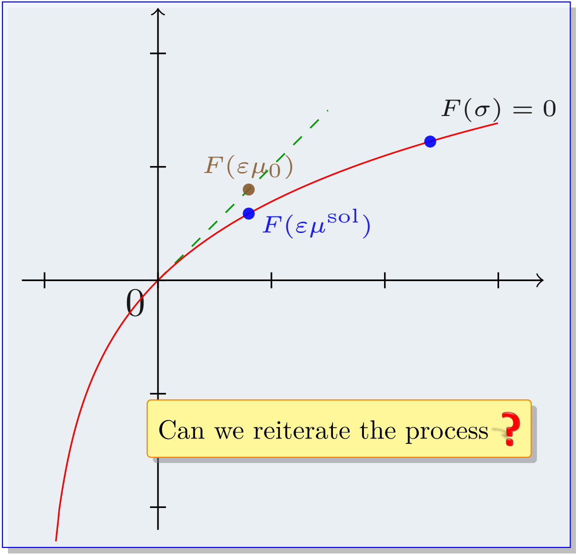

Numerical results to impose $s^- = 0$

- Same setting, **3 steps of iterations.**

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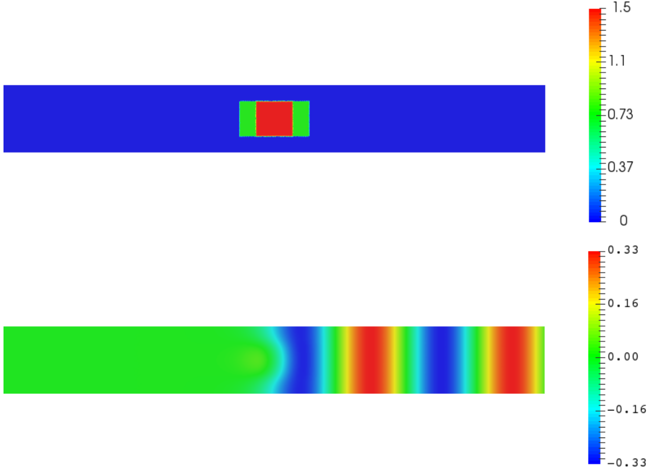

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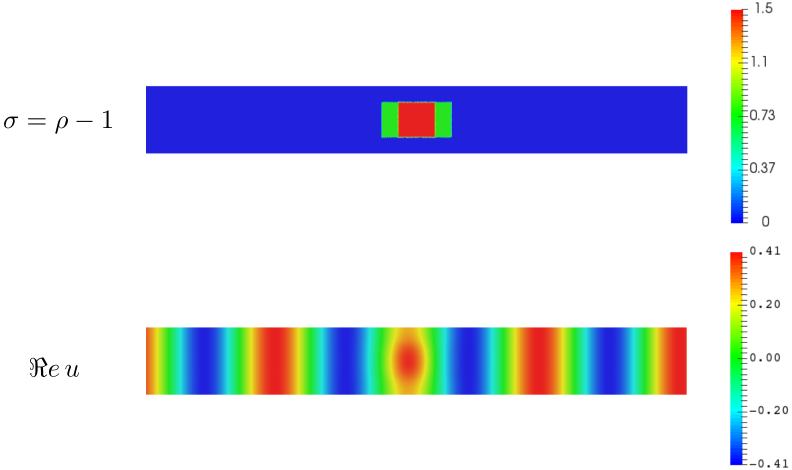

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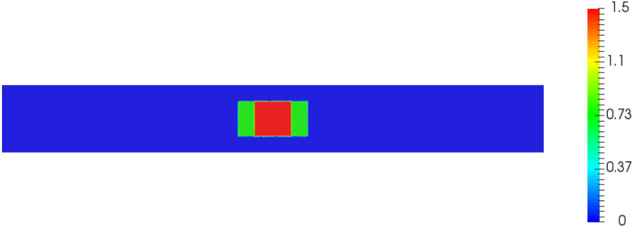

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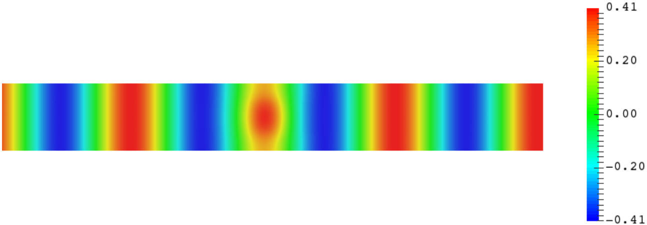
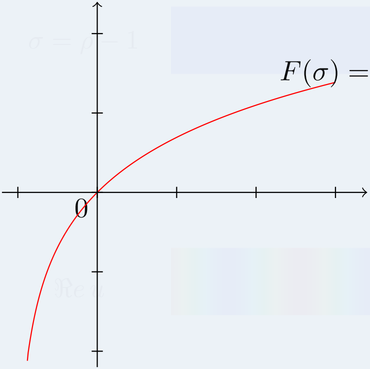


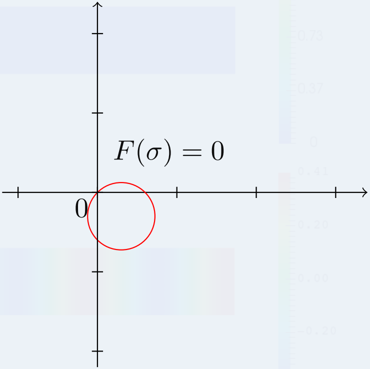
$\rightarrow$ First results are encouraging. Still some questions: at each step, how to choose the new directions?

Numerical results to impose $s^- = 0$

Depending on the directions, we may have



or



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Numerical results to impose $s^- = 0$

- Same setting, **3 steps of iterations.**

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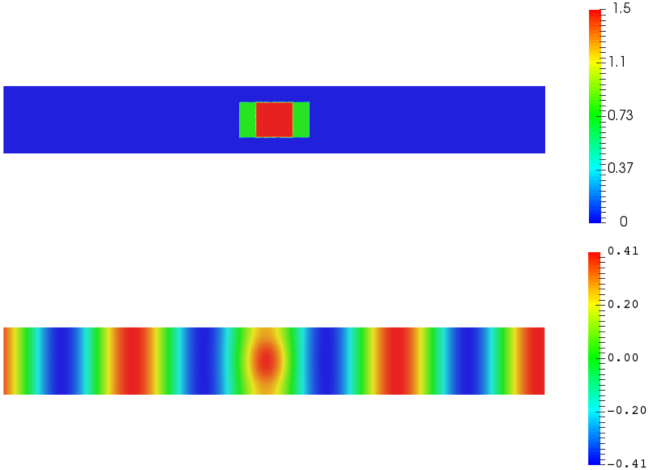

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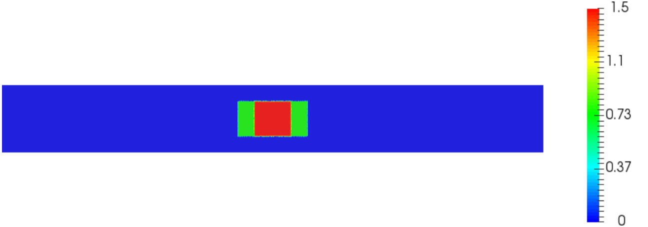

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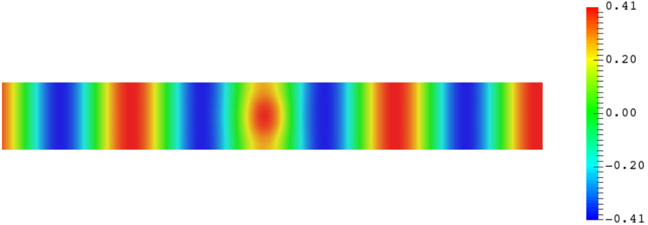
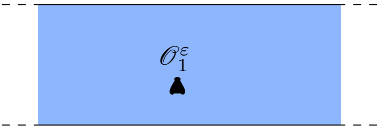


$\rightarrow$ First results are encouraging. Still some questions: at each step, how to choose the new directions?

$\rightarrow$ We are not able to prove that $ds^-(\sigma) : \mathrm{L}^\infty(\mathcal{D}) \rightarrow \mathbb{C}$ is onto for $\sigma \not\equiv 0$.

Small Dirichlet obstacle

- Can one hide a small Dirichlet obstacle $\mathscr{O}_1^\varepsilon = M_1 + \varepsilon\mathscr{O}$ centered at M_1?



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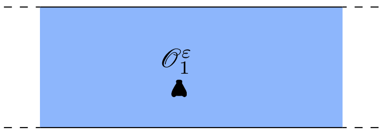

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Due to Dirichlet B.C., $w^\pm$ are not the same as previously (but this is not important).

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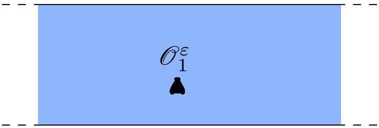

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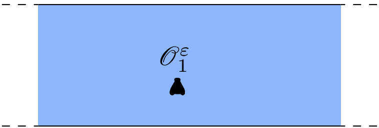

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Non zero terms!
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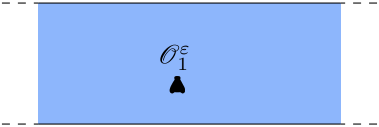

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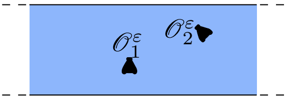
$\Rightarrow$ One single small obstacle cannot even be non reflective.

Small Dirichlet obstacles



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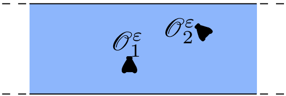

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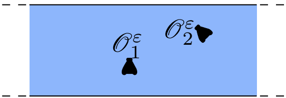

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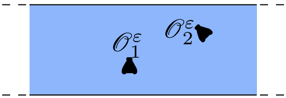

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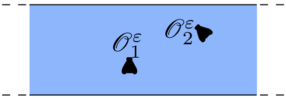

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Comments:

→ Hard part is to justify the asymptotics for the fixed point problem.

→ We cannot impose $s^+ = 0$ with this strategy.

→ When there are more propagative waves, we need more obstacles.

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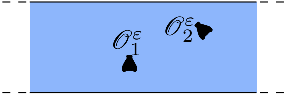

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Acting as a team, flies can become invisible!

Can one hide a perturbation of the wall?

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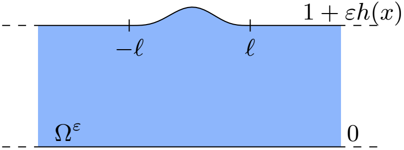

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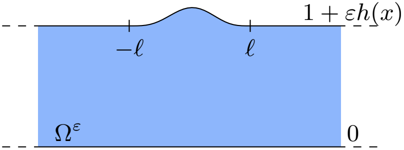

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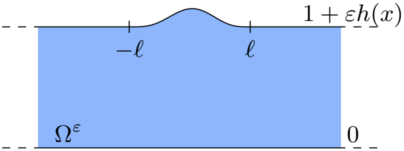

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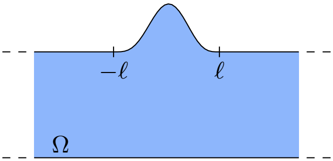
$\Rightarrow$ With this approach, we can impose $s^- = 0$ but not $s^+ = 0$.

Obstruction to invisibility

- More generally, for any Neumann wave-guide, one can show that $s^+ = 0$ implies
$$\int_\Omega |\nabla u_s|^2 - k^2|u_s|^2 \, d\boldsymbol{x} = 0.$$



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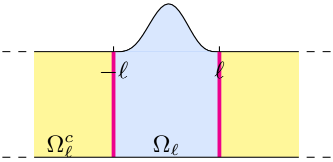

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$$\boxed{\int_{\Omega_\ell^c} |\nabla u_{\mathrm{s}}|^2 - k^2 |u_{\mathrm{s}}|^2 \, d\boldsymbol{x} \geq 0.}$$

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$$\boxed{\lambda_\dagger := \inf_{v \in \mathrm{Y}\setminus\{0\}} \left(\int_{\Omega_\ell} |\nabla v|^2 \, d\boldsymbol{x} \right) \Big/ \left(\int_{\Omega_\ell} |v|^2 \, d\boldsymbol{x} \right) > 0.}$$

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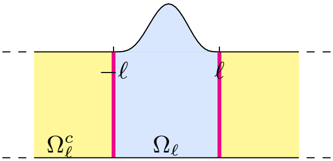

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Proposition: For a given shape, $s^+ = 0$ cannot hold for $k^2 \in (0; \lambda_\dagger)$.

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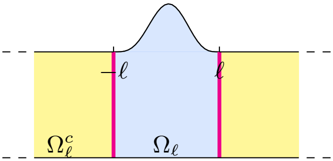

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Proposition: For a given shape, $s^+ = 0$ cannot hold for $k^2 \in (0; \lambda_\dagger)$.

$\rightarrow$ For a small smooth perturbation of amplitude εh, one finds $|\lambda_\dagger - \pi^2| \leq C\,\varepsilon$.

Obstruction to invisibility

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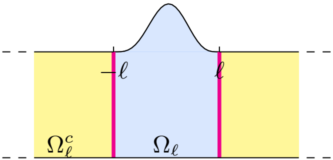

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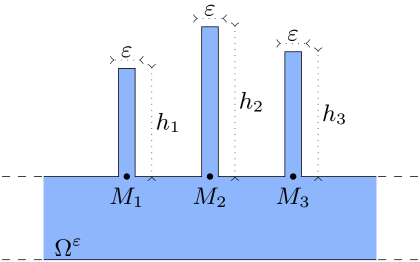
PROPOSITION: For a given shape, $s^+ = 0$ cannot hold for $k^2 \in (0; \lambda_\dagger)$.

$\rightarrow$ To impose invisibility at low frequency, we need to work with special shapes.

Non smooth perturbation of the wall

- We study the same problem in the geometry Ω^ε



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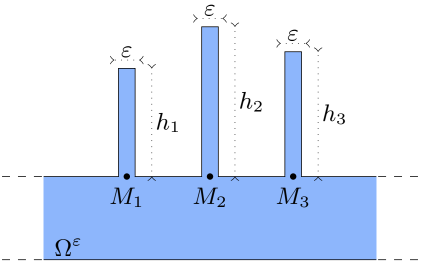

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1) We can find M_n, h_n such that $s^- = O(\varepsilon^2)$ *and* $s^+ = O(\varepsilon^2)$.

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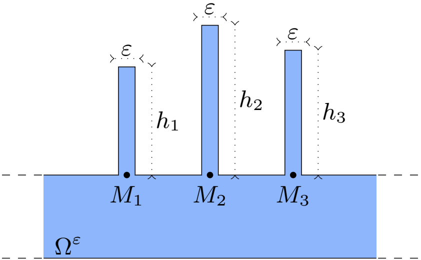

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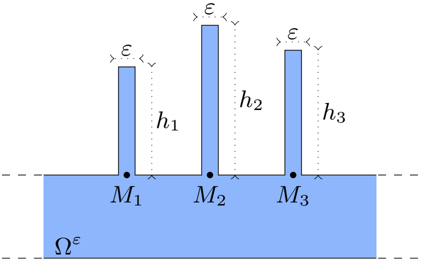
$$\begin{aligned} s^- &= 0+\varepsilon\left(ik\sum_{n=1}^{3}(w^+(M_n))^2\tan(kh_n)\right)+O(\varepsilon^2) \\ s^+ &= 0+\varepsilon\left(i/2\sum_{n=1}^{3}\tan(kh_n)\right)+O(\varepsilon^2) \end{aligned}$$

1) We can find M_n, h_n such that $s^-=O(\varepsilon^2)$ and $s^+=O(\varepsilon^2)$.

2) Then changing h_n into $h_n+\tau_n$, and choosing a good $\tau=(\tau_1,\tau_2,\tau_3)\in\mathbb{R}^3$ (fixed point), we can get $s^-=0$ and $\Im m\, s^+=0$.

Non smooth perturbation of the wall

- We study the same problem in the geometry Ω^ε



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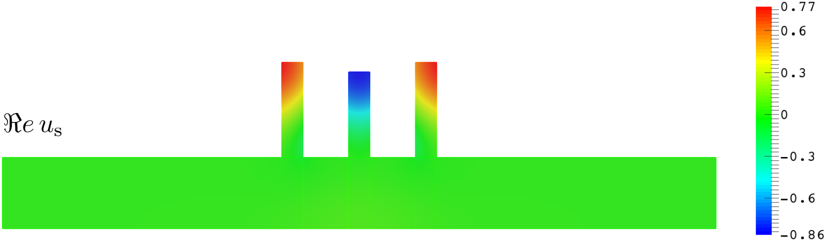
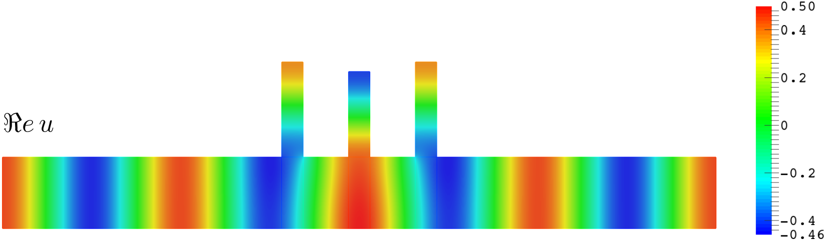
3) Energy conservation + $[s^+=O(\varepsilon)]$ $\Rightarrow$ $s^+=0$.

Geometry at the end of the numerical process

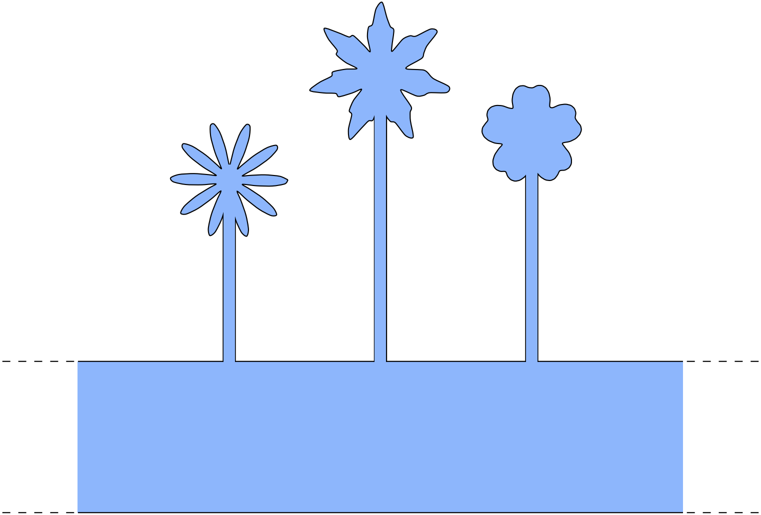
- We set $k = 0.8\pi$. We start with $h_1^0 = h_2^0 = h_3^0 = \pi/k$. At each step $j \geq 0$ of the fixed point loop, we change the geometry modifying h_1^j, h_2^j, h_3^j.





Remark

- We could also have worked with gardens of flowers!



1. Construction of invisible penetrable defects

2. Can one hide a small Dirichlet obstacle?

3. Can one hide a perturbation of the wall?

Conclusion

What we did

♠ We explained how to construct invisible perturbations of a reference situation in a setting with a finite number of measurements.

Future work

1) We want to continue the analysis of the reiteration process to construct large invisible defects of the reference medium.

2) It would be interesting to consider other models (Maxwell, elasticity, ...) and to investigate cases where the differential is not onto.

3) For a given perturbation, can we study the frequencies (invisible modes) such that invisibility holds?

4) We wish to better understand the link between the invisible modes and the so-called trapped modes in waveguides.

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$$\left|\begin{array}{lcll} (k,u_{\mathrm{s}}) \in \mathbb{R}\times \mathrm{H}^1(\Omega)\setminus\{0\} \text{ s. t.} & & & \\ -\Delta u_{\mathrm{s}} & = & k^2 u_{\mathrm{s}} & \text{in } \Omega, \\ \partial_n u_{\mathrm{s}} & = & -\partial_n u_{\mathrm{i}} & \text{on } \partial\Omega \end{array}\right.$$

Invisible mode

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Trapped mode

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Thank you for your attention!!!

1. Construction of invisible penetrable defects

2. Can one hide a small Dirichlet obstacle?

3. Can one hide a perturbation of the wall?

4. Can one construct a completely reflective defect?

Completely reflective defects

DEFINITION: Defect is said completely reflective if $s^+ = -1$.

- In these waveguides, $u = \chi^+(1+s^+)w^+ + \chi^-(w^+ + s^-w^-) + \tilde{u}$ is exponentially decaying at $+\infty$.

Can we construct such defects ?

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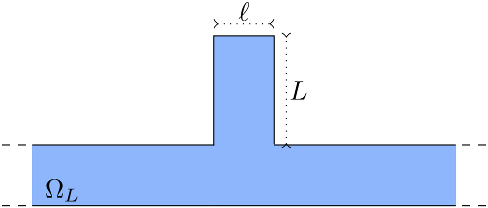

- Consider the Helmholtz problem with Dirichlet B.C. in Ω_L. It admits a solution $u = \chi^+(1+s_L^+)w^+ + \chi^-(w^+ + s_L^-w^-) + \tilde{u}$.

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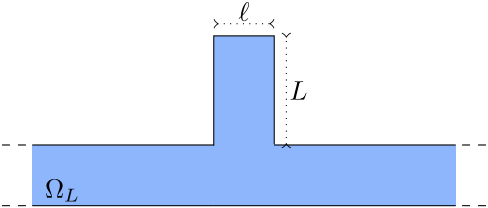

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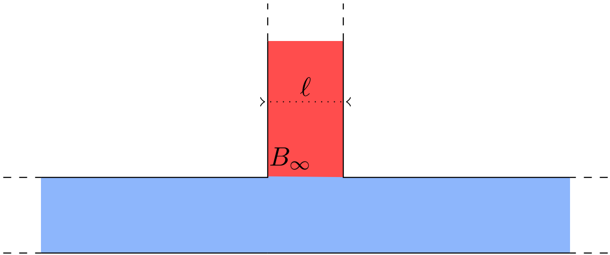
$\rightarrow$ For a given $\ell > 0$, we compute numerically $s_L^\pm$ as $L \rightarrow +\infty$

Asymptotic behaviours of $s_L^{\pm}$ as $L \to +\infty$ 1/2

Result depend on the nb. of prop. modes in the vertical branch B_∞



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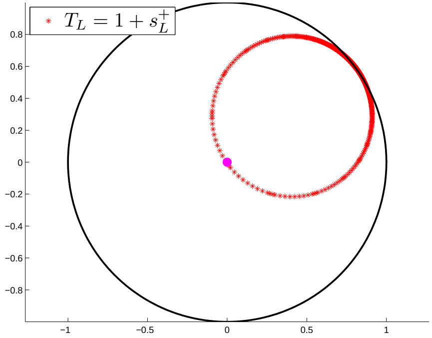

Figure: Numerical approximation of T_L, R_L for $L \in (0.4; 10)$, $\ell = 1$.

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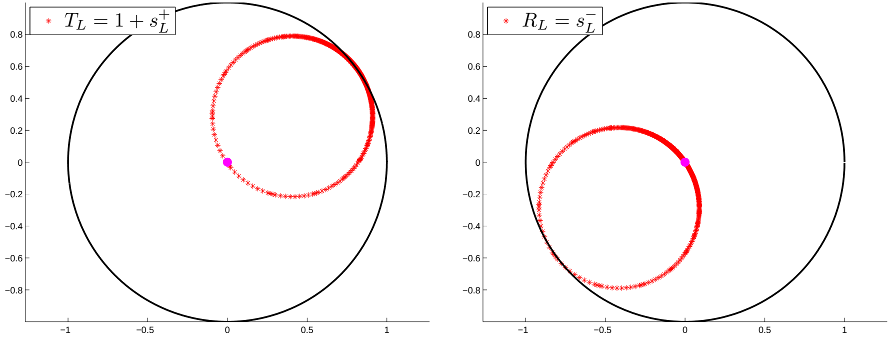

Figure: Numerical approximation of T_L, R_L for $L \in (0.4; 10)$, $\ell = 1$.

⋆ For $\boldsymbol{\ell \in (\pi/k; 2\pi/k)}$ (2 propagative modes $w_\circ^\pm$ in B_∞), we can prove that

$$|R_L - R_{\text{asy}}(L)| \leq C\, e^{-\alpha L} \quad \text{and} \quad |T_L - T_{\text{asy}}(L)| \leq C\, e^{-\alpha L}, \quad (C,\, \alpha > 0)$$

where $R_{\text{asy}}(L)$, $T_{\text{asy}}(L)$ run on circles passing period. through 0 as $L \to +\infty$.

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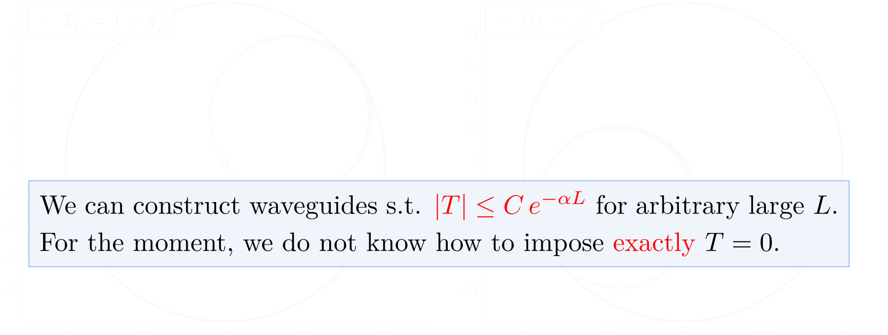

We can construct waveguides s.t. $|T| \leq C\, e^{-\alpha L}$ for arbitrary large L.
For the moment, we do not know how to impose exactly $T = 0$.

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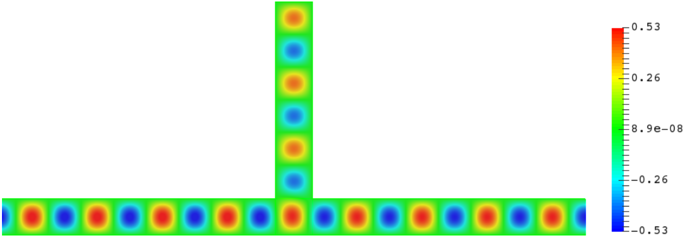
Numerical results

- Approximation of $t \mapsto \Re e\left(e^{-i\omega t} u(\boldsymbol{x})\right)$ for some arbitrary L.



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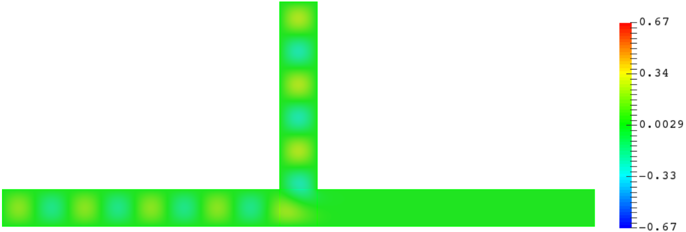

Asymptotic behaviours of $s_L^{\pm}$ as $L \to +\infty$ 2/2

$\star$ For $\boldsymbol{\ell \in (2\pi/k; 3\pi/k)}$ (4 propagative modes $w_\circ^{\pm}$, $w_\bullet^{\pm}$ in B_∞), one has

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but behaviours of $R_{\text{asy}}(L)$, $T_{\text{asy}}(L)$ can be more complex...

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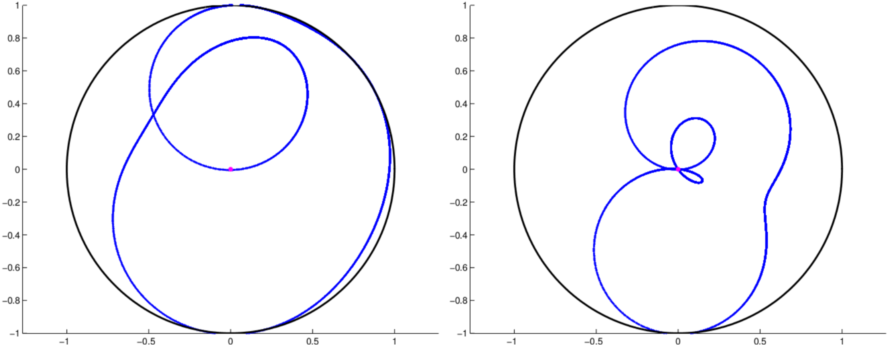

Figure: Curves $L \mapsto T_{\mathrm{asy}}(L)$, $R_{\mathrm{asy}}(L)$ for $L \in (0; +\infty)$, $\ell = \frac{\pi}{k}\sqrt{\frac{4m^2-1}{m^2-1}}$, $m = 2$.

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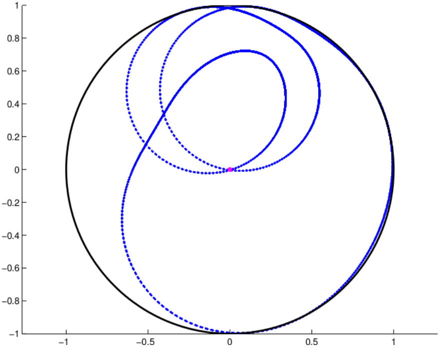

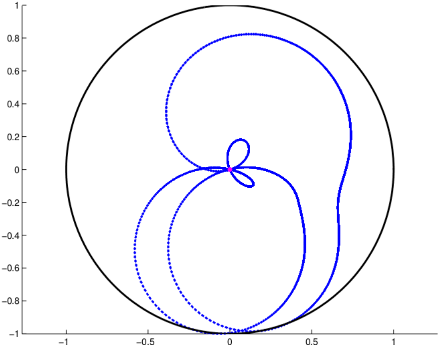

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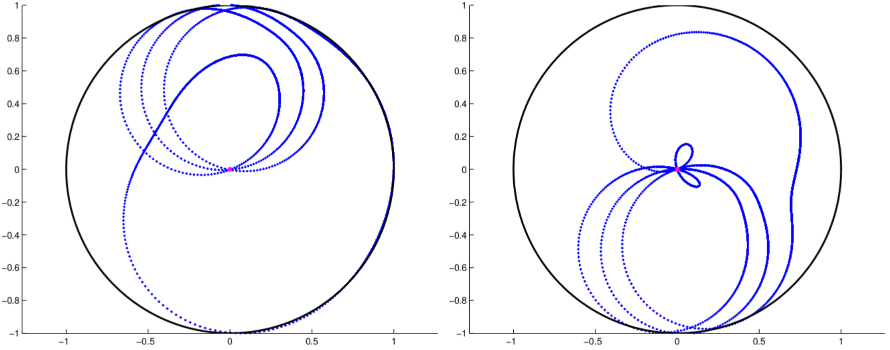

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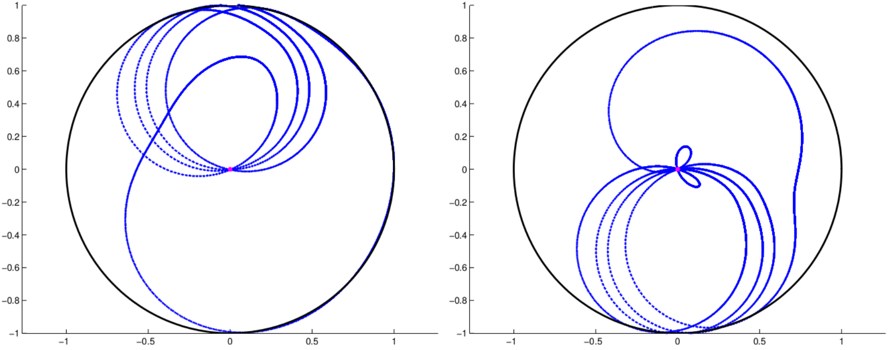

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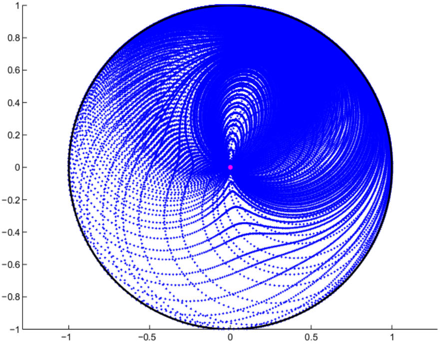

Figure: Curves $L \mapsto T_{\text{asy}}(L)$, $R_{\text{asy}}(L)$ for $L \in (0; 100)$, $\ell = 1.4$.

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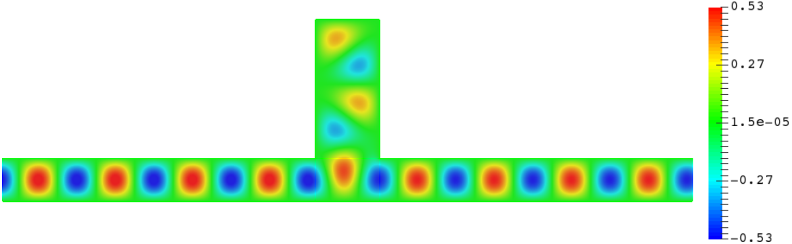

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