Investigation of some transmission problems with sign changing coefficients. Application to metamaterials.

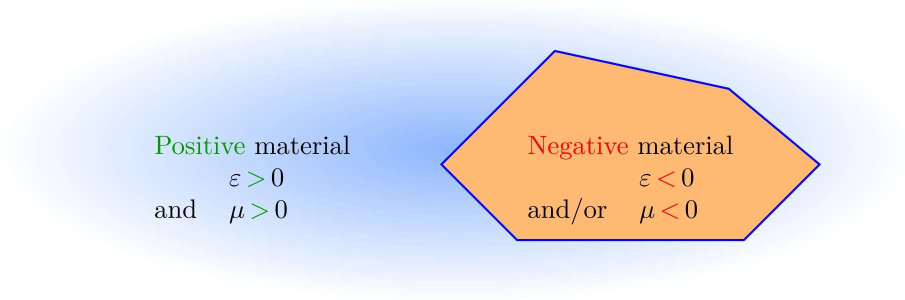
Lucas Chesnel

Supervisors: A.-S. Bonnet-Ben Dhia and P. Ciarlet
`UMA Ensta ParisTech, POems team`

Ensta ParisTech, Palaiseau, France, October 12, 2012

Introduction: objective

Scattering by a negative material in electromagnetism in 3D in time-harmonic regime (at a given frequency):



Positive material

$\varepsilon > 0$

and $\mu > 0$

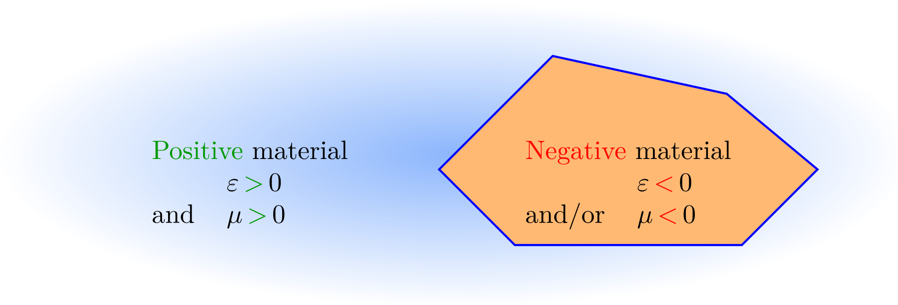
Negative material

$\varepsilon < 0$

and/or $\mu < 0$

Introduction: objective

Scattering by a negative material in electromagnetism in 3D in time-harmonic regime (at a given frequency):



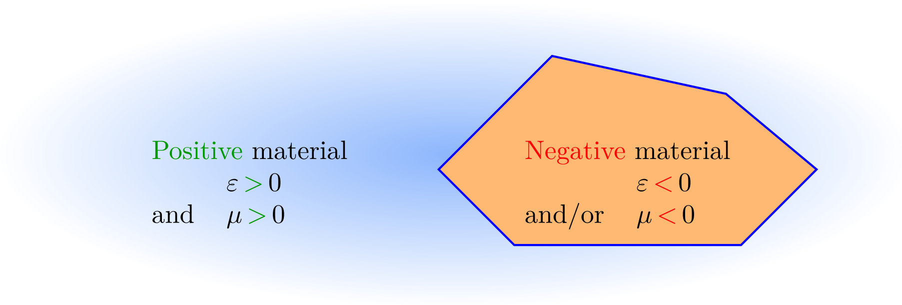
Positive material
$\varepsilon > 0$
and $\mu > 0$

Negative material
$\varepsilon < 0$
and/or $\mu < 0$

Do such negative materials occur in practice?

Introduction: objective

Scattering by a negative material in electromagnetism in 3D in time-harmonic regime (at a given frequency):



Positive material
$\varepsilon > 0$
and $\mu > 0$

Negative material
$\varepsilon < 0$
and/or $\mu < 0$

Do such negative materials occur in practice?

- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.

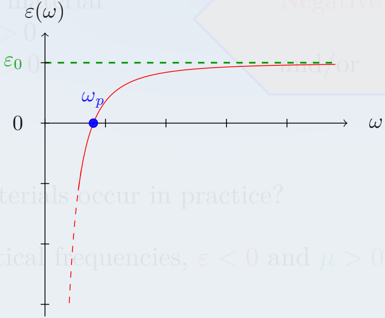
Introduction: objective

Scattering by a negative material in electromagnetism in 3D in time-harmonic regime (at a given frequency):

Drude model for a metal (high frequency):

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{{\omega_p}^2}{\omega^2} \right),$$

where ω_p is the plasma frequency.



Positive material
$\varepsilon > 0$
and $\mu > 0$

Negative material
$\varepsilon < 0$
and/or $\mu < 0$

Do such negative materials occur in practice?

- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.

Introduction: objective

Scattering by a negative material in electromagnetism in 3D in time-harmonic regime (at a given frequency):

Positive material	Negative material
$\varepsilon > 0$	$\varepsilon < 0$
and $\mu > 0$	and/or $\mu < 0$

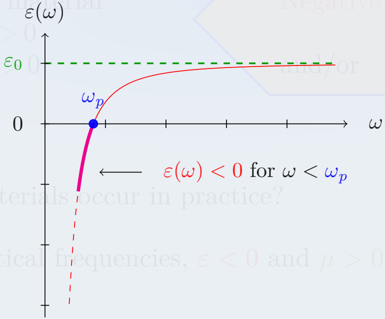
Do such negative materials occur in practice?

- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.

Drude model for a metal (high frequency):

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{{\omega_p}^2}{\omega^2}\right),$$

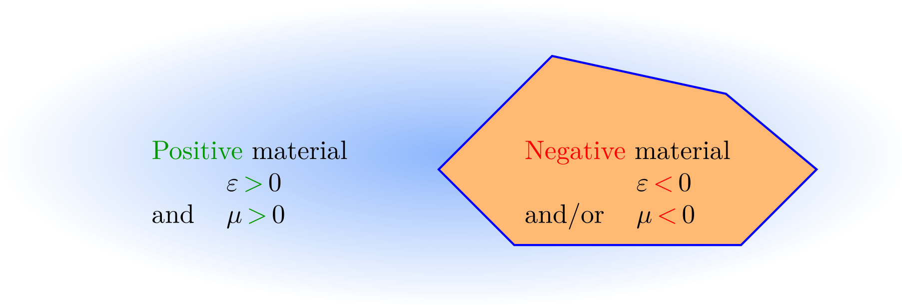
where ω_p is the plasma frequency.



$\varepsilon(\omega) < 0$ for $\omega < \omega_p$

Introduction: objective

Scattering by a negative material in electromagnetism in 3D in time-harmonic regime (at a given frequency):



Positive material
$\varepsilon > 0$
and $\mu > 0$

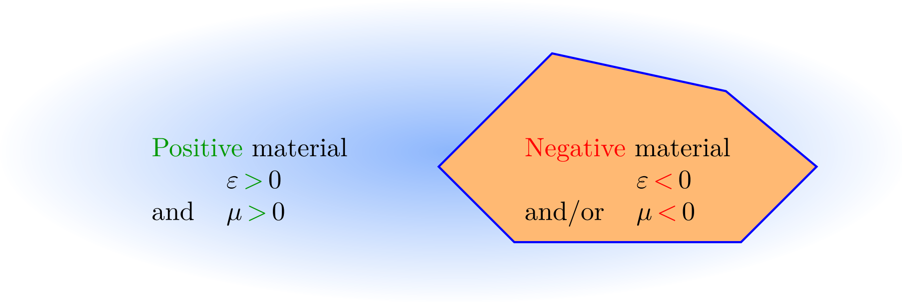
Negative material
$\varepsilon < 0$
and/or $\mu < 0$

Do such negative materials occur in practice?

- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.

Introduction: objective

Scattering by a negative material in electromagnetism in 3D in time-harmonic regime (at a given frequency):



Positive material
$\varepsilon > 0$
and $\mu > 0$

Negative material
$\varepsilon < 0$
and/or $\mu < 0$

Do such negative materials occur in practice?

- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.
- Recently, artificial metamaterials have been realized which can be modelled (at some frequency of interest) by $\varepsilon < 0$ and $\mu < 0$.

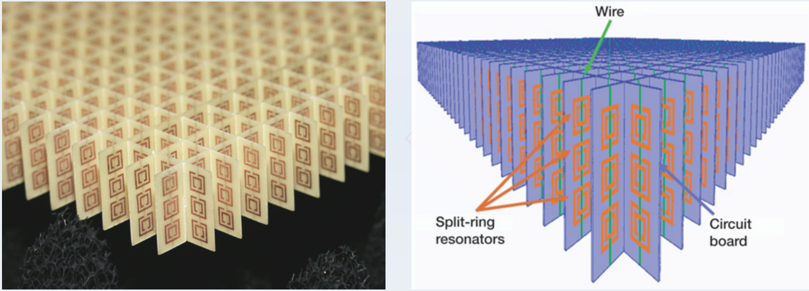
Introduction: objective

Scattering by a negative material in electromagnetism in 3D in time-harmonic regime (at a given frequency):

Do such negative materials occur in practice?

- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.
- Recently, artificial metamaterials have been realized which can be modelled (at some frequency of interest) by $\varepsilon < 0$ and $\mu < 0$.

Zoom on a metamaterial: practical realizations of metamaterials are achieved by a periodic assembly of small resonators.

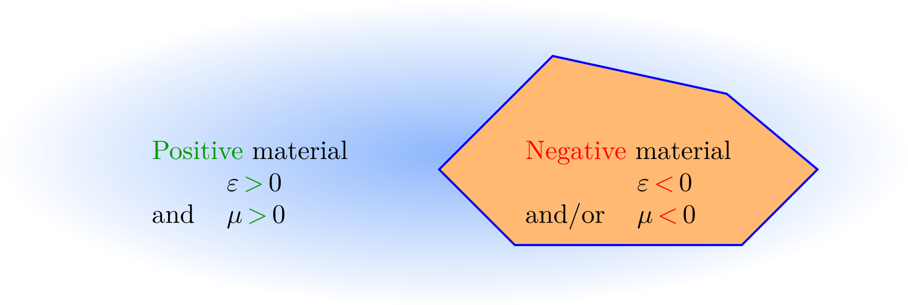


EXAMPLE OF METAMATERIAL (NASA)

Mathematical justification of the homogenized model (`Bouchitté, Bourel, Felbacq 09`).

Introduction: objective

Scattering by a negative material in electromagnetism in 3D in time-harmonic regime (at a given frequency):



Positive material
$\varepsilon > 0$
and $\mu > 0$

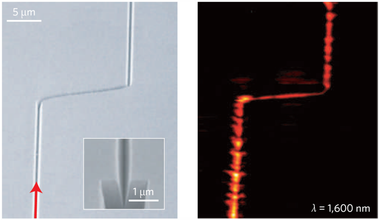
Negative material
$\varepsilon < 0$
and/or $\mu < 0$

Do such negative materials occur in practice?

- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.
- Recently, artificial metamaterials have been realized which can be modelled (at some frequency of interest) by $\varepsilon < 0$ and $\mu < 0$.

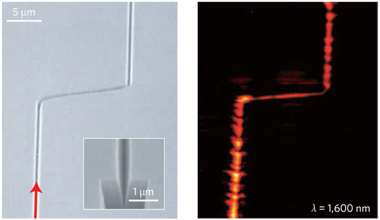
Introduction: applications

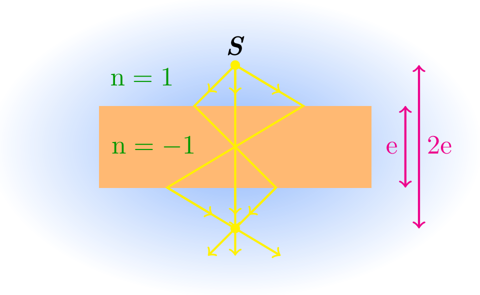
- Surface Plasmons Polaritons that propagate at the interface between a metal and a dielectric can help reducing the size of computer chips.



Introduction: applications

- Surface Plasmons Polaritons that propagate at the interface between a metal and a dielectric can help reducing the size of computer chips.

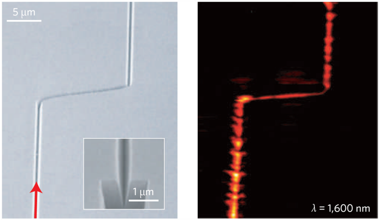


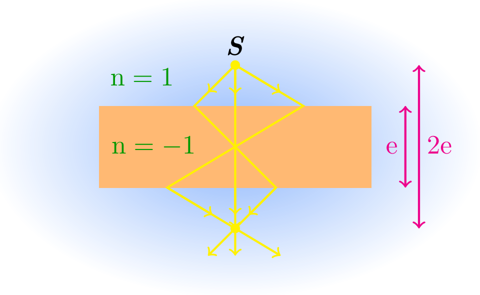


- The negative refraction at the interface metamaterial/dielectric could allow the realization of perfect lenses (`Pendry 00`), photonic traps ...

Introduction: applications

- Surface Plasmons Polaritons that propagate at the interface between a metal and a dielectric can help reducing the size of computer chips.



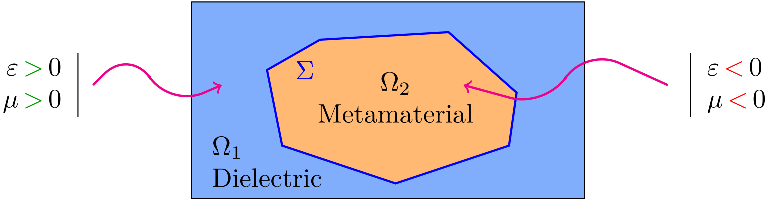


- The negative refraction at the interface metamaterial/dielectric could allow the realization of perfect lenses (`Pendry 00`), photonic traps ...

Interfaces between negative materials and dielectrics occur in all (exciting) applications...

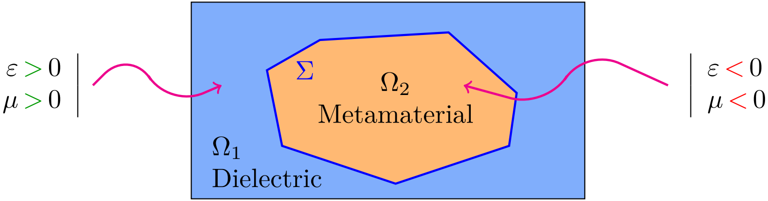
Introduction: in this talk

Problem set in a bounded domain $\Omega \subset \mathbb{R}^3$:



Introduction: in this talk

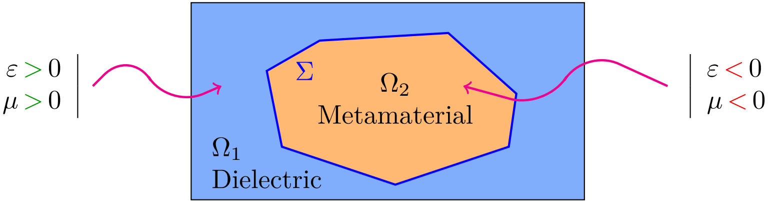
Problem set in a bounded domain $\Omega \subset \mathbb{R}^3$:



- Unusual transmission problem because the sign of the coefficients ε and μ changes through the interface Σ.

Introduction: in this talk

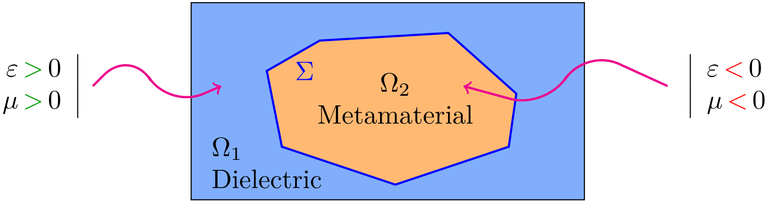
Problem set in a bounded domain $\Omega \subset \mathbb{R}^3$:



- Unusual transmission problem because the sign of the coefficients ε and μ changes through the interface Σ.
- Well-posedness is recovered by the presence of dissipation: $\Im m\, \varepsilon,\, \mu > 0$.

Introduction: in this talk

Problem set in a bounded domain $\Omega \subset \mathbb{R}^3$:

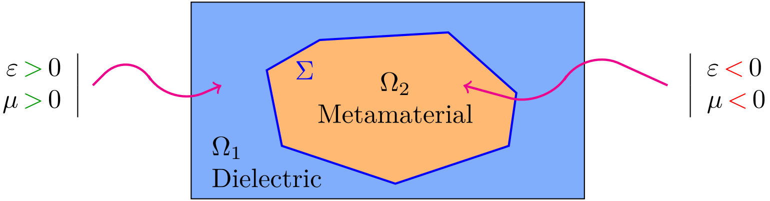


- Unusual transmission problem because the sign of the coefficients ε and μ changes through the interface Σ.

- Well-posedness is recovered by the presence of dissipation: $\Im m\, \varepsilon,\, \mu > 0$. But interesting phenomena occur for almost dissipationless materials.

Introduction: in this talk

Problem set in a bounded domain $\Omega \subset \mathbb{R}^3$:

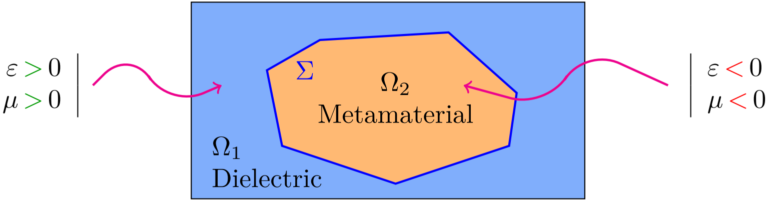


- Unusual transmission problem because the sign of the coefficients ε and μ changes through the interface Σ.
- Well-posedness is recovered by the presence of dissipation: $\Im m\, \varepsilon,\, \mu > 0$. But interesting phenomena occur for almost dissipationless materials.

The relevant question is then: what happens if dissipation is neglected ?

Introduction: in this talk

Problem set in a bounded domain $\Omega \subset \mathbb{R}^3$:



- Unusual transmission problem because the sign of the coefficients ε and μ changes through the interface Σ.

- Well-posedness is recovered by the presence of dissipation: $\Im m\, \varepsilon,\, \mu > 0$. But interesting phenomena occur for almost dissipationless materials.

The relevant question is then: what happens if dissipation is neglected ?

?

- Does well-posedness still hold?
- What is the appropriate functional framework?
- What about the convergence of approximation methods?

Outline of the talk

1. The coerciveness issue for the scalar case

> We develop a `T`-coercivity method based on geometrical transformations to study $\mathbf{div}(\boldsymbol{\mu}^{-1}\boldsymbol{\nabla}\cdot) : \mathbf{H}^1_0(\boldsymbol{\Omega}) \to \mathbf{H}^{-1}(\boldsymbol{\Omega})$ (improvement over `Bonnet-Ben Dhia` *`et al.`* `10`, `Zwölf 08`).

Outline of the talk

1. The coerciveness issue for the scalar case

We develop a T-coercivity method based on geometrical transformations to study $\mathbf{div}(\mu^{-1}\nabla\cdot) : \mathbf{H}^1_0(\Omega) \rightarrow \mathbf{H}^{-1}(\Omega)$ (improvement over `Bonnet-Ben Dhia` *`et al.`* `10`, `Zwölf 08`).

2. A new functional framework in the critical interval

We propose a new functional framework when $\mathbf{div}(\mu^{-1}\nabla\cdot) : \mathbf{X} \rightarrow \mathbf{Y}$ is not Fredholm for $\mathrm{X} = \mathrm{H}^1_0(\Omega)$ and $\mathrm{Y} = \mathrm{H}^{-1}(\Omega)$ (extension of `Dauge, Texier 97`, `Ramdani 99`).

Outline of the talk

1. The coerciveness issue for the scalar case

   We develop a T-coercivity method based on geometrical transformations to study $\mathbf{div}(\boldsymbol{\mu}^{-1}\boldsymbol{\nabla}\cdot) : \mathbf{H}^1_0(\boldsymbol{\Omega}) \to \mathbf{H}^{-1}(\boldsymbol{\Omega})$ (improvement over `Bonnet-Ben Dhia et al. 10`, `Zwölf 08`).

2. A new functional framework in the critical interval

   We propose a new functional framework when $\mathbf{div}(\boldsymbol{\mu}^{-1}\boldsymbol{\nabla}\cdot) : \mathbf{X} \to \mathbf{Y}$ is not Fredholm for $\mathrm{X} = \mathrm{H}^1_0(\Omega)$ and $\mathrm{Y} = \mathrm{H}^{-1}(\Omega)$ (extension of `Dauge, Texier 97`, `Ramdani 99`).

3. Study of Maxwell's equations

   We develop a T-coercivity method based on potentials to study $\mathbf{curl}\,(\boldsymbol{\varepsilon}^{-1}\mathbf{curl}\,\cdot) : \mathbf{V}_{\boldsymbol{T}}(\boldsymbol{\mu};\,\boldsymbol{\Omega}) \to \mathbf{V}_{\boldsymbol{T}}(\boldsymbol{\mu};\,\boldsymbol{\Omega})^*$.

Outline of the talk

1. The coerciveness issue for the scalar case

We develop a T-coercivity method based on geometrical transformations to study $\mathbf{div}(\mu^{-1}\nabla\cdot) : \mathbf{H}^1_0(\Omega) \to \mathbf{H}^{-1}(\Omega)$ (improvement over Bonnet-Ben Dhia *et al.* 10, Zwölf 08).

2. A new functional framework in the critical interval

We propose a new functional framework when $\mathbf{div}(\mu^{-1}\nabla\cdot) : \mathbf{X} \to \mathbf{Y}$ is not Fredholm for $\mathrm{X} = \mathrm{H}^1_0(\Omega)$ and $\mathrm{Y} = \mathrm{H}^{-1}(\Omega)$ (extension of Dauge, Texier 97, Ramdani 99).

3. Study of Maxwell's equations

We develop a T-coercivity method based on potentials to study $\mathbf{curl}\,(\varepsilon^{-1}\mathbf{curl}\,\cdot) : \mathbf{V}_T(\mu;\,\Omega) \to \mathbf{V}_T(\mu;\,\Omega)^*$.

4. The T-coercivity method for the Interior Transmission Problem

We study $\boldsymbol{\Delta}(\sigma\boldsymbol{\Delta}\cdot) : \mathbf{H}^2_0(\Omega) \to \mathbf{H}^{-2}(\Omega)$.

1. The coerciveness issue for the scalar case

2. A new functional framework in the critical interval

3. Study of Maxwell's equations

4. The `T`-coercivity method for the Interior Transmission Problem

A scalar model problem

Problem for E_z in 2D in case of an invariance with respect to z:

$$\left| \begin{array}{l} \text{Find } E_z \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \mathrm{div}(\mu^{-1}\,\nabla E_z) + \omega^2 \varepsilon E_z = -f \quad \text{ in } \Omega. \end{array} \right.$$

A scalar model problem

Problem for E_z in 2D in case of an invariance with respect to z:

$$\left|\begin{array}{l}\text{Find } E_z \in \mathrm{H}^1_0(\Omega) \text{ such that:}\\ \operatorname{div}(\mu^{-1}\,\nabla E_z) + \omega^2 \varepsilon E_z = -f \qquad \text{in } \Omega.\end{array}\right.$$

- $\mathrm{H}^1_0(\Omega) = \{v \in \mathrm{L}^2(\Omega) \,|\, \nabla v \in \mathrm{L}^2(\Omega);\ v|_{\partial\Omega} = 0\}$
- f is the source term in $\mathrm{H}^{-1}(\Omega)$

A scalar model problem

Problem for E_z in 2D in case of an invariance with respect to z:

$$\left| \begin{array}{l} \text{Find } E_z \in \mathrm{H}_0^1(\Omega) \text{ such that:} \\ \operatorname{div}(\mu^{-1}\,\nabla E_z) + \omega^2 \varepsilon E_z = -f \quad \text{ in } \Omega. \end{array} \right.$$

- $\mathrm{H}_0^1(\Omega) = \{v \in \mathrm{L}^2(\Omega) \,|\, \nabla v \in \mathrm{L}^2(\Omega);\; v|_{\partial\Omega} = 0\}$
- f is the source term in $\mathrm{H}^{-1}(\Omega)$

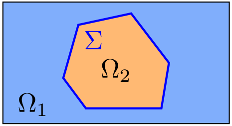
Since $\mathrm{H}_0^1(\Omega) \subset\subset \mathrm{L}^2(\Omega)$, we focus on the principal part.

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}_0^1(\Omega) \text{ s.t.:} \\ \operatorname{div}(\mu^{-1}\,\nabla u) = -f \text{ in } \Omega. \end{array} \right.$$

A scalar model problem

Problem for E_z in 2D in case of an invariance with respect to z:

$$\left|\begin{array}{l}\text{Find } E_z \in \mathrm{H}^1_0(\Omega) \text{ such that:}\\ \operatorname{div}(\mu^{-1}\nabla E_z) + \omega^2 \varepsilon E_z = -f \quad \text{in } \Omega.\end{array}\right.$$



- $\mathrm{H}^1_0(\Omega) = \{v \in \mathrm{L}^2(\Omega)\,|\,\nabla v \in \mathrm{L}^2(\Omega);\ v|_{\partial\Omega} = 0\}$
- f is the source term in $\mathrm{H}^{-1}(\Omega)$

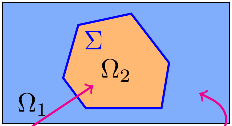
Since $\mathrm{H}^1_0(\Omega) \subset\subset \mathrm{L}^2(\Omega)$, we focus on the principal part.

$$(\mathscr{P})\ \left|\begin{array}{l}\text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:}\\ \operatorname{div}(\mu^{-1}\nabla u) = -f \text{ in } \Omega.\end{array}\right.$$

A scalar model problem

Problem for E_z in 2D in case of an invariance with respect to z:

$$\left| \begin{array}{l} \text{Find } E_z \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \mathrm{div}(\mu^{-1}\,\nabla E_z) + \omega^2 \varepsilon E_z = -f \quad \text{ in } \Omega. \end{array} \right.$$



$\mu_1 = \mu|_{\Omega_1} > 0$
$\mu_2 = \mu|_{\Omega_2} < 0$
(constant)

- $\mathrm{H}^1_0(\Omega) = \{v \in \mathrm{L}^2(\Omega) \,|\, \nabla v \in \mathrm{L}^2(\Omega);\ v|_{\partial\Omega} = 0\}$
- f is the source term in $\mathrm{H}^{-1}(\Omega)$

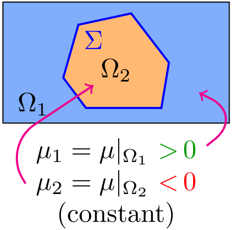
Since $\mathrm{H}^1_0(\Omega) \subset\subset \mathrm{L}^2(\Omega)$, we focus on the principal part.

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ \mathrm{div}(\mu^{-1}\,\nabla u) = -f \text{ in } \Omega. \end{array} \right.$$

A scalar model problem

Problem for E_z in 2D in case of an invariance with respect to z:

$$\left| \begin{array}{l} \text{Find } E_z \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \operatorname{div}(\mu^{-1}\,\nabla E_z) + \omega^2 \varepsilon E_z = -f \quad \text{in } \Omega. \end{array} \right.$$



$\mu_1 = \mu|_{\Omega_1} > 0$
$\mu_2 = \mu|_{\Omega_2} < 0$
(constant)

- $\mathrm{H}^1_0(\Omega) = \{v \in \mathrm{L}^2(\Omega) \,|\, \nabla v \in \mathrm{L}^2(\Omega);\ v|_{\partial\Omega} = 0\}$
- f is the source term in $\mathrm{H}^{-1}(\Omega)$

Since $\mathrm{H}^1_0(\Omega) \subset\subset \mathrm{L}^2(\Omega)$, we focus on the principal part.

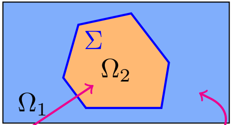
$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ \operatorname{div}(\mu^{-1}\,\nabla u) = -f \text{ in } \Omega. \end{array} \right. \quad \Leftrightarrow \quad (\mathscr{P}_V) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ a(u,v) = l(v),\ \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

with $a(u,v) = \displaystyle\int_\Omega \mu^{-1}\,\nabla u \cdot \nabla v$ and $l(v) = \langle f, v \rangle_\Omega$.

A scalar model problem

Problem for E_z in 2D in case of an invariance with respect to z:

$$\left|\begin{array}{l}\text{Find } E_z \in \mathrm{H}^1_0(\Omega) \text{ such that:}\\ \mathrm{div}(\mu^{-1}\,\nabla E_z) + \omega^2\varepsilon E_z = -f \quad \text{in } \Omega.\end{array}\right.$$



$\mu_1 = \mu|_{\Omega_1} > 0$
$\mu_2 = \mu|_{\Omega_2} < 0$
(constant)

- $\mathrm{H}^1_0(\Omega) = \{v \in \mathrm{L}^2(\Omega)\,|\,\nabla v \in \mathrm{L}^2(\Omega);\ v|_{\partial\Omega} = 0\}$
- f is the source term in $\mathrm{H}^{-1}(\Omega)$

Since $\mathrm{H}^1_0(\Omega) \subset\subset \mathrm{L}^2(\Omega)$, we focus on the principal part.

$$(\mathscr{P})\left|\begin{array}{l}\text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:}\\ \mathrm{div}(\mu^{-1}\,\nabla u) = -f \text{ in } \Omega.\end{array}\right. \quad\Leftrightarrow\quad (\mathscr{P}_V)\left|\begin{array}{l}\text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:}\\ a(u,v) = l(v),\ \forall v \in \mathrm{H}^1_0(\Omega).\end{array}\right.$$

with $a(u,v) = \displaystyle\int_\Omega \mu^{-1}\,\nabla u\cdot\nabla v$ and $l(v) = \langle f, v\rangle_\Omega$.

DEFINITION. We will say that the problem $(\mathscr{P})$ is well-posed if the operator $A = \mathrm{div}\,(\mu^{-1}\nabla\cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$.

Mathematical difficulty

- Classical case $\mu > 0$ everywhere:

$$a(u,u) = \int_\Omega \mu^{-1}\,|\nabla u|^2 \geq \min(\mu^{-1})\,\|u\|^2_{\mathrm{H}^1_0(\Omega)} \qquad \text{coercivity}$$

Lax-Milgram theorem ⇒ $(\mathscr{P})$ well-posed.

Mathematical difficulty

- Classical case $\mu > 0$ everywhere:

$$a(u,u) = \int_\Omega \mu^{-1} |\nabla u|^2 \geq \min(\mu^{-1}) \|u\|^2_{\mathrm{H}^1_0(\Omega)}$$

coercivity

Lax-Milgram theorem ⇒ $(\mathscr{P})$ well-posed.

VS.

- The case μ changes sign:

$$\xcancel{a(u,u) = \int_\Omega \mu^{-1} |\nabla u|^2 \geq C \|u\|^2_{\mathrm{H}^1_0(\Omega)}}$$

loss of coercivity

Mathematical difficulty

- Classical case $\mu > 0$ everywhere:

$$a(u,u) = \int_\Omega \mu^{-1} |\nabla u|^2 \geq \min(\mu^{-1}) \, \|u\|^2_{H^1_0(\Omega)} \qquad \text{coercivity}$$

Lax-Milgram theorem ⇒ $(\mathscr{P})$ well-posed.

VS.

- The case μ changes sign:

$$\xcancel{a(u,u) = \int_\Omega \mu^{-1} |\nabla u|^2 \geq C \, \|u\|^2_{H^1_0(\Omega)}} \qquad \text{loss of coercivity}$$

  - When $\mu_2 = -\mu_1$, $(\mathscr{P})$ is always ill-posed (`Costabel-Stephan 85`). For a symmetric domain (w.r.t. Σ) we can build a kernel of infinite dimension.

Idea of the `T`-coercivity 1/2

Let `T` be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$$(\mathscr{P}) \Leftrightarrow (\mathscr{P}_V) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(u,v) = l(v),\ \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Idea of the `T`-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$$(\mathscr{P}) \Leftrightarrow (\mathscr{P}_V) \Leftrightarrow (\mathscr{P}^{\mathtt{T}}_V) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(u, \mathtt{T}v) = l(\mathtt{T}v),\ \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Idea of the $\mathtt{T}$-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}_0^1(\Omega)$.

$$(\mathscr{P}) \Leftrightarrow (\mathscr{P}_V) \Leftrightarrow (\mathscr{P}_V^{\mathtt{T}}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}_0^1(\Omega) \text{ such that:} \\ a(u, \mathtt{T}v) = l(\mathtt{T}v), \ \forall v \in \mathrm{H}_0^1(\Omega). \end{array} \right.$$

Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \mu^{-1} \nabla u \cdot \nabla(\mathtt{T}u) \geq C \|u\|^2_{\mathrm{H}_0^1(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

Idea of the T-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$$(\mathscr{P}) \Leftrightarrow \ (\mathscr{P}_V) \ \Leftrightarrow (\mathscr{P}_V^{\mathtt{T}}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(u, \mathtt{T}v) = l(\mathtt{T}v),\ \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \mu^{-1}\, \nabla u \cdot \nabla(\mathtt{T}u) \geq C\, \|u\|^2_{\mathrm{H}^1_0(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

---

① Define $\mathtt{T}_1 u = \left| \begin{array}{ll} u_1 & \text{in } \Omega_1 \\ -u_2 + ... & \text{in } \Omega_2 \end{array} \right.$

Idea of the $\mathtt{T}$-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$$(\mathscr{P}) \Leftrightarrow (\mathscr{P}_V) \Leftrightarrow (\mathscr{P}_V^{\mathtt{T}}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(u, \mathtt{T}v) = l(\mathtt{T}v),\ \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

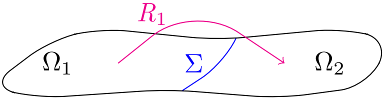
Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \mu^{-1}\, \nabla u \cdot \nabla(\mathtt{T}u) \geq C\, \|u\|^2_{\mathrm{H}^1_0(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

---

① Define $\mathtt{T}_1 u = \left| \begin{array}{ll} u_1 & \text{in } \Omega_1 \\ -u_2 + 2R_1 u_1 & \text{in } \Omega_2 \end{array} \right.$, with

R_1 transfer/extension operator



Idea of the T-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$$(\mathscr{P}) \Leftrightarrow (\mathscr{P}_V) \Leftrightarrow (\mathscr{P}_V^{\mathtt{T}}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(u, \mathtt{T}v) = l(\mathtt{T}v),\ \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

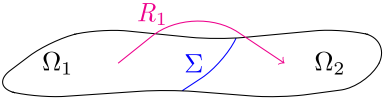
Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \mu^{-1}\,\nabla u \cdot \nabla(\mathtt{T}u) \geq C\,\|u\|^2_{\mathrm{H}^1_0(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

---

❶ Define $\mathtt{T}_1 u = \left| \begin{array}{ll} u_1 & \text{in } \Omega_1 \\ -u_2 + 2R_1 u_1 & \text{in } \Omega_2 \end{array} \right.$, with

R_1 transfer/extension operator continuous from Ω_1 to Ω_2



$$\left| \begin{array}{lcll} R_1 u_1 & = & u_1 & \text{on } \Sigma \\ R_1 u_1 & = & 0 & \text{on } \partial\Omega_2 \setminus \Sigma \end{array} \right.$$

Idea of the $\mathtt{T}$-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

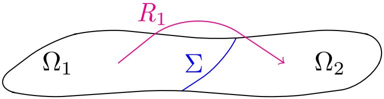
$$(\mathscr{P}) \Leftrightarrow \ (\mathscr{P}_V) \ \Leftrightarrow (\mathscr{P}_V^{\mathtt{T}}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(u, \mathtt{T}v) = l(\mathtt{T}v),\ \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \mu^{-1}\,\nabla u \cdot \nabla(\mathtt{T}u) \geq C\,\|u\|^2_{\mathrm{H}^1_0(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

① Define $\mathtt{T}_1 u = \left| \begin{array}{ll} u_1 & \text{in } \Omega_1 \\ -u_2 + 2R_1 u_1 & \text{in } \Omega_2 \end{array} \right.$, with

R_1 transfer/extension operator continuous from Ω_1 to Ω_2



$$\left| \begin{array}{lcll} R_1 u_1 & = & u_1 & \text{on } \Sigma \\ R_1 u_1 & = & 0 & \text{on } \partial\Omega_2 \setminus \Sigma \end{array} \right.$$

On Σ, we have $-u_2 + 2R_1 u_1 = -u_2 + 2u_1 = u_1 \Rightarrow \mathtt{T}_1 u \in \mathrm{H}^1_0(\Omega)$.

Idea of the T-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

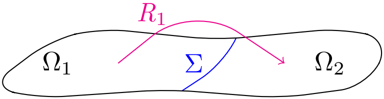
$$(\mathscr{P}) \Leftrightarrow \ (\mathscr{P}_V) \ \Leftrightarrow (\mathscr{P}_V^{\mathtt{T}}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(u, \mathtt{T}v) = l(\mathtt{T}v), \ \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \mu^{-1} \nabla u \cdot \nabla(\mathtt{T}u) \geq C \, \|u\|^2_{\mathrm{H}^1_0(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

① Define $\mathtt{T}_1 u = \left| \begin{array}{ll} u_1 & \text{in } \Omega_1 \\ -u_2 + 2R_1 u_1 & \text{in } \Omega_2 \end{array} \right.$, with

R_1 transfer/extension operator continuous from Ω_1 to Ω_2



$$\left| \begin{array}{lll} R_1 u_1 & = u_1 & \text{on } \Sigma \\ R_1 u_1 & = 0 & \text{on } \partial\Omega_2 \setminus \Sigma \end{array} \right.$$

② $\mathtt{T}_1 \circ \mathtt{T}_1 = Id$ so $\mathtt{T}_1$ is an isomorphism of $\mathrm{H}^1_0(\Omega)$

Idea of the $\mathtt{T}$-coercivity 2/2

❸ One has $a(u, \mathtt{T}_1 u) = \displaystyle\int_\Omega |\mu|^{-1} |\nabla u|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla u \cdot \nabla (R_1\, u_1)$

Idea of the T-coercivity 2/2

❸ One has $a(u, \mathtt{T}_1 u) = \int_\Omega |\mu|^{-1} |\nabla u|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla u \cdot \nabla(R_1 u_1)$

Young's inequality ⇒ a is T-coercive when $|\mu_2| > \|R_1\|^2 \mu_1$.

Idea of the T-coercivity 2/2

❸ One has $a(u, \mathtt{T}_1 u) = \displaystyle\int_\Omega |\mu|^{-1} |\nabla u|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla u \cdot \nabla (R_1\, u_1)$

Young's inequality $\Rightarrow$ a is T-coercive when $|\mu_2| > \|R_1\|^2\, \mu_1$.

❹ Working with $\mathtt{T}_2 u = \left| \begin{array}{ll} u_1 - 2 R_2 u_2 & \text{in } \Omega_1 \\ -u_2 & \text{in } \Omega_2 \end{array} \right.$, where $R_2 : \Omega_2 \to \Omega_1$, one proves that a is T-coercive when $\mu_1 > \|R_2\|^2\, |\mu_2|$.

Idea of the T-coercivity 2/2

③ One has $a(u, \mathtt{T}_1 u) = \int_\Omega |\mu|^{-1} |\nabla u|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla u \cdot \nabla(R_1\, u_1)$

Young's inequality ⇒ a is T-coercive when $|\mu_2| > \|R_1\|^2\, \mu_1$.

④ Working with $\mathtt{T}_2 u = \left| \begin{array}{ll} u_1 - 2R_2 u_2 & \text{in } \Omega_1 \\ -u_2 & \text{in } \Omega_2 \end{array} \right.$, where $R_2 : \Omega_2 \to \Omega_1$, one proves that a is T-coercive when $\mu_1 > \|R_2\|^2\, |\mu_2|$.

⑤ Conclusion:

THEOREM. If the contrast $\kappa_\mu = \mu_2/\mu_1 \notin [-\|R_1\|^2; -1/\|R_2\|^2]$, then the operator $\operatorname{div}(\mu^{-1} \nabla \cdot)$ is an isomorphism from $\mathrm{H}_0^1(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$.

Idea of the T-coercivity 2/2

③ One has $a(u, \mathtt{T}_1 u) = \displaystyle\int_\Omega |\mu|^{-1} |\nabla u|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla u \cdot \nabla(R_1 u_1)$

Young's inequality ⇒ a is T-coercive when $|\mu_2| > \|R_1\|^2 \mu_1$.

④ Working with $\mathtt{T}_2 u = \left| \begin{array}{ll} u_1 - 2R_2 u_2 & \text{in } \Omega_1 \\ -u_2 & \text{in } \Omega_2 \end{array} \right.$, where $R_2 : \Omega_2 \to \Omega_1$, one proves that a is T-coercive when $\mu_1 > \|R_2\|^2 |\mu_2|$.

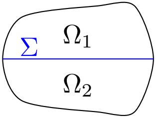
⑤ Conclusion:

The interval depends on the norms of the transfer operators

THEOREM. If the contrast $\kappa_\mu = \mu_2/\mu_1 \notin [-\|R_1\|^2; -1/\|R_2\|^2]$ then the operator $\operatorname{div}(\mu^{-1} \nabla \cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$.

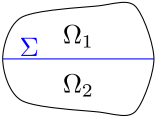
Choice of R_1,R_2?

- A simple case: symmetric domain



Choice of R_1,R_2?

- A simple case: symmetric domain



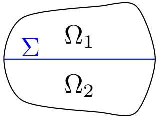
$$R_1 = R_2 = S_{\Sigma}$$

so that $\|R_1\| = \|R_2\| = 1$

$(\mathscr{P})$ well-posed $\Leftrightarrow \kappa_\mu \neq -1$

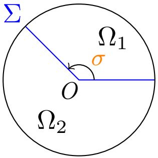
Choice of R_1,R_2?

- A simple case: symmetric domain



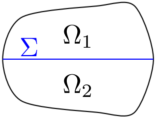
$$R_1 = R_2 = S_\Sigma$$
so that $\|R_1\| = \|R_2\| = 1$
$(\mathscr{P})$ well-posed $\Leftrightarrow \kappa_\mu \neq -1$

- Interface with a 2D corner



Choice of R_1,R_2?

- A simple case: symmetric domain

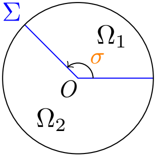


$$R_1 = R_2 = S_{\Sigma}$$

so that $\|R_1\| = \|R_2\| = 1$

$(\mathscr{P})$ well-posed $\Leftrightarrow \kappa_\mu \neq -1$

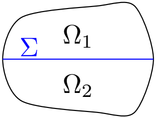
- Interface with a 2D corner



Action of R_1:

Choice of R_1,R_2?

- A simple case: symmetric domain

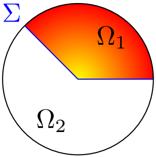


$$R_1 = R_2 = S_{\Sigma}$$

so that $\|R_1\| = \|R_2\| = 1$

$(\mathscr{P})$ well-posed $\Leftrightarrow$ $\kappa_\mu \neq -1$

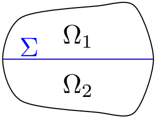
- Interface with a 2D corner



Action of R_1:

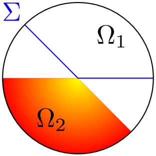
Choice of R_1,R_2?

- A simple case: symmetric domain



$$R_1 = R_2 = S_\Sigma$$
$$\text{so that } \|R_1\| = \|R_2\| = 1$$
$$(\mathscr{P}) \text{ well-posed} \Leftrightarrow \kappa_\mu \neq -1$$

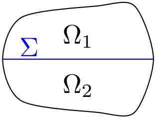
- Interface with a 2D corner



Action of R_1: symmetry w.r.t θ

Choice of R_1,R_2?

- A simple case: symmetric domain

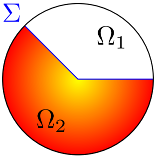


$$R_1 = R_2 = S_\Sigma$$

so that $\|R_1\| = \|R_2\| = 1$

$(\mathscr{P})$ well-posed $\Leftrightarrow$ $\kappa_\mu \neq -1$

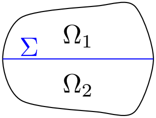
- Interface with a 2D corner



Action of R_1: symmetry + dilatation w.r.t θ

Choice of R_1,R_2?

- A simple case: symmetric domain

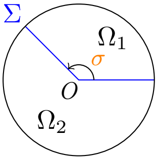


$$R_1 = R_2 = S_\Sigma$$

so that $\|R_1\| = \|R_2\| = 1$

$(\mathscr{P})$ well-posed $\Leftrightarrow$ $\kappa_\mu \neq -1$

- Interface with a 2D corner

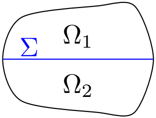


Action of R_1: symmetry + dilatation w.r.t θ

$$\|R_1\|^2 \qquad = \mathcal{R}_\sigma := (2\pi - \sigma)/\sigma$$

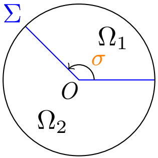
Choice of R_1,R_2?

- A simple case: symmetric domain



$$R_1 = R_2 = S_\Sigma$$
so that $\|R_1\| = \|R_2\| = 1$
$(\mathscr{P})$ well-posed $\Leftrightarrow$ $\kappa_\mu \neq -1$

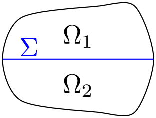
- Interface with a 2D corner



Action of R_1: symmetry + dilatation w.r.t θ
Action of R_2: symmetry + dilatation w.r.t θ
$$\|R_1\|^2 = \|R_2\|^2 = \mathcal{R}_\sigma := (2\pi - \sigma)/\sigma$$

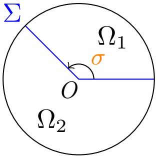
Choice of R_1,R_2?

- A simple case: symmetric domain



$$R_1 = R_2 = S_{\Sigma}$$
so that $\|R_1\| = \|R_2\| = 1$
$(\mathscr{P})$ well-posed $\Leftrightarrow \kappa_\mu \neq -1$

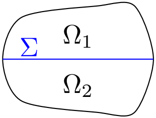
- Interface with a 2D corner



Action of R_1: symmetry + dilatation w.r.t θ
Action of R_2: symmetry + dilatation w.r.t θ
$$\|R_1\|^2 = \|R_2\|^2 = \mathcal{R}_\sigma := (2\pi - \sigma)/\sigma$$
$(\mathscr{P})$ well-posed $\Leftarrow \kappa_\mu \notin [-\mathcal{R}_\sigma; -1/\mathcal{R}_\sigma]$

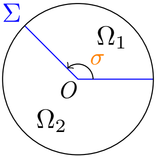
Choice of R_1,R_2?

- A simple case: symmetric domain



$$R_1 = R_2 = S_\Sigma$$
so that $\|R_1\| = \|R_2\| = 1$
$(\mathscr{P})$ well-posed $\Leftrightarrow$ $\kappa_\mu \neq -1$

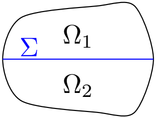
- Interface with a 2D corner



Action of R_1: symmetry + dilatation w.r.t θ
Action of R_2: symmetry + dilatation w.r.t θ
$\|R_1\|^2 = \|R_2\|^2 = \mathcal{R}_\sigma := (2\pi - \sigma)/\sigma$
$(\mathscr{P})$ well-posed $\Leftrightarrow$ $\kappa_\mu \notin [-\mathcal{R}_\sigma; -1/\mathcal{R}_\sigma]$

Choice of R_1,R_2?

- A simple case: symmetric domain

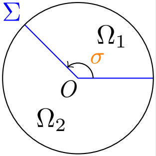


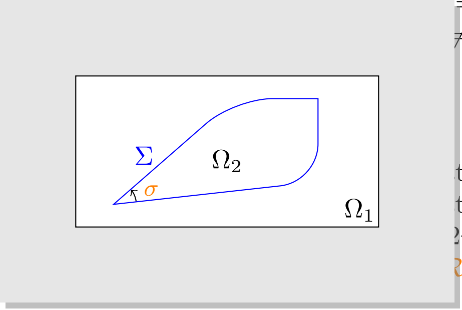
$$R_1 = R_2 = S_\Sigma$$

so that $\|R \| \quad \|R \| = 1$

$\neq -1$

- Interface with





tation w.r.t θ

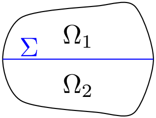
tation w.r.t θ

$2\pi - \sigma)/\sigma$

$\mathcal{R}_\sigma; -1/\mathcal{R}_\sigma]$

Choice of R_1,R_2?

- A simple case: symmetric domain

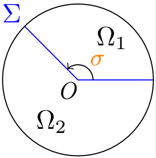


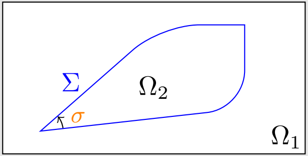
$$R_1 = R_2 = S_\Sigma$$

$= 1$

$\neq -1$

- Interface with





tation w.r.t θ

tation w.r.t θ

$2\pi - \sigma)/\sigma$

$\mathcal{R}_\sigma; -1/\mathcal{R}_\sigma]$

- By localization techniques, we prove

> **PROPOSITION.** ($\mathscr{P}$) is well-posed in the Fredholm sense for a curvilinear polygonal interface iff $\kappa_\mu \notin [-\mathcal{R}_\sigma; -1/\mathcal{R}_\sigma]$ where σ is the smallest angle.

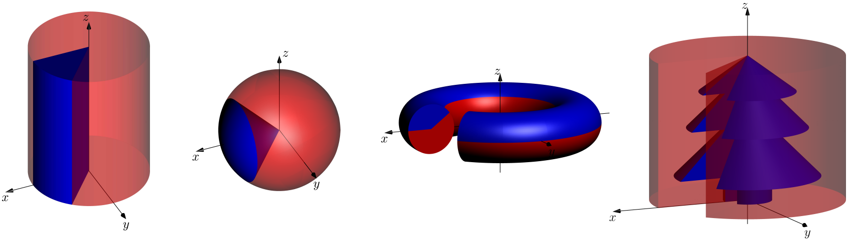
⇒When Σ is smooth, ($\mathscr{P}$) is well-posed in the Fredholm sense iff $\kappa_\mu \neq -1$.

Extensions for the scalar case

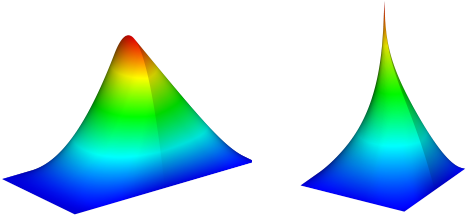
- The `T`-coercivity approach can be used to deal with non constant μ_1, μ_2 and with the Neumann problem.

Extensions for the scalar case

- The `T`-coercivity approach can be used to deal with non constant μ_1, μ_2 and with the Neumann problem.
- 3D geometries can be handled in the same way.

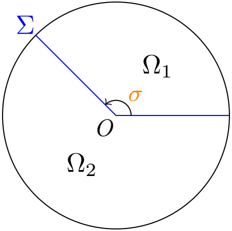


- The `T`-coercivity technique allows to justify convergence of standard finite element method for simple meshes (`Bonnet-Ben Dhia` *`et al.`* `10`, `Nicaise, Venel 11`, `Chesnel, Ciarlet 12`).



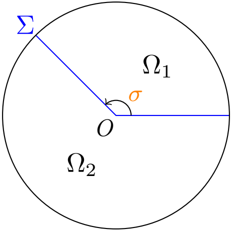
Transition: from variational methods to Fourier/Mellin techniques

❓ For the corner case, what happens when the contrast lies inside the criticial interval, *i.e.* when $\kappa_\mu \in [-\mathcal{R}_\sigma; -1/\mathcal{R}_\sigma]$???



Transition: from variational methods to Fourier/Mellin techniques

? For the corner case, what happens when the contrast lies inside the criticial interval, *i.e.* when $\kappa_\mu \in [-\mathcal{R}_\sigma; -1/\mathcal{R}_\sigma]$???



Idea: we will study precisely the regularity of the "solutions" using the Kondratiev's tools, *i.e.* the Fourier/Mellin transform (`Dauge, Texier 97`, `Nazarov, Plamenevsky 94`).

1. The coerciveness issue for the scalar case

2. A new functional framework in the critical interval
   ⇒ collaboration with `X. Claeys` (LJLL Paris VI).

3. Study of Maxwell's equations

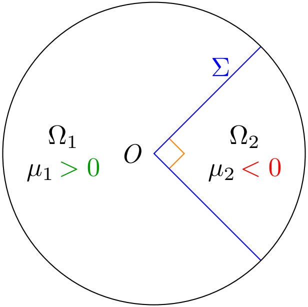
4. The `T`-coercivity method for the Interior Transmission Problem

Problem considered in this section

- We recall the problem under consideration

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ -\mathrm{div}(\mu^{-1}\nabla u) = f \quad \text{ in } \Omega. \end{array} \right.$$

- To simplify the presentation, we work on a particular configuration.

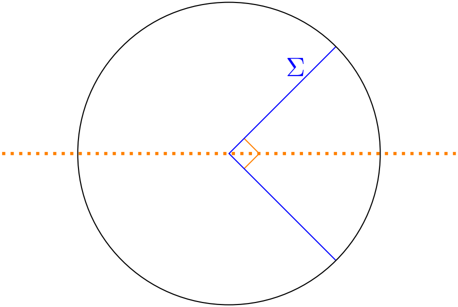


Problem considered in this section

- We recall the problem under consideration

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ -\mathrm{div}(\mu^{-1}\nabla u) = f \quad \text{ in } \Omega. \end{array} \right.$$

- To simplify the presentation, we work on a particular configuration.

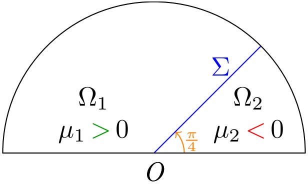


Problem considered in this section

- We recall the problem under consideration

$$(\mathscr{P})\ \left|\ \begin{array}{l}\text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:}\\ -\mathrm{div}(\mu^{-1}\nabla u) = f \quad \text{ in } \Omega.\end{array}\right.$$

- To simplify the presentation, we work on a particular configuration.

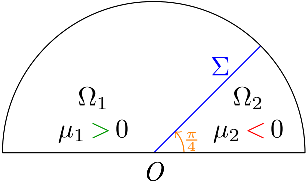


Problem considered in this section

- We recall the problem under considération

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ -\mathrm{div}(\mu^{-1}\nabla u) = f \quad \text{in } \Omega. \end{array} \right.$$

- To simplify the presentation, we work on a particular configuration.



- Using the variational method of the previous section, we prove the

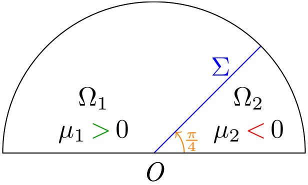
Proposition. The problem $(\mathscr{P})$ is well-posed as soon as the contrast $\kappa_\mu = \mu_2/\mu_1$ satisfies $\kappa_\mu \notin [-3; -1]$.

Problem considered in this section

- We recall the problem under consideration

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ -\mathrm{div}(\mu^{-1}\nabla u) = f \quad \text{ in } \Omega. \end{array} \right.$$

- To simplify the presentation, we work on a particular configuration.



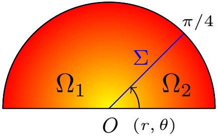
- Using the variational method of the previous section, we prove the

Proposition. The problem $(\mathscr{P})$ is well-posed as soon as the contrast $\kappa_\mu = \mu_2/\mu_1$ satisfies $\kappa_\mu \notin [-3; -1]$.

What happens when $\kappa_\mu \in [-3; -1)$?

Analogy with a waveguide problem

- Bounded sector Ω

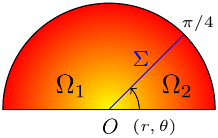


- Equation:

$$\underbrace{-\mathrm{div}(\mu^{-1}\,\nabla u)}_{-r^{-2}(\mu^{-1}(r\partial_r)^2+\partial_\theta\mu^{-1}\partial_\theta)u} \qquad = f$$

Analogy with a waveguide problem

- Bounded sector Ω



- Equation:

$$\underbrace{-\mathrm{div}(\mu^{-1}\,\nabla u)}_{-r^{-2}(\mu^{-1}(r\partial_r)^2+\partial_\theta\mu^{-1}\partial_\theta)u} \qquad = f$$

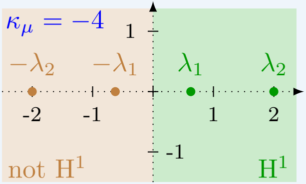
- Singularities in the sector

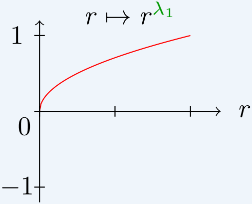
$s(r,\theta) = r^{\lambda}\varphi(\theta)$

Analogy with a waveguide problem

We compute the singularities $s(r,\theta) = r^{\lambda}\varphi(\theta)$ and we observe two cases:

- Outside the critical interval

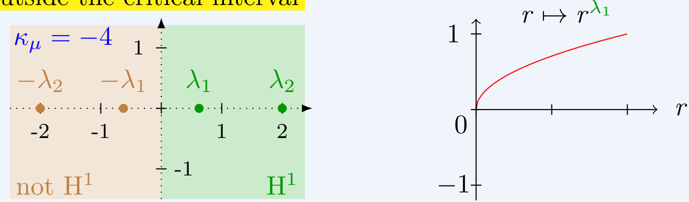




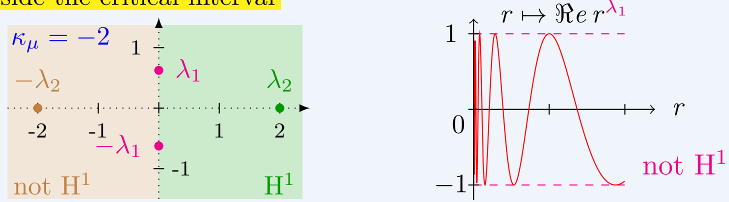
Analogy with a waveguide problem

We compute the singularities $s(r,\theta) = r^{\lambda}\varphi(\theta)$ and we observe two cases:

- **Outside the critical interval**



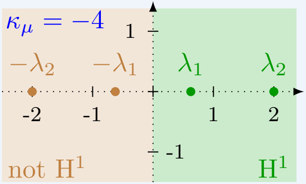
- **Inside the critical interval**

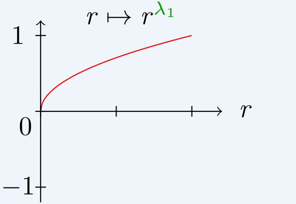


Analogy with a waveguide problem

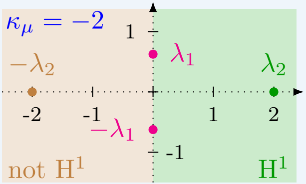
We compute the singularities $s(r,\theta) = r^{\lambda}\varphi(\theta)$ and we observe two cases:

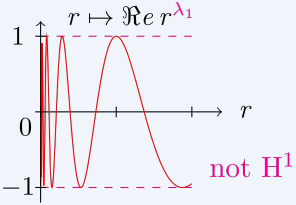
- Outside the critical interval





- Inside the critical interval

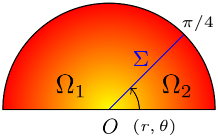




How to deal with the propagative singularities inside the critical interval?

Analogy with a waveguide problem

- Bounded sector Ω



- Equation:

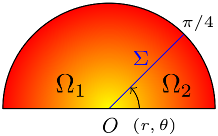
$$\underbrace{-\mathrm{div}(\mu^{-1}\,\nabla u)}_{-r^{-2}(\mu^{-1}(r\partial_r)^2+\partial_\theta\mu^{-1}\partial_\theta)u} \qquad = f$$

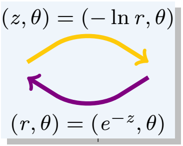
- Singularities in the sector

$s(r,\theta) = r^\lambda \varphi(\theta)$

Analogy with a waveguide problem

- Bounded sector Ω

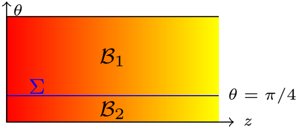




$(z,\theta) = (-\ln r, \theta)$

$(r,\theta) = (e^{-z}, \theta)$

- Half-strip $\mathcal{B}$



- Equation:

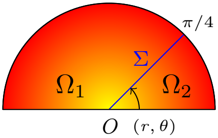
$$\underbrace{-\mathrm{div}(\mu^{-1}\,\nabla u)}_{-r^{-2}(\mu^{-1}(r\partial_r)^2+\partial_\theta\mu^{-1}\partial_\theta)u} = f$$

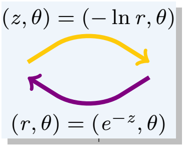
- Singularities in the sector

$s(r,\theta) = r^{\lambda}\varphi(\theta)$

Analogy with a waveguide problem

- Bounded sector Ω

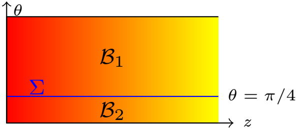




$(z,\theta) = (-\ln r, \theta)$

$(r,\theta) = (e^{-z}, \theta)$

- Half-strip $\mathcal{B}$



- Equation:

$$\underbrace{-\mathrm{div}(\mu^{-1}\nabla u)}_{-r^{-2}(\mu^{-1}(r\partial_r)^2+\partial_\theta\mu^{-1}\partial_\theta)u} = f$$

- Equation:

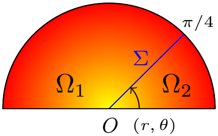
$$\underbrace{-\mathrm{div}(\mu^{-1}\nabla u)}_{-(\mu^{-1}\partial_z^2+\partial_\theta\mu^{-1}\partial_\theta)u} = e^{-2z} f$$

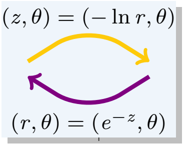
- Singularities in the sector

$s(r,\theta) = r^\lambda \varphi(\theta)$

Analogy with a waveguide problem

- Bounded sector Ω





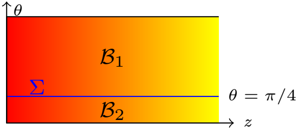
- Equation:

$$\underbrace{-\mathrm{div}(\mu^{-1}\,\nabla u)}_{-r^{-2}(\mu^{-1}(r\partial_r)^2+\partial_\theta\mu^{-1}\partial_\theta)u} = f$$

- Singularities in the sector

$$s(r,\theta) = r^{\lambda}\varphi(\theta)$$

- Half-strip $\mathcal{B}$



- Equation:

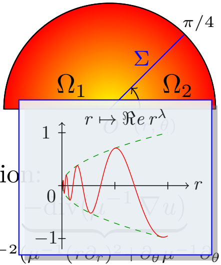
$$\underbrace{-\mathrm{div}(\mu^{-1}\nabla u)}_{-(\mu^{-1}\partial_z^2+\partial_\theta\mu^{-1}\partial_\theta)u} = e^{-2z} f$$

- Modes in the strip

$$m(z,\theta) = e^{-\lambda z}\varphi(\theta)$$

Analogy with a waveguide problem

- Bounded sector Ω

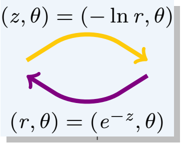


- Equation:

$$\underbrace{-\mathrm{div}(\mu^{-1}\nabla u)}_{-r^{-2}(\mu^{-1}(r\partial_r)^2+\partial_\theta\mu^{-1}\partial_\theta)u} = f$$

- Singularities in the sector

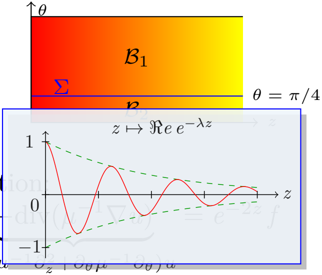
$s(r,\theta) = r^{\lambda}\varphi(\theta)$



$(z,\theta) = (-\ln r, \theta)$

$(r,\theta) = (e^{-z}, \theta)$

- Half-strip $\mathcal{B}$



- Equation:

$$\underbrace{-\mathrm{div}(\mu^{-1}\nabla u)}_{-(\mu^{-1}\partial_z^2+\partial_\theta\mu^{-1}\partial_\theta)u} = e^{-2z} f$$

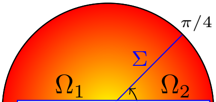
- Modes in the strip

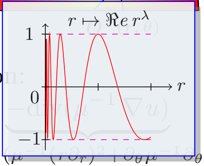
$m(z,\theta) = e^{-\lambda z}\varphi(\theta)$

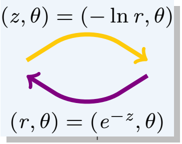
$s \in \mathrm{H}^1(\Omega)$ $\Re e\,\lambda > 0$ m is evanescent

Analogy with a waveguide problem

- Bounded sector Ω



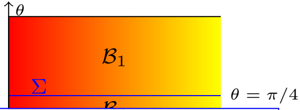


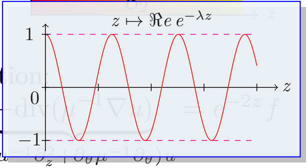


$(z,\theta) = (-\ln r, \theta)$

$(r,\theta) = (e^{-z}, \theta)$

- Half-strip $\mathcal{B}$





- Equation:

$$\underbrace{-\mathrm{div}(\mu^{-1}\nabla u)}_{-r^{-2}(\mu^{-1}(r\partial_r)^2+\partial_\theta\mu^{-1}\partial_\theta)u} = f$$

- Equation:

$$\underbrace{-\mathrm{div}(\mu^{-1}\nabla u)}_{-(\mu^{-1}\partial_z^2+\partial_\theta\mu^{-1}\partial_\theta)u} = e^{-2z} f$$

- Singularities in the sector

$$\begin{aligned} s(r,\theta) &= r^{\lambda}\varphi(\theta) \\ &= \cancel{r^{a}}\,(\cos b\ln r + i\sin b\ln r)\varphi(\theta) \end{aligned}$$

- Modes in the strip

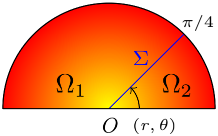
$$\begin{aligned} m(z,\theta) &= e^{-\lambda z}\varphi(\theta) \\ &= \cancel{e^{-az}}\,(\cos bz - i\sin bz)\varphi(\theta) \end{aligned}$$

$(\Re e\,\lambda = a,\ \Im m\,\lambda = b)$

$s \in \mathrm{H}^1(\Omega)$	$\Re e\,\lambda > 0$	m is evanescent
$s \notin \mathrm{H}^1(\Omega)$	$\Re e\,\lambda = 0$	m is propagative

Analogy with a waveguide problem

- Bounded sector Ω

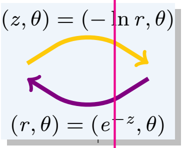


- Equation:

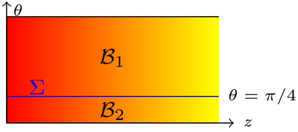
$$\underbrace{-\mathrm{div}(\mu^{-1}\nabla u)}_{-r^{-2}(\mu^{-1}(r\partial_r)^2+\partial_\theta\mu^{-1}\partial_\theta)u} = f$$

- Singularities in the sector

$$\begin{aligned} s(r,\theta) &= r^\lambda\varphi(\theta) \\ &= \cancel{r^a}\,(\cos b\ln r + i\sin b\ln r)\varphi(\theta) \end{aligned}$$



- Half-strip $\mathcal{B}$



- Equation:

$$\underbrace{-\mathrm{div}(\mu^{-1}\nabla u)}_{-(\mu^{-1}\partial_z^2+\partial_\theta\mu^{-1}\partial_\theta)u} = e^{-2z}f$$

- Modes in the strip

$$\begin{aligned} m(z,\theta) &= e^{-\lambda z}\varphi(\theta) \\ &= \cancel{e^{-az}}\,(\cos bz - i\sin bz)\varphi(\theta) \end{aligned}$$

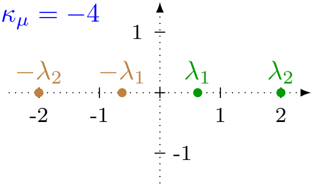
$(\Re e\,\lambda = a,\ \Im m\,\lambda = b)$

$s \in \mathrm{H}^1(\Omega)$ $\Re e\,\lambda > 0$ m is evanescent

$s \notin \mathrm{H}^1(\Omega)$ $\Re e\,\lambda = 0$ m is propagative

- ▶ This encourages us to use modal decomposition in the half-strip.

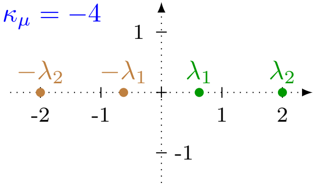
Modal analysis in the waveguide



- Outside the critical interval. All the modes are exponentially growing or decaying.

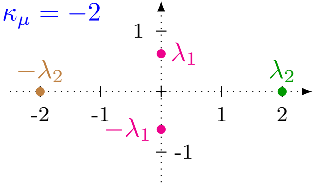
→ We look for an exponentially decaying solution. $\boxed{\mathrm{H}^1 \text{ framework}}$

Modal analysis in the waveguide



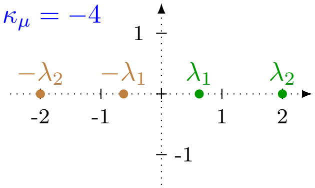
- **Outside the critical interval**. All the modes are exponentially growing or decaying.

→ We look for an exponentially decaying solution. $\boxed{\mathrm{H}^1 \text{ framework}}$



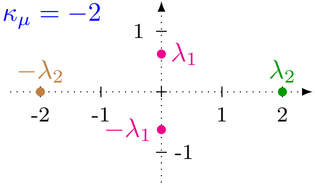
- **Inside the critical interval**. There are exactly two propagative modes.

Modal analysis in the waveguide



- Outside the critical interval. All the modes are exponentially growing or decaying.

→ We look for an exponentially decaying solution. $\boxed{\mathrm{H}^1 \text{ framework}}$



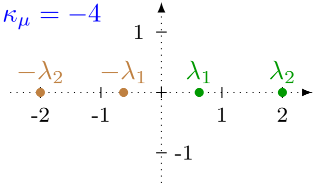
- Inside the critical interval. There are exactly two propagative modes.

→ The decomposition on the outgoing modes leads to look for a solution of the form

$$u = \underbrace{c_1\,\varphi_1\,e^{\lambda_1 z}}_{\text{propagative part}} + \underbrace{u_e.}_{\text{evanescent part}}$$

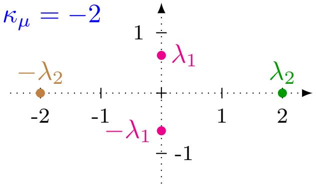
$$\boxed{\text{non } \mathrm{H}^1 \text{ framework}}$$

Modal analysis in the waveguide



- Outside the critical interval. All the modes are exponentially growing or decaying.

→ We look for an exponentially decaying solution. $\boxed{\mathrm{H}^1 \text{ framework}}$



- Inside the critical interval. There are exactly two propagative modes.

→ The decomposition on the outgoing modes leads to look for a solution of the form

$$u = \underbrace{c_1\,\varphi_1\,e^{\lambda_1 z}}_{\text{propagative part}} + \underbrace{u_e.}_{\text{evanescent part}}$$

$\boxed{\text{non } \mathrm{H}^1 \text{ framework}}$

... but the modal decomposition is not easy to justify because two sign-changing appear in the transverse problem: $\partial_\theta \sigma \partial_\theta \varphi = -\sigma \lambda^2 \varphi$.

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$\mathrm{W}_{-\beta} = \{v \,|\, e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\}$ space of exponentially decaying functions

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$\mathrm{W}_{-\beta} = \{v \,|\, e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\}$ space of exponentially decaying functions

$\mathrm{W}_{\beta} = \{v \,|\, e^{-\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\}$ space of exponentially growing functions

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$\mathrm{W}_{-\beta} = \{v \mid e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\}$ space of exponentially decaying functions

$\mathrm{W}^{+} = \mathrm{span}(\zeta\varphi_1\, e^{\lambda_1 z}) \oplus \mathrm{W}_{-\beta}$ propagative part + evanescent part

$\mathrm{W}_{\beta} = \{v \mid e^{-\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\}$ space of exponentially growing functions

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$$\begin{array}{lll} \mathrm{W}_{-\beta} & = \{v \mid e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} & \text{space of exponentially decaying functions} \\ \cap & & \\ \mathrm{W}^{+} & = \mathrm{span}(\zeta \varphi_1 \, e^{\lambda_1 z}) \oplus \mathrm{W}_{-\beta} & \text{propagative part} + \text{evanescent part} \\ \cap & & \\ \mathrm{W}_{\beta} & = \{v \mid e^{-\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} & \text{space of exponentially growing functions} \end{array}$$

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$$\begin{array}{lll} \mathrm{W}_{-\beta} & = \{ v \,|\, e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B}) \} & \text{space of exponentially decaying functions} \\ \cap & & \\ \mathrm{W}^{+} & = \mathrm{span}(\zeta \varphi_1 \, e^{\lambda_1 z}) \oplus \mathrm{W}_{-\beta} & \text{propagative part} + \text{evanescent part} \\ \cap & & \\ \mathrm{W}_{\beta} & = \{ v \,|\, e^{-\beta z} v \in \mathrm{H}^1_0(\mathcal{B}) \} & \text{space of exponentially growing functions} \end{array}$$

THEOREM. Let $\kappa_\mu \in (-3; -1)$ and $0 < \beta < 2$. The operator $A^+ : \mathrm{div}(\mu^{-1} \nabla \cdot)$ from W^+ to W^*_β is an isomorphism.

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$$\begin{array}{lll} \underset{\cap}{\mathrm{W}_{-\beta}} & = \{v \,|\, e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} & \text{space of exponentially decaying functions} \\ \underset{\cap}{\mathrm{W}^+} & = \mathrm{span}(\zeta \varphi_1 \, e^{\lambda_1 z}) \oplus \mathrm{W}_{-\beta} & \text{propagative part + evanescent part} \\ \mathrm{W}_{\beta} & = \{v \,|\, e^{-\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} & \text{space of exponentially growing functions} \end{array}$$

THEOREM. Let $\kappa_\mu \in (-3; -1)$ and $0 < \beta < 2$. The operator A^+ : $\mathrm{div}(\mu^{-1}\nabla\cdot)$ from W^+ to W^*_β is an isomorphism.

IDEAS OF THE PROOF:

1. $A_{-\beta}$: $\mathrm{div}(\mu^{-1}\nabla\cdot)$ from $\mathrm{W}_{-\beta}$ to W^*_β is injective but not surjective.

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$$\begin{array}{lll} \underset{\cap}{\mathrm{W}_{-\beta}} & = \{v \,|\, e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} & \text{space of exponentially decaying functions} \\ \underset{\cap}{\mathrm{W}^+} & = \mathrm{span}(\zeta\varphi_1\, e^{\lambda_1 z}) \oplus \mathrm{W}_{-\beta} & \text{propagative part + evanescent part} \\ \mathrm{W}_{\beta} & = \{v \,|\, e^{-\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} & \text{space of exponentially growing functions} \end{array}$$

THEOREM. Let $\kappa_\mu \in (-3;-1)$ and $0 < \beta < 2$. The operator $A^+ : \mathrm{div}(\mu^{-1}\nabla\cdot)$ from W^+ to W^*_β is an isomorphism.

IDEAS OF THE PROOF:

1. $A_{-\beta} : \mathrm{div}(\mu^{-1}\nabla\cdot)$ from $\mathrm{W}_{-\beta}$ to W^*_β is injective but not surjective.
2. $A_{\beta} : \mathrm{div}(\mu^{-1}\nabla\cdot)$ from W_{β} to $\mathrm{W}^*_{-\beta}$ is surjective but not injective.

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$$\begin{array}{lll} \mathrm{W}_{-\beta} & = \{v \,|\, e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} & \text{space of exponentially decaying functions} \\ \cap & & \\ \mathrm{W}^{+} & = \mathrm{span}(\zeta \varphi_1 \, e^{\lambda_1 z}) \oplus \mathrm{W}_{-\beta} & \text{propagative part} + \text{evanescent part} \\ \cap & & \\ \mathrm{W}_{\beta} & = \{v \,|\, e^{-\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} & \text{space of exponentially growing functions} \end{array}$$

THEOREM. Let $\kappa_\mu \in (-3; -1)$ and $0 < \beta < 2$. The operator $A^+ : \mathrm{div}(\mu^{-1}\nabla \cdot)$ from W^+ to W^*_β is an isomorphism.

IDEAS OF THE PROOF:

1. $A_{-\beta} : \mathrm{div}(\mu^{-1}\nabla \cdot)$ from $\mathrm{W}_{-\beta}$ to W^*_β is injective but not surjective.
2. $A_{\beta} : \mathrm{div}(\mu^{-1}\nabla \cdot)$ from W_{β} to $\mathrm{W}^*_{-\beta}$ is surjective but not injective.
3. The intermediate operator $A^+ : \mathrm{W}^+ \to \mathrm{W}^*_\beta$ is injective (energy integral) and surjective (residue theorem).

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$$\begin{array}{lll} \mathrm{W}_{-\beta} & = \{v \,|\, e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} & \text{space of exponentially decaying functions} \\ \cap & & \\ \mathrm{W}^{+} & = \mathrm{span}(\zeta \varphi_1 \, e^{\lambda_1 z}) \oplus \mathrm{W}_{-\beta} & \text{propagative part} + \text{evanescent part} \\ \cap & & \\ \mathrm{W}_{\beta} & = \{v \,|\, e^{-\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\} & \text{space of exponentially growing functions} \end{array}$$

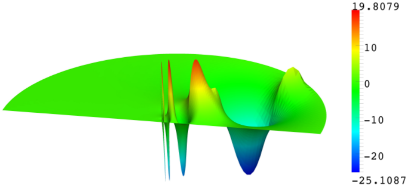
THEOREM. Let $\kappa_\mu \in (-3; -1)$ and $0 < \beta < 2$. The operator $A^+ : \mathrm{div}(\mu^{-1}\nabla\cdot)$ from W^+ to W^*_β is an isomorphism.

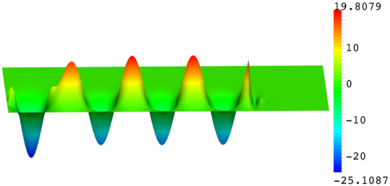
IDEAS OF THE PROOF:

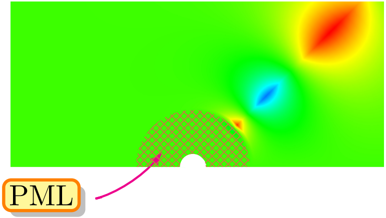
1. $A_{-\beta} : \mathrm{div}(\mu^{-1}\nabla\cdot)$ from $\mathrm{W}_{-\beta}$ to W^*_β is injective but not surjective.
2. $A_\beta : \mathrm{div}(\mu^{-1}\nabla\cdot)$ from W_β to $\mathrm{W}^*_{-\beta}$ is surjective but not injective.
3. The intermediate operator $A^+ : \mathrm{W}^+ \to \mathrm{W}^*_\beta$ is injective (energy integral) and surjective (residue theorem).
4. Limiting absorption principle to select the outgoing mode.

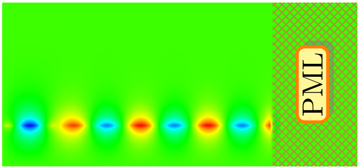
A funny use of PMLs

- We use a PML (*Perfectly Matched Layer*) to bound the domain $\mathcal{B}$ + finite elements in the truncated strip









Contrast $\kappa_\mu = -1.001 \in (-3; -1)$.

A black hole phenomenon

- The same phenomenon occurs for the Helmholtz equation.

$$(\boldsymbol{x}, t) \mapsto \Re e\,(u(\boldsymbol{x})e^{-i\omega t}) \quad \text{for } \kappa_\mu = -1.3 \in (-3; -1)$$

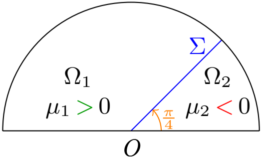
$(\dots)$ $(\dots)$

- Analogous phenomena occur in cuspidal domains in the theory of water-waves and in elasticity (`Cardone, Nazarov, Taskinen`).
- On going work for a general domain (`C. Carvalho`).

Summary of the results for the scalar problem

Problem

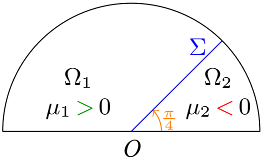
$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ -\mathrm{div}\,(\mu^{-1}\nabla u) = f \quad \text{ in } \Omega. \end{array} \right.$$



Summary of the results for the scalar problem

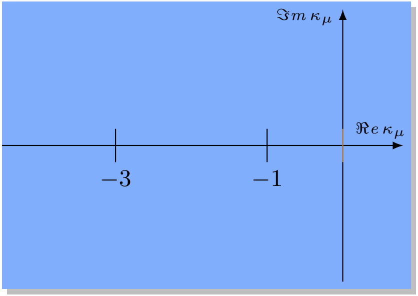
Problem

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ -\mathrm{div}\,(\mu^{-1}\nabla u) = f \quad \text{ in } \Omega. \end{array} \right.$$



Results

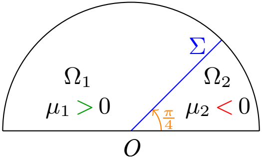
🟦 For $\kappa_\mu \in \mathbb{C}\backslash\mathbb{R}_-$, $(\mathscr{P})$ well-posed in $\mathrm{H}^1_0(\Omega)$ (Lax-Milgram)



Summary of the results for the scalar problem

Problem

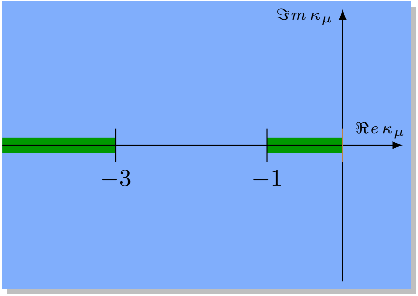
$$(\mathscr{P})\ \left|\ \begin{array}{l}\text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:}\\ -\mathrm{div}\,(\mu^{-1}\nabla u) = f \quad \text{ in } \Omega.\end{array}\right.$$



Results

🟦 For $\kappa_\mu \in \mathbb{C}\backslash\mathbb{R}_-$, $(\mathscr{P})$ well-posed in $\mathrm{H}^1_0(\Omega)$ (Lax-Milgram)

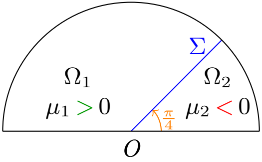
🟩 For $\kappa_\mu \in \mathbb{R}^*_-\backslash[-3;-1]$, $(\mathscr{P})$ well-posed in $\mathrm{H}^1_0(\Omega)$ (`T`-coercivity)



Summary of the results for the scalar problem

Problem

$$(\mathscr{P})\quad\left|\quad\begin{array}{l}\text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:}\\ -\mathrm{div}\,(\mu^{-1}\nabla u) = f \qquad \text{in } \Omega.\end{array}\right.$$

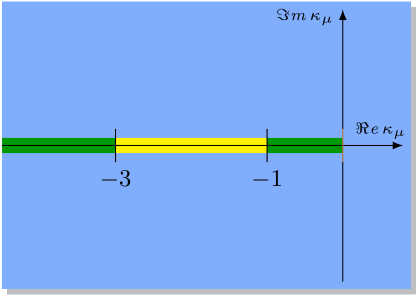


Results

🟦 For $\kappa_\mu \in \mathbb{C}\backslash\mathbb{R}_-$, $(\mathscr{P})$ well-posed in $\mathrm{H}^1_0(\Omega)$ (Lax-Milgram)

🟩 For $\kappa_\mu \in \mathbb{R}^*_-\backslash[-3;-1]$, $(\mathscr{P})$ well-posed in $\mathrm{H}^1_0(\Omega)$ (T-coercivity)

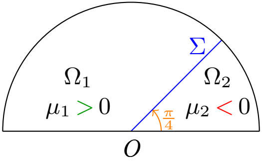
🟨 For $\kappa_\mu \in (-3;-1)$, $(\mathscr{P})$ is not well-posed in the Fredholm sense in $\mathrm{H}^1_0(\Omega)$ but well-posed in V^+ (PMLs)



Summary of the results for the scalar problem

Problem

$$(\mathscr{P}) \left|\ \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ -\mathrm{div}\,(\mu^{-1}\nabla u) = f \quad \text{in } \Omega. \end{array}\right.$$



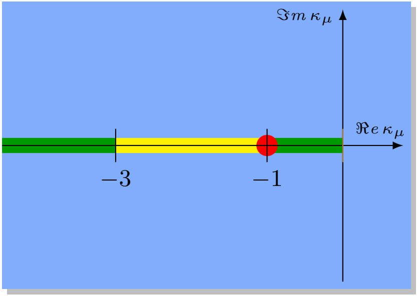
Results

🟦 For $\kappa_\mu \in \mathbb{C}\backslash\mathbb{R}_-$, $(\mathscr{P})$ well-posed in $\mathrm{H}^1_0(\Omega)$ (Lax-Milgram)

🟩 For $\kappa_\mu \in \mathbb{R}^*_-\backslash[-3;-1]$, $(\mathscr{P})$ well-posed in $\mathrm{H}^1_0(\Omega)$ (T-coercivity)

🟨 For $\kappa_\mu \in (-3;-1)$, $(\mathscr{P})$ is not well-posed in the Fredholm sense in $\mathrm{H}^1_0(\Omega)$ but well-posed in V^+ (PMLs)

🔴 $\kappa_\mu = -1$, $(\mathscr{P})$ ill-posed in $\mathrm{H}^1_0(\Omega)$



$\mathbb{T}$-coercivity in the vector case 1/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

$$\left|\begin{array}{l} \text{Find } \boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega) \text{ such that for all } \boldsymbol{H}' \in \mathbf{V}_T(\mu;\Omega) : \\ \underbrace{\int_\Omega \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')} - \omega^2 \underbrace{\int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')} = \underbrace{\int_\Omega \boldsymbol{F} \cdot \boldsymbol{H}'}_{l(\boldsymbol{H}')}, \end{array}\right.$$

with $\mathbf{V}_T(\mu;\Omega) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\Omega) \,|\, \mathrm{div}\,(\mu\boldsymbol{u}) = 0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

$\mathbb{T}$-coercivity in the vector case 1/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

$$\left|\begin{array}{l}\text{Find } \boldsymbol{H} \in \mathbf{V}_T(\mu;\,\Omega) \text{ such that for all } \boldsymbol{H}' \in \mathbf{V}_T(\mu;\,\Omega) : \\ \underbrace{\displaystyle\int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')} - \omega^2 \underbrace{\displaystyle\int_\Omega \mu\boldsymbol{H}\cdot\boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')} = \underbrace{\displaystyle\int_\Omega \boldsymbol{F}\cdot\boldsymbol{H}'}_{l(\boldsymbol{H}')},\end{array}\right.$$

with $\mathbf{V}_T(\mu;\,\Omega) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\,\Omega)\,|\,\mathrm{div}\,(\mu\boldsymbol{u}) = 0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

By analogy with the scalar case, we look for $\mathbb{T} \in \mathcal{L}(\mathbf{V}_T(\mu;\,\Omega))$ such that $a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}') = \displaystyle\int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,(\mathbb{T}\boldsymbol{H}')$ is coercive on $\mathbf{V}_T(\mu;\,\Omega)$.

$\mathbb{T}$-coercivity in the vector case 1/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

$$\left|\begin{array}{l}\text{Find } \boldsymbol{H} \in \mathbf{V}_T(\mu;\,\Omega) \text{ such that for all } \boldsymbol{H}' \in \mathbf{V}_T(\mu;\,\Omega) :\\ \underbrace{\int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')} - \omega^2 \underbrace{\int_\Omega \mu\boldsymbol{H}\cdot\boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')} = \underbrace{\int_\Omega \boldsymbol{F}\cdot\boldsymbol{H}'}_{l(\boldsymbol{H}')},\end{array}\right.$$

with $\mathbf{V}_T(\mu;\,\Omega) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\,\Omega)\,|\,\mathrm{div}\,(\mu\boldsymbol{u}) = 0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

By analogy with the scalar case, we look for $\mathbb{T} \in \mathcal{L}(\mathbf{V}_T(\mu;\,\Omega))$ such that $a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}') = \displaystyle\int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,(\mathbb{T}\boldsymbol{H}')$ is coercive on $\mathbf{V}_T(\mu;\,\Omega)$.

Scalar approach

$\mathbb{T}$-coercivity in the vector case 1/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

$$\left| \begin{array}{l} \text{Find } \boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega) \text{ such that for all } \boldsymbol{H}' \in \mathbf{V}_T(\mu;\Omega) : \\ \underbrace{\int_\Omega \varepsilon^{-1} \mathbf{curl}\, \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')} - \omega^2 \underbrace{\int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')} = \underbrace{\int_\Omega \boldsymbol{F} \cdot \boldsymbol{H}'}_{l(\boldsymbol{H}')}, \end{array} \right.$$

with $\mathbf{V}_T(\mu;\Omega) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\Omega) \,|\, \mathrm{div}\,(\mu \boldsymbol{u}) = 0,\ \mu \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

By analogy with the scalar case, we look for $\mathbb{T} \in \mathcal{L}(\mathbf{V}_T(\mu;\Omega))$ such that $a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}') = \int_\Omega \varepsilon^{-1} \mathbf{curl}\, \boldsymbol{H} \cdot \mathbf{curl}\, (\mathbb{T}\boldsymbol{H}')$ is coercive on $\mathbf{V}_T(\mu;\Omega)$.

Scalar approach

Let us try $\mathbb{T}\boldsymbol{H} = \left| \begin{array}{ll} \boldsymbol{H}_1 & \text{in } \Omega_1 \\ -\boldsymbol{H}_2 + 2R_1 \boldsymbol{H}_1 & \text{in } \Omega_2 \end{array} \right.$,

$\mathbb{T}$-coercivity in the vector case 1/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

$$\left|\begin{array}{l}\text{Find } \boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega) \text{ such that for all } \boldsymbol{H}' \in \mathbf{V}_T(\mu;\Omega) : \\ \underbrace{\int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')} - \omega^2 \underbrace{\int_\Omega \mu\boldsymbol{H}\cdot\boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')} = \underbrace{\int_\Omega \boldsymbol{F}\cdot\boldsymbol{H}'}_{l(\boldsymbol{H}')},\end{array}\right.$$

with $\mathbf{V}_T(\mu;\Omega) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\Omega)\,|\,\mathrm{div}\,(\mu\boldsymbol{u}) = 0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

By analogy with the scalar case, we look for $\mathbb{T} \in \mathcal{L}(\mathbf{V}_T(\mu;\Omega))$ such that $a(\boldsymbol{H},\mathbb{T}\boldsymbol{H}') = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,(\mathbb{T}\boldsymbol{H}')$ is coercive on $\mathbf{V}_T(\mu;\Omega)$.

Scalar approach

Let us try $\mathbb{T}\boldsymbol{H} = \left|\begin{array}{ll}\boldsymbol{H}_1 & \text{in } \Omega_1 \\ -\boldsymbol{H}_2 + 2R_1\boldsymbol{H}_1 & \text{in } \Omega_2\end{array}\right.$, with R_1 such that

$$\left\{\begin{array}{rcll}(R_1\boldsymbol{H}_1)\times\boldsymbol{n} & = & \boldsymbol{H}_2\times\boldsymbol{n} & \text{on } \Sigma \\ \mu_1(R_1\boldsymbol{H}_1)\cdot\boldsymbol{n} & = & \mu_2\boldsymbol{H}_2\cdot\boldsymbol{n} & \text{on } \Sigma\end{array}\right.$$

$\mathbb{T}$-coercivity in the vector case 1/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

$$\boxed{\left|\begin{array}{l} \text{Find } \boldsymbol{H}\in \mathbf{V}_T(\mu;\Omega) \text{ such that for all } \boldsymbol{H}'\in\mathbf{V}_T(\mu;\Omega):\\ \underbrace{\int_\Omega \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')} - \omega^2 \underbrace{\int_\Omega \mu\boldsymbol{H}\cdot\boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')} = \underbrace{\int_\Omega \boldsymbol{F}\cdot\boldsymbol{H}'}_{l(\boldsymbol{H}')}, \end{array}\right.}$$

with $\mathbf{V}_T(\mu;\Omega) := \{\boldsymbol{u}\in\mathbf{H}(\mathbf{curl}\,;\Omega)\,|\,\mathrm{div}\,(\mu\boldsymbol{u})=0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega\}$.

By analogy with the scalar case, we look for $\mathbb{T}\in\mathcal{L}(\mathbf{V}_T(\mu;\Omega))$ such that $a(\boldsymbol{H},\mathbb{T}\boldsymbol{H}') = \int_\Omega \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,(\mathbb{T}\boldsymbol{H}')$ is coercive on $\mathbf{V}_T(\mu;\Omega)$.

Scalar approach

Let us try $\mathbb{T}\boldsymbol{H} = \left|\begin{array}{ll} \boldsymbol{H}_1 & \text{in } \Omega_1\\ -\boldsymbol{H}_2 + 2R_1\boldsymbol{H}_1 & \text{in } \Omega_2 \end{array}\right.$, with R_1 such that

$$\xcancel{\left\{\begin{array}{lcll} (R_1\boldsymbol{H}_1)\times\boldsymbol{n} & = & \boldsymbol{H}_2\times\boldsymbol{n} & \text{on } \Sigma\\ \mu_1(R_1\boldsymbol{H}_1)\cdot\boldsymbol{n} & = & \mu_2\boldsymbol{H}_2\cdot\boldsymbol{n} & \text{on } \Sigma \end{array}\right.}$$

Not possible!

$\mathbb{T}$-coercivity in the vector case 2/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

$$\left|\begin{array}{l}\text{Find } \boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega) \text{ such that for all } \boldsymbol{H}' \in \mathbf{V}_T(\mu;\Omega) : \\ \underbrace{\int_\Omega \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')} - \omega^2 \underbrace{\int_\Omega \mu\boldsymbol{H}\cdot\boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')} = \underbrace{\int_\Omega \boldsymbol{F}\cdot\boldsymbol{H}'}_{l(\boldsymbol{H}')},\end{array}\right.$$

with $\mathbf{V}_T(\mu;\Omega) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\Omega)\,|\,\mathrm{div}\,(\mu\boldsymbol{u}) = 0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

By analogy with the scalar case, we look for $\mathbb{T} \in \mathcal{L}(\mathbf{V}_T(\mu;\Omega))$ such that $a(\boldsymbol{H},\mathbb{T}\boldsymbol{H}') = \int_\Omega \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,(\mathbb{T}\boldsymbol{H}')$ is coercive on $\mathbf{V}_T(\mu;\Omega)$.

Maxwell approach

$\mathbb{T}$-coercivity in the vector case 2/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

$$\boxed{\left|\begin{array}{l}\text{Find } \boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega) \text{ such that for all } \boldsymbol{H}' \in \mathbf{V}_T(\mu;\Omega) : \\ \underbrace{\int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')} - \omega^2 \underbrace{\int_\Omega \mu\boldsymbol{H}\cdot\boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')} = \underbrace{\int_\Omega \boldsymbol{F}\cdot\boldsymbol{H}'}_{l(\boldsymbol{H}')},\end{array}\right.}$$

with $\mathbf{V}_T(\mu;\Omega) := \{\boldsymbol{u}\in\mathbf{H}(\mathbf{curl}\,;\Omega)\,|\,\mathrm{div}\,(\mu\boldsymbol{u}) = 0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

By analogy with the scalar case, we look for $\mathbb{T}\in\mathcal{L}(\mathbf{V}_T(\mu;\Omega))$ such that $a(\boldsymbol{H},\mathbb{T}\boldsymbol{H}') = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,(\mathbb{T}\boldsymbol{H}')$ is coercive on $\mathbf{V}_T(\mu;\Omega)$.

Maxwell approach

Let us try to define $\mathbb{T}\boldsymbol{H}\in\mathbf{V}_T(\mu;\Omega)$ as "the function satisfying"

$\mathbb{T}$-coercivity in the vector case 2/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

$$\left|\begin{array}{l}\text{Find } \boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega) \text{ such that for all } \boldsymbol{H}' \in \mathbf{V}_T(\mu;\Omega) : \\ \underbrace{\int_\Omega \varepsilon^{-1} \mathbf{curl}\, \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')} - \omega^2 \underbrace{\int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')} = \underbrace{\int_\Omega \boldsymbol{F} \cdot \boldsymbol{H}'}_{l(\boldsymbol{H}')},\end{array}\right.$$

with $\mathbf{V}_T(\mu;\Omega) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\Omega) \,|\, \mathrm{div}\,(\mu\boldsymbol{u}) = 0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

By analogy with the scalar case, we look for $\mathbb{T} \in \mathcal{L}(\mathbf{V}_T(\mu;\Omega))$ such that $a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}') = \displaystyle\int_\Omega \varepsilon^{-1} \mathbf{curl}\, \boldsymbol{H} \cdot \mathbf{curl}\,(\mathbb{T}\boldsymbol{H}')$ is coercive on $\mathbf{V}_T(\mu;\Omega)$.

Maxwell approach

Let us try to define $\mathbb{T}\boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega)$ as "the function satisfying"

$$\mathbf{curl}\,(\mathbb{T}\boldsymbol{H}) = \varepsilon\, \mathbf{curl}\, \boldsymbol{H} \quad \text{in } \Omega$$

$\mathbb{T}$-coercivity in the vector case 2/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

$$\left|\begin{array}{l} \text{Find } \boldsymbol{H} \in \mathbf{V}_T(\mu;\,\Omega) \text{ such that for all } \boldsymbol{H}' \in \mathbf{V}_T(\mu;\,\Omega): \\ \underbrace{\int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')} - \omega^2 \underbrace{\int_\Omega \mu\boldsymbol{H}\cdot\boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')} = \underbrace{\int_\Omega \boldsymbol{F}\cdot\boldsymbol{H}'}_{l(\boldsymbol{H}')}, \end{array}\right.$$

with $\mathbf{V}_T(\mu;\,\Omega) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\,\Omega)\,|\,\mathrm{div}\,(\mu\boldsymbol{u}) = 0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

By analogy with the scalar case, we look for $\mathbb{T} \in \mathcal{L}(\mathbf{V}_T(\mu;\,\Omega))$ such that $a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}') = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,(\mathbb{T}\boldsymbol{H}')$ is coercive on $\mathbf{V}_T(\mu;\,\Omega)$.

Maxwell approach

Let us try to define $\mathbb{T}\boldsymbol{H} \in \mathbf{V}_T(\mu;\,\Omega)$ as "the function satisfying"

$$\mathbf{curl}\,(\mathbb{T}\boldsymbol{H}) = \varepsilon\,\mathbf{curl}\,\boldsymbol{H} \quad \text{in } \Omega \qquad \text{so that} \quad a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}) = \int_\Omega |\mathbf{curl}\,\boldsymbol{H}|^2.$$

$\mathbb{T}$-coercivity in the vector case 2/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

$$\left|\begin{array}{l}\text{Find } \boldsymbol{H}\in\mathbf{V}_T(\mu;\,\Omega) \text{ such that for all } \boldsymbol{H}'\in\mathbf{V}_T(\mu;\,\Omega):\\ \underbrace{\int_\Omega \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')}-\omega^2\underbrace{\int_\Omega \mu\boldsymbol{H}\cdot\boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')}=\underbrace{\int_\Omega \boldsymbol{F}\cdot\boldsymbol{H}'}_{l(\boldsymbol{H}')},\end{array}\right.$$

with $\mathbf{V}_T(\mu;\,\Omega):=\{\boldsymbol{u}\in\mathbf{H}(\mathbf{curl}\,;\,\Omega)\,|\,\mathrm{div}\,(\mu\boldsymbol{u})=0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n}=0\text{ on }\partial\Omega\}$.

By analogy with the scalar case, we look for $\mathbb{T}\in\mathcal{L}(\mathbf{V}_T(\mu;\,\Omega))$ such that $a(\boldsymbol{H},\mathbb{T}\boldsymbol{H}')=\displaystyle\int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,(\mathbb{T}\boldsymbol{H}')$ is coercive on $\mathbf{V}_T(\mu;\,\Omega)$.

Maxwell approach

Let us try to define $\mathbb{T}\boldsymbol{H}\in\mathbf{V}_T(\mu;\,\Omega)$ as "the function satisfying"

$$\mathbf{curl}\,(\mathbb{T}\boldsymbol{H})=\varepsilon\,\mathbf{curl}\,\boldsymbol{H}\quad\text{in }\Omega\qquad\text{so that}\qquad a(\boldsymbol{H},\mathbb{T}\boldsymbol{H})=\int_\Omega|\mathbf{curl}\,\boldsymbol{H}|^2.$$

♠ Impossible because $\mathrm{div}\,(\varepsilon\,\mathbf{curl}\,\boldsymbol{H})\neq 0$.

$\mathbb{T}$-coercivity in the vector case 2/3

Let us consider the problem for the magnetic field $\boldsymbol{H}$:

Find $\boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega)$ such that for all $\boldsymbol{H}' \in \mathbf{V}_T(\mu;\Omega)$:

$$\underbrace{\int_\Omega \varepsilon^{-1} \mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\boldsymbol{H}'}_{a(\boldsymbol{H},\boldsymbol{H}')} - \omega^2 \underbrace{\int_\Omega \mu \boldsymbol{H} \cdot \boldsymbol{H}'}_{c(\boldsymbol{H},\boldsymbol{H}')} = \underbrace{\int_\Omega \boldsymbol{F} \cdot \boldsymbol{H}'}_{l(\boldsymbol{H}')},$$

with $\mathbf{V}_T(\mu;\Omega) := \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}\,;\Omega) \,|\, \mathrm{div}\,(\mu\boldsymbol{u}) = 0,\ \mu\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

By analogy with the scalar case, we look for $\mathbb{T} \in \mathcal{L}(\mathbf{V}_T(\mu;\Omega))$ such that $a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}') = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,(\mathbb{T}\boldsymbol{H}')$ is coercive on $\mathbf{V}_T(\mu;\Omega)$.

Maxwell approach

Let us try to define $\mathbb{T}\boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega)$ as "the function satisfying"

$$\mathbf{curl}\,(\mathbb{T}\boldsymbol{H}) = \varepsilon\,\mathbf{curl}\,\boldsymbol{H} \quad \text{in } \Omega \qquad \text{so that} \qquad a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}) = \int_\Omega |\mathbf{curl}\,\boldsymbol{H}|^2.$$

♠ Impossible because $\mathrm{div}\,(\varepsilon\,\mathbf{curl}\,\boldsymbol{H}) \neq 0$.

Idea: add a gradient...

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega)$.

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega)$.

❶ Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ s.t. $\operatorname{div}(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)) = 0$.

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega)$.

① Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ s.t. $\operatorname{div}(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)) = 0$.

✔ Ok if $(\varphi,\varphi') \mapsto \displaystyle\int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'$ is T-coercive on $\mathrm{H}^1_0(\Omega)$. $(\mathcal{A}_\varepsilon)$

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega)$.

① Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ s.t. $\operatorname{div}(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)) = 0$.

✓ Ok if $(\varphi,\varphi') \mapsto \int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'$ is T-coercive on $\mathrm{H}^1_0(\Omega)$. $(\mathcal{A}_\varepsilon)$

② Introduce $\boldsymbol{u} \in \mathbf{V}_T(1;\Omega)$ (`Amrouche et al. 98`) the function satisfying

$$\mathbf{curl}\,\boldsymbol{u} = \varepsilon\,(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi) \quad \text{in } \Omega.$$

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\Omega)$.

❶ Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ s.t. $\mathrm{div}\,(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)) = 0$.

✓ Ok if $(\varphi,\varphi') \mapsto \displaystyle\int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'$ is $\mathtt{T}$-coercive on $\mathrm{H}^1_0(\Omega)$. $(\mathcal{A}_\varepsilon)$

❷ Introduce $\boldsymbol{u} \in \mathbf{V}_T(1;\Omega)$ (`Amrouche et al. 98`) the function satisfying

$$\mathbf{curl}\,\boldsymbol{u} = \varepsilon\,(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi) \quad \text{in } \Omega.$$

❸ Introduce $\psi \in \mathrm{H}^1(\Omega)/\mathbb{R}$ s.t. $\boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\Omega)$ $(\mathrm{div}\,(\mu(\boldsymbol{u} - \nabla\psi)) = 0)$.

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\,\Omega)$.

① Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ s.t. $\operatorname{div}(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)) = 0$.

✓ Ok if $(\varphi,\varphi') \mapsto \int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'$ is T-coercive on $\mathrm{H}^1_0(\Omega)$. $(\mathcal{A}_\varepsilon)$

② Introduce $\boldsymbol{u} \in \mathbf{V}_T(1;\,\Omega)$ (`Amrouche et al. 98`) the function satisfying

$$\mathbf{curl}\,\boldsymbol{u} = \varepsilon\,(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi) \quad \text{in } \Omega.$$

③ Introduce $\psi \in \mathrm{H}^1(\Omega)/\mathbb{R}$ s.t. $\boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$ $(\operatorname{div}(\mu(\boldsymbol{u} - \nabla\psi)) = 0)$.

✓ Ok if $(\psi,\psi') \mapsto \int_\Omega \mu\nabla\psi\cdot\nabla\psi'$ is T-coercive on $\mathrm{H}^1(\Omega)/\mathbb{R}$. $(\mathcal{A}_\mu)$

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu; \Omega)$.

① Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ s.t. $\operatorname{div}(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)) = 0$.

✓ Ok if $(\varphi, \varphi') \mapsto \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi'$ is T-coercive on $\mathrm{H}^1_0(\Omega)$. $(\mathcal{A}_\varepsilon)$

② Introduce $\boldsymbol{u} \in \mathbf{V}_T(1; \Omega)$ (`Amrouche et al. 98`) the function satisfying

$$\mathbf{curl}\,\boldsymbol{u} = \varepsilon\,(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi) \quad \text{in } \Omega.$$

③ Introduce $\psi \in \mathrm{H}^1(\Omega)/\mathbb{R}$ s.t. $\boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu; \Omega)$ $(\operatorname{div}(\mu(\boldsymbol{u} - \nabla\psi)) = 0)$.

✓ Ok if $(\psi, \psi') \mapsto \int_\Omega \mu \nabla\psi \cdot \nabla\psi'$ is T-coercive on $\mathrm{H}^1(\Omega)/\mathbb{R}$. $(\mathcal{A}_\mu)$

④ Finally, define $\mathbb{T}\boldsymbol{H} := \boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu; \Omega)$.

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\,\Omega)$.

❶ Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ s.t. $\operatorname{div}(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)) = 0$.

✓ Ok if $(\varphi, \varphi') \mapsto \displaystyle\int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'$ is T-coercive on $\mathrm{H}^1_0(\Omega)$. $(\mathcal{A}_\varepsilon)$

❷ Introduce $\boldsymbol{u} \in \mathbf{V}_T(1;\,\Omega)$ (`Amrouche et al. 98`) the function satisfying

$$\mathbf{curl}\,\boldsymbol{u} = \varepsilon\,(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi) \quad \text{in } \Omega.$$

❸ Introduce $\psi \in \mathrm{H}^1(\Omega)/\mathbb{R}$ s.t. $\boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$ $(\operatorname{div}(\mu(\boldsymbol{u} - \nabla\psi)) = 0)$.

✓ Ok if $(\psi, \psi') \mapsto \displaystyle\int_\Omega \mu\nabla\psi\cdot\nabla\psi'$ is T-coercive on $\mathrm{H}^1(\Omega)/\mathbb{R}$. $(\mathcal{A}_\mu)$

❹ Finally, define $\mathbb{T}\boldsymbol{H} := \boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$. There holds:

$$a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}) = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,(\mathbb{T}\boldsymbol{H})$$

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\,\Omega)$.

① Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ s.t. $\operatorname{div}\left(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)\right) = 0$.

✓ Ok if $(\varphi,\varphi') \mapsto \int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'$ is T-coercive on $\mathrm{H}^1_0(\Omega)$. $(\mathcal{A}_\varepsilon)$

② Introduce $\boldsymbol{u} \in \mathbf{V}_T(1;\,\Omega)$ (`Amrouche et al. 98`) the function satisfying

$$\mathbf{curl}\,\boldsymbol{u} = \varepsilon\left(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi\right) \quad \text{in } \Omega.$$

③ Introduce $\psi \in \mathrm{H}^1(\Omega)/\mathbb{R}$ s.t. $\boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$ $(\operatorname{div}\left(\mu(\boldsymbol{u} - \nabla\psi)\right) = 0)$.

✓ Ok if $(\psi,\psi') \mapsto \int_\Omega \mu\nabla\psi\cdot\nabla\psi'$ is T-coercive on $\mathrm{H}^1(\Omega)/\mathbb{R}$. $(\mathcal{A}_\mu)$

④ Finally, define $\mathbb{T}\boldsymbol{H} := \boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$. There holds:

$$a(\boldsymbol{H},\mathbb{T}\boldsymbol{H}) = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{u}$$

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\,\Omega)$.

① Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ s.t. $\operatorname{div}(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)) = 0$.

✓ Ok if $(\varphi, \varphi') \mapsto \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi'$ is T-coercive on $\mathrm{H}^1_0(\Omega)$. $(\mathcal{A}_\varepsilon)$

② Introduce $\boldsymbol{u} \in \mathbf{V}_T(1;\,\Omega)$ (`Amrouche et al. 98`) the function satisfying

$$\mathbf{curl}\,\boldsymbol{u} = \varepsilon\,(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi) \quad \text{in } \Omega.$$

③ Introduce $\psi \in \mathrm{H}^1(\Omega)/\mathbb{R}$ s.t. $\boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$ $(\operatorname{div}(\mu(\boldsymbol{u} - \nabla\psi)) = 0)$.

✓ Ok if $(\psi, \psi') \mapsto \int_\Omega \mu \nabla\psi \cdot \nabla\psi'$ is T-coercive on $\mathrm{H}^1(\Omega)/\mathbb{R}$. $(\mathcal{A}_\mu)$

④ Finally, define $\mathbb{T}\boldsymbol{H} := \boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$. There holds:

$$a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}) = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\boldsymbol{u} = \int_\Omega \mathbf{curl}\,\boldsymbol{H} \cdot (\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)$$

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\,\Omega)$.

❶ Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ s.t. $\mathrm{div}\,(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)) = 0$.

✔ Ok if $(\varphi, \varphi') \mapsto \displaystyle\int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi'$ is T-coercive on $\mathrm{H}^1_0(\Omega)$. $(\mathcal{A}_\varepsilon)$

❷ Introduce $\boldsymbol{u} \in \mathbf{V}_T(1;\,\Omega)$ (`Amrouche` *`et al.`* `98`) the function satisfying

$$\mathbf{curl}\,\boldsymbol{u} = \varepsilon\,(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi) \quad \text{in } \Omega.$$

❸ Introduce $\psi \in \mathrm{H}^1(\Omega)/\mathbb{R}$ s.t. $\boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$ $(\mathrm{div}\,(\mu(\boldsymbol{u} - \nabla\psi)) = 0)$.

✔ Ok if $(\psi, \psi') \mapsto \displaystyle\int_\Omega \mu \nabla\psi \cdot \nabla\psi'$ is T-coercive on $\mathrm{H}^1(\Omega)/\mathbb{R}$. $(\mathcal{A}_\mu)$

❹ Finally, define $\mathbb{T}\boldsymbol{H} := \boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$. There holds:

$$a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}) = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\boldsymbol{u} = \int_\Omega |\mathbf{curl}\,\boldsymbol{H}|^2.$$

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

Consider $\boldsymbol{H} \in \mathbf{V}_T(\mu;\,\Omega)$.

① Introduce $\varphi \in \mathrm{H}^1_0(\Omega)$ s.t. $\operatorname{div}(\varepsilon(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi)) = 0$.

✓ Ok if $(\varphi, \varphi') \mapsto \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi'$ is T-coercive on $\mathrm{H}^1_0(\Omega)$. $(\mathcal{A}_\varepsilon)$

② Introduce $\boldsymbol{u} \in \mathbf{V}_T(1;\,\Omega)$ (`Amrouche` *et al.* `98`) the function satisfying

$$\mathbf{curl}\,\boldsymbol{u} = \varepsilon\,(\mathbf{curl}\,\boldsymbol{H} - \nabla\varphi) \quad \text{in } \Omega.$$

③ Introduce $\psi \in \mathrm{H}^1(\Omega)/\mathbb{R}$ s.t. $\boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$ $(\operatorname{div}(\mu(\boldsymbol{u} - \nabla\psi)) = 0)$.

✓ Ok if $(\psi, \psi') \mapsto \int_\Omega \mu \nabla\psi \cdot \nabla\psi'$ is T-coercive on $\mathrm{H}^1(\Omega)/\mathbb{R}$. $(\mathcal{A}_\mu)$

④ Finally, define $\mathbb{T}\boldsymbol{H} := \boldsymbol{u} - \nabla\psi \in \mathbf{V}_T(\mu;\,\Omega)$. There holds:

$$a(\boldsymbol{H}, \mathbb{T}\boldsymbol{H}) = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\boldsymbol{u} = \int_\Omega |\mathbf{curl}\,\boldsymbol{H}|^2.$$

☛ Use the results of the previous section to check $(\mathcal{A}_\varepsilon)$ and $(\mathcal{A}_\mu)$.

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

- Using this idea, we prove that the embedding of $\mathbf{V}_T(\mu;\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact when $(\mathcal{A}_\mu)$ is true (extension of `Weber 80`'s result).

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

- Using this idea, we prove that the embedding of $\mathbf{V}_T(\mu;\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact when $(\mathcal{A}_\mu)$ is true (extension of `Weber 80`'s result).
- We deduce that $a(\cdot,\mathbb{T}\cdot)$ is coercive on $\mathbf{V}_T(\mu;\Omega)\times\mathbf{V}_T(\mu;\Omega)$ when $(\mathcal{A}_\varepsilon)$ and $(\mathcal{A}_\mu)$ are true.

$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

- Using this idea, we prove that the embedding of $\mathbf{V}_T(\mu;\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact when $(\mathcal{A}_\mu)$ is true (extension of `Weber 80`'s result).
- We deduce that $a(\cdot,\mathbb{T}\cdot)$ is coercive on $\mathbf{V}_T(\mu;\Omega)\times\mathbf{V}_T(\mu;\Omega)$ when $(\mathcal{A}_\varepsilon)$ and $(\mathcal{A}_\mu)$ are true.

Refinements are necessary when:

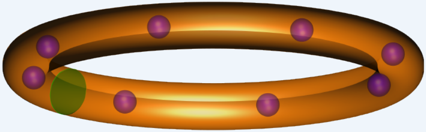
$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

- Using this idea, we prove that the embedding of $\mathbf{V}_T(\mu;\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact when $(\mathcal{A}_\mu)$ is true (extension of `Weber 80`'s result).
- We deduce that $a(\cdot, \mathbb{T}\cdot)$ is coercive on $\mathbf{V}_T(\mu;\Omega) \times \mathbf{V}_T(\mu;\Omega)$ when $(\mathcal{A}_\varepsilon)$ and $(\mathcal{A}_\mu)$ are true.

⚠ Refinements are necessary when:

- The geometry is non trivial (Ω non simply connected and/or $\partial\Omega$ non connected).



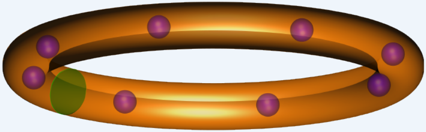
$\mathbb{T}$-coercivity in the vector case 3/3

Maxwell approach

- Using this idea, we prove that the embedding of $\mathbf{V}_T(\mu;\Omega)$ in $\mathbf{L}^2(\Omega)$ is compact when $(\mathcal{A}_\mu)$ is true (extension of `Weber 80`'s result).
- We deduce that $a(\cdot, \mathbb{T}\cdot)$ is coercive on $\mathbf{V}_T(\mu;\Omega) \times \mathbf{V}_T(\mu;\Omega)$ when $(\mathcal{A}_\varepsilon)$ and $(\mathcal{A}_\mu)$ are true.

⚠ Refinements are necessary when:

- ▶ The geometry is non trivial (Ω non simply connected and/or $\partial\Omega$ non connected).



- ▶ The scalar problems are Fredholm with a non trivial kernel.

The result for the magnetic field

Consider $\boldsymbol{F} \in \mathbf{L}^2(\Omega)$ such that $\operatorname{div} \boldsymbol{F} \in \mathbf{L}^2(\Omega)$.

Theorem. Suppose

$$(\varphi, \varphi') \mapsto \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' \text{ is T-coercive on } \mathrm{H}^1_0(\Omega); \qquad (\mathcal{A}_\varepsilon)$$

$$(\varphi, \varphi') \mapsto \int_\Omega \mu \nabla\varphi \cdot \nabla\varphi' \text{ is T-coercive on } \mathrm{H}^1(\Omega)/\mathbb{R}. \qquad (\mathcal{A}_\mu)$$

Then, the problem for the magnetic field

$$\left| \begin{array}{ll} \text{Find } \boldsymbol{H} \in \mathbf{H}(\mathbf{curl}\,; \Omega) \text{ such that:} & \\ \mathbf{curl}\,(\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}) - \omega^2 \mu \boldsymbol{H} = \boldsymbol{F} & \text{in } \Omega \\ \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega \\ \mu \boldsymbol{H} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{array} \right.$$

is well-posed for all $\omega \in \mathbb{C} \backslash \mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

The result for the magnetic field

Consider $\boldsymbol{F} \in \mathbf{L}^2(\Omega)$ such that $\operatorname{div} \boldsymbol{F} \in \mathbf{L}^2(\Omega)$.

Theorem. Suppose

$$(\varphi, \varphi') \mapsto \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' \text{ is T-coercive on } \mathrm{H}^1_0(\Omega); \qquad (\mathcal{A}_\varepsilon)$$

$$(\varphi, \varphi') \mapsto \int_\Omega \mu \nabla\varphi \cdot \nabla\varphi' \text{ is T-coercive on } \mathrm{H}^1(\Omega)/\mathbb{R}. \qquad (\mathcal{A}_\mu)$$

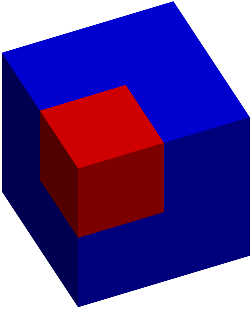
Then, the problem for the magnetic field

$$\left| \begin{array}{ll} \text{Find } \boldsymbol{H} \in \mathbf{H}(\mathbf{curl}\,; \Omega) \text{ such that:} & \\ \mathbf{curl}\,(\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H}) - \omega^2\mu\boldsymbol{H} = \boldsymbol{F} & \text{in } \Omega \\ \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{H} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega \\ \mu\boldsymbol{H} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{array} \right.$$

is well-posed for all $\omega \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

- This result (with the same assumptions) is also true for the problem for the electric field.

Application to the Fichera's corner



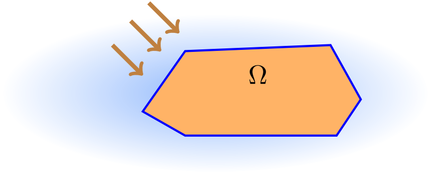
Proposition. Suppose

$$\kappa_\varepsilon \notin [-7; -\frac{1}{7}] \qquad \text{and} \qquad \kappa_\mu \notin [-7; -\frac{1}{7}] \,. \quad ✻$$

Then, the problems for the electric and magnetic fields are well-posed for all $\omega \in \mathbb{C} \backslash \mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

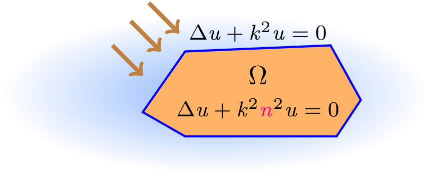
✻ *Note that 7 is the ratio of the blue volume over the red volume...*

The ITEP in three words



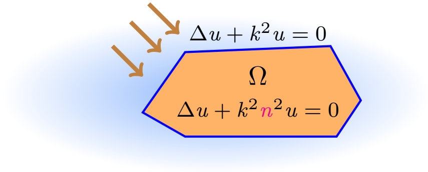
- We want to determine the support of an inclusion Ω embedded in a reference medium ($\mathbb{R}^2$) using the Linear Sampling Method.

The ITEP in three words



- We want to determine the support of an inclusion Ω embedded in a reference medium ($\mathbb{R}^2$) using the Linear Sampling Method.

The ITEP in three words

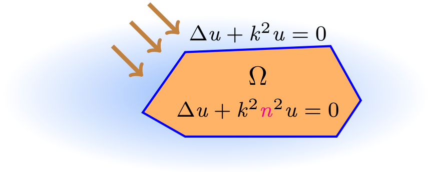


- We want to determine the support of an inclusion Ω embedded in a reference medium $(\mathbb{R}^2)$ using the Linear Sampling Method.

- We can use the method when k is not an eigenvalue of the Interior Transmission Eigenvalue Problem:

$$\left|\begin{array}{l}\text{Find } (k, v) \in \mathbb{C} \times \mathrm{H}_0^2(\Omega) \setminus \{0\} \text{ such that:}\\ \displaystyle\int_\Omega \frac{1}{1-n^2}(\Delta v + k^2 n^2 v)(\Delta v' + k^2 v') = 0, \quad \forall v' \in \mathrm{H}_0^2(\Omega).\end{array}\right.$$

The ITEP in three words



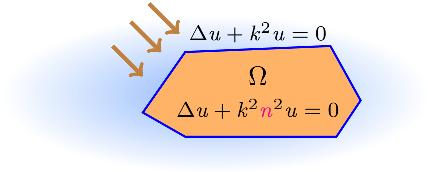
- We want to determine the support of an inclusion Ω embedded in a reference medium ($\mathbb{R}^2$) using the Linear Sampling Method.

- We can use the method when k is not an eigenvalue of the Interior Transmission Eigenvalue Problem:

$$\left| \begin{array}{l} \text{Find } (k, v) \in \mathbb{C} \times \mathrm{H}_0^2(\Omega) \setminus \{0\} \text{ such that:} \\ \displaystyle\int_\Omega \frac{1}{1 - n^2} (\Delta v + k^2 n^2 v)(\Delta v' + k^2 v') = 0, \quad \forall v' \in \mathrm{H}_0^2(\Omega). \end{array} \right.$$

- One of the goals is to prove that the set of transmission eigenvalues is at most discrete.

The ITEP in three words



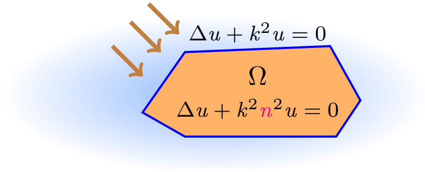
- We want to determine the support of an inclusion Ω embedded in a reference medium ($\mathbb{R}^2$) using the Linear Sampling Method.

- We can use the method when k is not an eigenvalue of the Interior Transmission Eigenvalue Problem:

$$\left|\begin{array}{l}\text{Find } (k, v) \in \mathbb{C} \times \mathrm{H}_0^2(\Omega) \setminus \{0\} \text{ such that:} \\ \displaystyle\int_\Omega \frac{1}{1-n^2}(\Delta v + k^2 n^2 v)(\Delta v' + k^2 v') = 0, \quad \forall v' \in \mathrm{H}_0^2(\Omega).\end{array}\right.$$

- One of the goals is to prove that the set of transmission eigenvalues is at most discrete.
- This problem has been widely studied since 1986-1988 (`Bellis, Cakoni, Colton, Gintides, Guzina, Haddar, Kirsch, Kress, Monk, Päivärinta, Rynne, Sleeman, Sylvester...`) when $n > 1$ on Ω or $n < 1$ on Ω.

The ITEP in three words



- We want to determine the support of an inclusion Ω embedded in a reference medium $(\mathbb{R}^2)$ using the Linear Sampling Method.

- We can use the method when k is not an eigenvalue of the Interior Transmission Eigenvalue Problem:

TRANSMISSION PROBLEM WITH A SIGN-CHANGING COEFFICIENT

$$\left|\begin{array}{l}\text{Find } (k, v) \in \mathbb{C} \times \mathrm{H}_0^2(\Omega) \setminus \{0\} \text{ such that:}\\ \displaystyle\int_\Omega \frac{1}{1-n^2}(\Delta v + k^2 n^2 v)(\Delta v' + k^2 v') = 0, \quad \forall v' \in \mathrm{H}_0^2(\Omega).\end{array}\right.$$

- One of the goals is to prove that the set of transmission eigenvalues is at most discrete.
- This problem has been widely studied since 1986-1988 (`Bellis, Cakoni, Colton, Gintides, Guzina, Haddar, Kirsch, Kress, Monk, Päivärinta, Rynne, Sleeman, Sylvester...`) when $n > 1$ on Ω or $n < 1$ on Ω.

What happens when $1 - n^2$ changes sign?

A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

$$(\mathscr{F}_V) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^2_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \sigma \Delta v \Delta v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}^2_0(\Omega). \end{array} \right.$$

A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

$$(\mathscr{F}_V) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^2_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \sigma \Delta v \Delta v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}^2_0(\Omega). \end{array} \right.$$

✉ Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\operatorname{div}(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

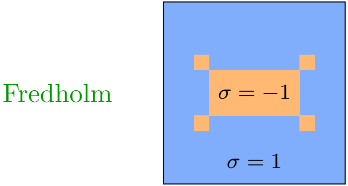
A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

$$(\mathscr{F}_V)\quad \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^2_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \sigma \Delta v \Delta v'}_{a(v,v')} = \underbrace{\langle f, v'\rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}^2_0(\Omega). \end{array}\right.$$

Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\mathrm{div}\,(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

THEOREM. The problem $(\mathscr{F}_V)$ is well-posed in the Fredholm sense as soon as σ does not change sign in a neighbourhood of $\partial\Omega$.



A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1 - \eta^2)^{-1}$ and we focus on the principal part:

$$(\mathscr{F}_V) \quad \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^2_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \sigma \Delta v \Delta v'}_{a(v,v')} = \underbrace{(f, v')_\Omega}_{l(v')} \quad \forall v' \in \mathrm{H}^2_0(\Omega). \end{array} \right.$$

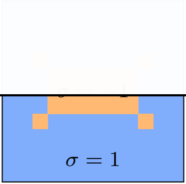
Ideas of the proof: We have

$$a(v, v) = (\sigma\Delta v, \Delta v)_\Omega.$$

✉ Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\mathrm{div}\,(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

Theorem. The problem $(\mathscr{F}_V)$ is well-posed in the Fredholm sense as soon as σ does not change sign in a neighbourhood of $\partial\Omega$.

Fredholm



A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

Find $v \in \mathrm{H}^2_0(\Omega)$ such that:

$$(\mathscr{F}_V) \quad \underbrace{\int_\Omega \sigma \Delta v \Delta v'}_{a(v,v')} = \underbrace{(f, v')_\Omega}_{\ell(v')}, \quad \forall v' \in \mathrm{H}^2_0(\Omega).$$

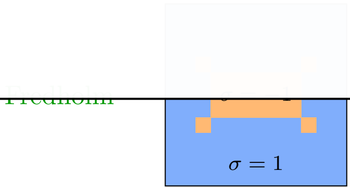
Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\operatorname{div}(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

THEOREM. The problem $(\mathscr{F}_V)$ is well-posed in the Fredholm sense as soon as σ does not change sign in a neighbourhood of $\partial\Omega$.

IDEAS OF THE PROOF: We have

$$a(v, v) = (\sigma\Delta v, \Delta v)_\Omega.$$

We would like to build $\mathtt{T} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that $\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v$



A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

IDEAS OF THE PROOF: We have

$$a(v,v) = (\sigma\Delta v, \Delta v)_\Omega.$$

We would like to build $\mathtt{T} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that $\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v$

so that $$a(v, \mathtt{T}v) = (\sigma\Delta v, \Delta(\mathtt{T}v))_\Omega = (\Delta v, \Delta v)_\Omega.$$

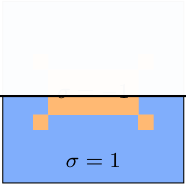
Find $v \in \mathrm{H}^2_0(\Omega)$ such that:

$$(\mathscr{F}_V) \quad \underbrace{\int_\Omega \sigma\Delta v\,\Delta v'}_{a(v,v')} = \underbrace{(f, v')_\Omega}_{\ell(v')} \quad \forall v' \in \mathrm{H}^2_0(\Omega).$$

Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\mathrm{div}\,(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

THEOREM. The problem $(\mathscr{F}_V)$ is well-posed in the Fredholm sense as soon as σ does not change sign in a neighbourhood of $\partial\Omega$.

Fredholm



A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

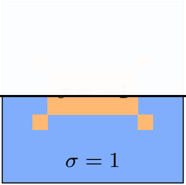
IDEAS OF THE PROOF: We have

$$a(v, v) = (\sigma\Delta v, \Delta v)_\Omega.$$

We would like to build $\mathtt{T} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that $\boxed{\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v}$

Not simple!

so that $$a(v, \mathtt{T}v) = (\sigma\Delta v, \Delta(\mathtt{T}v))_\Omega = (\Delta v, \Delta v)_\Omega.$$



A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1 - \eta^2)^{-1}$ and we focus on the principal part:

IDEAS OF THE PROOF: We have

$$a(v,v) = (\sigma\Delta v, \Delta v)_\Omega.$$

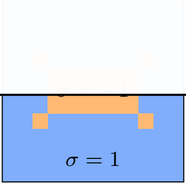
Not simple!

We would like to build $\mathtt{T} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that $\boxed{\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v}$

so that $$a(v,\mathtt{T}v) = (\sigma\Delta v, \Delta(\mathtt{T}v))_\Omega = (\Delta v, \Delta v)_\Omega.$$

① Let $w \in \mathrm{H}^1_0(\Omega)$ such that $\Delta w = \sigma^{-1}\Delta v$.

Fredholm



A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1-\eta^2)^{-1}$ and we focus on the principal part:

IDEAS OF THE PROOF: We have

$$a(v,v) = (\sigma\Delta v, \Delta v)_\Omega.$$

Not simple!

We would like to build $\mathtt{T} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that $\boxed{\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v}$

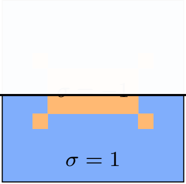
so that $$a(v,\mathtt{T}v) = (\sigma\Delta v, \Delta(\mathtt{T}v))_\Omega = (\Delta v, \Delta v)_\Omega.$$

---

❶ Let $w \in \mathrm{H}^1_0(\Omega)$ such that $\Delta w = \sigma^{-1}\Delta v$.

❷ Let $\zeta \in \mathscr{C}^\infty_0(\Omega)$. Define $\mathtt{T}v = \zeta w + (1-\zeta)v \in \mathrm{H}^2_0(\Omega)$.

Fredholm



A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

IDEAS OF THE PROOF: We have

$$a(v,v) = (\sigma\Delta v, \Delta v)_\Omega.$$

Not simple!

We would like to build $\mathtt{T} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that $\boxed{\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v}$

so that

$$a(v,\mathtt{T}v) = (\sigma\Delta v, \Delta(\mathtt{T}v))_\Omega = (\Delta v, \Delta v)_\Omega.$$

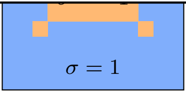
① Let $w \in \mathrm{H}^1_0(\Omega)$ such that $\Delta w = \sigma^{-1}\Delta v$.

② Let $\zeta \in \mathscr{C}^\infty_0(\Omega)$. Define $\mathtt{T}v = \zeta w + (1-\zeta)v \in \mathrm{H}^2_0(\Omega)$.

③ We find $a(v,\mathtt{T}v) = ([\zeta + \sigma(1-\zeta)]\Delta v, \Delta v)_\Omega + (Kv, v)_{\mathrm{H}^2_0(\Omega)}$

where $K : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ is compact.

Fredholm



A bilaplacian with a sign-changing coefficient

- We define $\sigma = (1-\eta^2)^{-1}$ and we focus on the principal part:

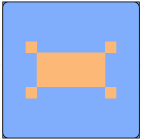
Ideas of the proof: We have

$$a(v,v) = (\sigma\Delta v, \Delta v)_\Omega.$$

Not simple!

We would like to build $\mathtt{T} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that $\boxed{\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v}$

so that $$a(v,\mathtt{T}v) = (\sigma\Delta v, \Delta(\mathtt{T}v))_\Omega = (\Delta v, \Delta v)_\Omega.$$

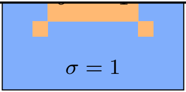


① Let $w \in \mathrm{H}^1_0(\Omega)$ such that $\Delta w = \sigma^{-1}\Delta v$.

② Let $\zeta \in \mathscr{C}^\infty_0(\Omega)$. Define $\mathtt{T}v = \zeta w + (1-\zeta)v \in \mathrm{H}^2_0(\Omega)$.

③ We find $a(v,\mathtt{T}v) = (\boxed{[\zeta + \sigma(1-\zeta)]}\Delta v, \Delta v)_\Omega + (Kv, v)_{\mathrm{H}^2_0(\Omega)}$

where $K : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ is compact.



A bilaplacian with a sign-changing coefficient

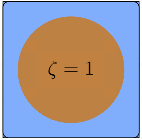
IDEAS OF THE PROOF: We have

$$a(v,v) = (\sigma\Delta v, \Delta v)_\Omega.$$

Not simple!

We would like to build $\mathtt{T} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that $\boxed{\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v}$

so that $$a(v,\mathtt{T}v) = (\sigma\Delta v, \Delta(\mathtt{T}v))_\Omega = (\Delta v, \Delta v)_\Omega.$$



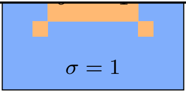
① Let $w \in \mathrm{H}^1_0(\Omega)$ such that $\Delta w = \sigma^{-1}\Delta v$.

② Let $\zeta \in \mathscr{C}^\infty_0(\Omega)$. Define $\mathtt{T}v = \zeta w + (1-\zeta)v \in \mathrm{H}^2_0(\Omega)$.

③ We find $a(v,\mathtt{T}v) = (\boxed{[\zeta + \sigma(1-\zeta)]}\Delta v, \Delta v)_\Omega + (Kv, v)_{\mathrm{H}^2_0(\Omega)}$

where $K : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ is compact.

Fredholm



A bilaplacian with a sign-changing coefficient

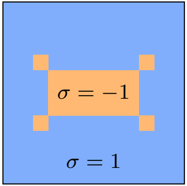
- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

$$(\mathscr{F}_V) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^2_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \sigma \Delta v \Delta v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}^2_0(\Omega). \end{array} \right.$$

Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\operatorname{div}(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

THEOREM. The problem $(\mathscr{F}_V)$ is well-posed in the Fredholm sense as soon as σ does not change sign in a neighbourhood of $\partial\Omega$.

Fredholm



A bilaplacian with a sign-changing coefficient

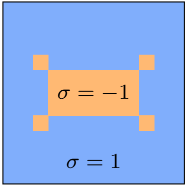
- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

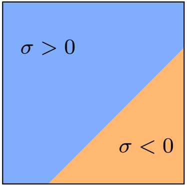
$$(\mathscr{F}_V)\ \left|\ \begin{array}{l}\text{Find } v \in \mathrm{H}^2_0(\Omega) \text{ such that:}\\ \underbrace{\int_\Omega \sigma \Delta v \Delta v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}^2_0(\Omega).\end{array}\right.$$

Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\mathrm{div}\,(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

... but $(\mathscr{F}_V)$ can be ill-posed (not Fredholm) when σ changes sign "on $\partial\Omega$" $\Rightarrow$ work with `J. Firozaly`.

Fredholm





Not always Fredholm

1. The coerciveness issue for the scalar case

2. A new functional framework in the critical interval

3. Study of Maxwell's equations

4. The $\mathtt{T}$-coercivity method for the Interior Transmission Problem

Conclusions

Scalar problem outside the critical interval $\mathrm{div}\,(\mu^{-1}\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$

- ♠ Concerning the approximation of the solution, in practice, usual methods converge. Only partial proofs are available.
- ♠ In 3D, are the interval obtained optimal?

Conclusions

Scalar problem outside the critical interval $\mathrm{div}\,(\mu^{-1}\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$

- ♠ Concerning the approximation of the solution, in practice, usual methods converge. Only partial proofs are available.
- ♠ In 3D, are the interval obtained optimal?

Scalar problem inside the critical interval $\mathrm{div}\,(\mu^{-1}\nabla\cdot) : \mathrm{V}^+(\Omega) \to \mathrm{V}_\beta(\Omega)^*$

- ♠ Interesting questions of numerical analysis to approximate the solution. What happens in 3D (edge, conical tip,...)? ⇒ PhD thesis of `C. Carvalho`.

Conclusions

Scalar problem outside the critical interval $\mathrm{div}\,(\mu^{-1}\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$

- ♠ Concerning the approximation of the solution, in practice, usual methods converge. Only partial proofs are available.
- ♠ In 3D, are the interval obtained optimal?

Scalar problem inside the critical interval $\mathrm{div}\,(\mu^{-1}\nabla\cdot) : \mathrm{V}^+(\Omega) \to \mathrm{V}_\beta(\Omega)^*$

- ♠ Interesting questions of numerical analysis to approximate the solution. What happens in 3D (edge, conical tip,...)? ⇒ PhD thesis of `C. Carvalho`.

Maxwell's equations $\mathbf{curl}\,(\varepsilon^{-1}\mathbf{curl}\,\cdot) : \mathbf{V}_T(\mu;\,\Omega) \to \mathbf{V}_T(\mu;\,\Omega)^*$

- ♠ Convergence of an edge element method has to be studied.
- ♠ Can we develop a new functional framework inside the critical interval?

Conclusions

Scalar problem outside the critical interval $\mathrm{div}\,(\mu^{-1}\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$

- ♠ Concerning the approximation of the solution, in practice, usual methods converge. Only partial proofs are available.
- ♠ In 3D, are the interval obtained optimal?

Scalar problem inside the critical interval $\mathrm{div}\,(\mu^{-1}\nabla\cdot) : \mathrm{V}^+(\Omega) \to \mathrm{V}_\beta(\Omega)^*$

- ♠ Interesting questions of numerical analysis to approximate the solution. What happens in 3D (edge, conical tip,...)? ⇒ PhD thesis of `C. Carvalho`.

Maxwell's equations $\mathbf{curl}\,(\varepsilon^{-1}\mathbf{curl}\,\cdot) : \mathbf{V}_T(\mu;\,\Omega) \to \mathbf{V}_T(\mu;\,\Omega)^*$

- ♠ Convergence of an edge element method has to be studied.
- ♠ Can we develop a new functional framework inside the critical interval?

Interior Transmission Eigenvalue Problem $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$

- ♠ Can we find a criterion on σ and on the geometry to ensure that $\Delta(\sigma\Delta\cdot)$ is Fredholm? Many questions remain open for the ITEP...

Open questions

- Our new model in the critical interval raises a lot of questions, related to the physics of plasmonics and metamaterials.

  Can we observe this black-hole effect in practice? For a rounded corner, "the solution" seems unstable with respect to the rounding parameter...

- The case $\kappa_\sigma = -1$ (the most interesting for applications) is not understood yet: singularities appear all over the interface.

  $\Rightarrow$ Is there a functional framework in which $(\mathscr{P})$ is well-posed?

- More generally, can we reconsider the homogenization process to take into account interfacial phenomena?

  $\Rightarrow$ *METAMATH project (ANR) directed by* `S. Fliss` and PhD thesis of `V. Vinoles`.

- What happens in time-domain regime? Is the limiting amplitude principle still valid?

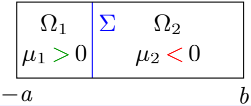
  $\Rightarrow$ PhD thesis of `M. Cassier`.

Thank you for your attention!!!

Summary of the results for the 2D cavity

Problem

$$(\mathscr{P}) \quad \left|\begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ -\mathrm{div}\,(\mu^{-1}\nabla u) = f \qquad \text{in } \Omega. \end{array}\right.$$



Proposition. The operator $A = \mathrm{div}\,(\mu^{-1}\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is an isomorphism if and only $\kappa_\mu \in \mathbb{C}^*\setminus\mathscr{S}$ with $\mathscr{S} = \{-\tanh(n\pi a)/\tanh(n\pi b),\ n \in \mathbb{N}^*\} \cup \{-1\}$. For $\kappa_\mu = -\tanh(n\pi a)/\tanh(n\pi b)$, we have $\ker A = \mathrm{span}\,\varphi_n$ with

$$\varphi_n(x,y) = \begin{cases} \sinh(n\pi(x+a))\sin(n\pi y) & \text{on } \Omega_1 \\ -\dfrac{\sinh(n\pi a)}{\sinh(n\pi b)}\sinh(n\pi(x-b))\sin(n\pi y) & \text{on } \Omega_2 \end{cases}.$$

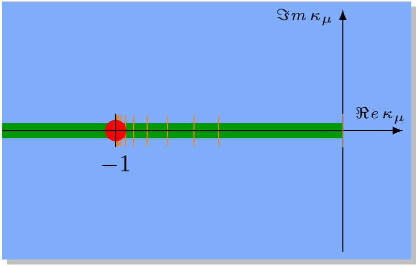
Results

🟦 For $\kappa_\mu \in \mathbb{C}\setminus\mathbb{R}_-$, $(\mathscr{P})$ well-posed (Lax-Milgram)

🟩 For $\kappa_\mu \in \mathbb{R}^*_-\setminus\mathscr{S}$, $(\mathscr{P})$ well-posed

🟧 For $\kappa_\mu \in \mathscr{S}\setminus\{-1\}$, $(\mathscr{P})$ is well-posed in the Fredholm sense with a one dimension kernel

● $\kappa_\mu = -1$, $(\mathscr{P})$ ill-posed in $\mathrm{H}^1_0(\Omega)$



The blinking eigenvalue

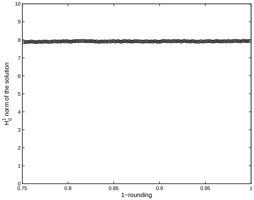
- We approximate by a FEM "the solution" of the problem

Find $u_\delta \in \mathrm{H}_0^1(\Omega_\delta)$ s.t.:
$-\mathrm{div}\,(\mu_\delta^{-1}\nabla u_\delta) = f \quad \text{in } \Omega_\delta.$.

$\kappa_\mu = -0.9999$ (outside the critical interval)

(. . .)

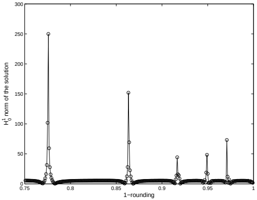
(. . .)



$\kappa_\mu = -1.0001$ (inside the critical interval)

(. . .)

(. . .)



The result for the electric field

Consider $\boldsymbol{F} \in \mathbf{L}^2(\Omega)$ such that $\operatorname{div} \boldsymbol{F} \in \mathbf{L}^2(\Omega)$.

THEOREM. Suppose

$$(\varphi, \varphi') \mapsto \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\varphi' \text{ is T-coercive on } \mathrm{H}^1_0(\Omega); \qquad (\mathcal{A}_\varepsilon)$$

$$(\varphi, \varphi') \mapsto \int_\Omega \mu \nabla\varphi \cdot \nabla\varphi' \text{ is T-coercive on } \mathrm{H}^1(\Omega)/\mathbb{R}. \qquad (\mathcal{A}_\mu)$$

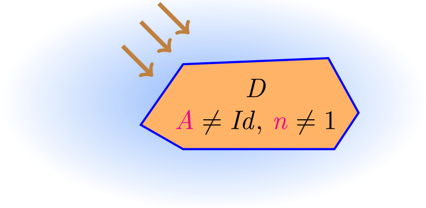
Then, the problem for the electric field

$$\left| \begin{array}{ll} \text{Find } \boldsymbol{E} \in \mathbf{H}(\mathbf{curl}\,;\Omega) \text{ such that:} & \\ \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}) - \omega^2 \varepsilon \boldsymbol{E} = \boldsymbol{F} & \text{in } \Omega \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{array} \right.$$

is well-posed for all $\omega \in \mathbb{C} \backslash \mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

What is the ITEP?

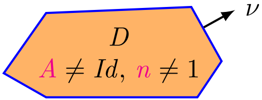
- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.



What is the ITEP?

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem:

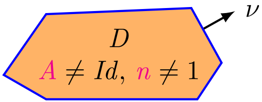


What is the ITEP?

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem:
  ☛ u is the total field in D

$$\left| \begin{array}{rcll} \mathrm{div}\,(A\nabla u) + k^2 n u & = & 0 & \text{in } D \\ & & & \\ & & & \end{array} \right.$$

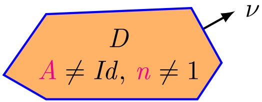


What is the ITEP?

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem:
  - ☛ u is the total field in D
  - ☛ w is the incident field in D

$$\left|\begin{array}{lcll} \operatorname{div}(A\nabla u)+k^2 n u & = & 0 & \text{in } D \\ \Delta w + k^2 w & = & 0 & \text{in } D \\ & & & \\ & & & \end{array}\right.$$

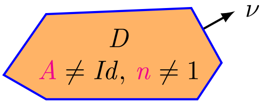


What is the ITEP?

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem:
  ☛ u is the total field in D ☛ w is the incident field in D

$$\left|\begin{array}{lcll} \operatorname{div}(A\nabla u) + k^2 n u & = & 0 & \text{in } D \\ \Delta w + k^2 w & = & 0 & \text{in } D \\ \boxed{\begin{array}{lcll} u - w & = & 0 & \text{on } \partial D \\ \nu \cdot A\nabla u - \nu \cdot \nabla w & = & 0 & \text{on } \partial D. \end{array}} \end{array}\right.$$



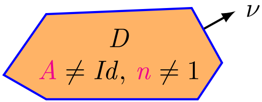
TRANSMISSION CONDITIONS ON ∂D

What is the ITEP?

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem:
  ☛ u is the total field in D ☛ w is the incident field in D

$$\left|\begin{array}{lcll} \multicolumn{4}{l}{\text{Find } (u,w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \ \text{ such that:}} \\ \mathrm{div}\,(A\nabla u) + k^2 n u & = & 0 & \text{in } D \\ \Delta w + k^2 w & = & 0 & \text{in } D \\ u - w & = & 0 & \text{on } \partial D \\ \nu \cdot A\nabla u - \nu \cdot \nabla w & = & 0 & \text{on } \partial D. \end{array}\right.$$



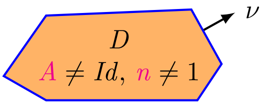
TRANSMISSION CONDITIONS ON ∂D

What is the ITEP?

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem:
  - ☛ u is the total field in D
  - ☛ w is the incident field in D

$$\left|\begin{array}{llll} \text{Find } (u,w)\in \mathrm{H}^1(D)\times\mathrm{H}^1(D) \text{ such that:} & & & \\ \operatorname{div}(A\nabla u)+k^2 n u & = & 0 & \text{in } D \\ \Delta w + k^2 w & = & 0 & \text{in } D \\ u-w & = & 0 & \text{on } \partial D \\ \nu\cdot A\nabla u - \nu\cdot\nabla w & = & 0 & \text{on } \partial D. \end{array}\right.$$



TRANSMISSION CONDITIONS ON ∂D

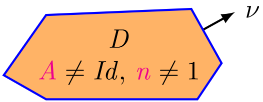
DEFINITION. Values of $k\in\mathbb{C}$ for which this problem has a nontrivial solution (u,w) are called transmission eigenvalues.

What is the ITEP?

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem:
  ☛ u is the total field in D ☛ w is the incident field in D

$$\left|\begin{array}{lcll} \text{Find } (u,w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \text{ such that:} & & & \\ \operatorname{div}(A\nabla u) + k^2 n u & = & 0 & \text{in } D \\ \Delta w + k^2 w & = & 0 & \text{in } D \\ \boxed{\begin{array}{lcll} u - w & = & 0 & \text{on } \partial D \\ \nu \cdot A\nabla u - \nu \cdot \nabla w & = & 0 & \text{on } \partial D. \end{array}} & & & \end{array}\right.$$



TRANSMISSION CONDITIONS ON ∂D

DEFINITION. Values of $k \in \mathbb{C}$ for which this problem has a nontrivial solution (u,w) are called transmission eigenvalues.

- One of the goals is to prove that the set of transmission eigenvalues is at most discrete.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X}\backslash\{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} \quad = \quad k^2 \quad \int_\Omega (n u\overline{u'} - w\overline{w'}),$$

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X}\backslash\{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\int_\Omega A \nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} \quad = \quad k^2 \quad \int_\Omega (n u \overline{u'} - w \overline{w'}),$$

with $\mathrm{X} = \{(u, w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \,|\, u - w \in \mathrm{H}^1_0(\Omega)\}$.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X}\backslash\{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\int_\Omega A \nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} \;=\; k^2 \int_\Omega (n u \overline{u'} - w \overline{w'}),$$

with $\mathrm{X} = \{(u, w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \,|\, u - w \in \mathrm{H}^1_0(\Omega)\}$.

- This is a non standard eigenvalue problem.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X}\backslash\{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\boxed{\int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'}} = k^2 \int_\Omega (n u\overline{u'} - w\overline{w'}),$$

not coercive on X

with $\mathrm{X} = \{(u, w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \,|\, u - w \in \mathrm{H}^1_0(\Omega)\}$.

- This is a non standard eigenvalue problem.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X}\backslash\{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\boxed{\int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'}} \;=\; k^2 \boxed{\int_\Omega (n u\overline{u'} - w\overline{w'})},$$

not coercive on X not an inner product on X

with $\mathrm{X} = \{(u, w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \,|\, u - w \in \mathrm{H}^1_0(\Omega)\}$.

- This is a non standard eigenvalue problem.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u,w) \in \mathrm{X}\backslash\{0\}$ such that, for all $(u',w') \in \mathrm{X}$,

$$\boxed{\int_\Omega A\nabla u\cdot\overline{\nabla u'} - \nabla w\cdot\overline{\nabla w'}} \;=\; k^2\,\boxed{\int_\Omega (n u\overline{u'} - w\overline{w'})},$$

not coercive on X　　　　not an inner product on X

with $\mathrm{X} = \{(u,w) \in \mathrm{H}^1(\Omega)\times\mathrm{H}^1(\Omega)\,|\,u-w \in \mathrm{H}^1_0(\Omega)\}$.

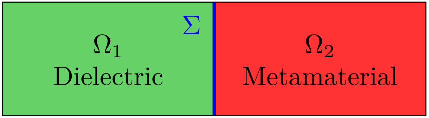
- This is a non standard eigenvalue problem.

- We want to highlight an

Idea: Analogy with the transmission problem between a dielectric and a double negative metamaterial...

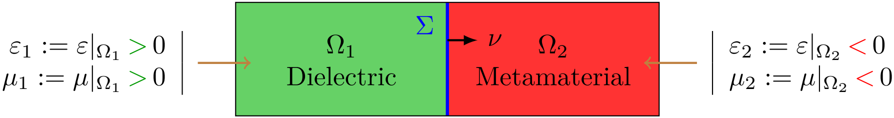
Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain Ω of $\mathbb{R}^2$:



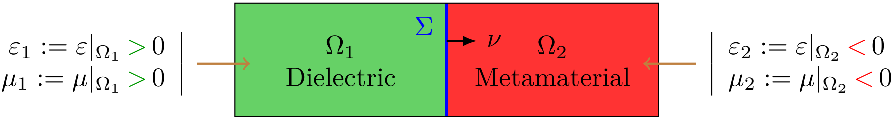
Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain Ω of $\mathbb{R}^2$:



Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain Ω of $\mathbb{R}^2$:



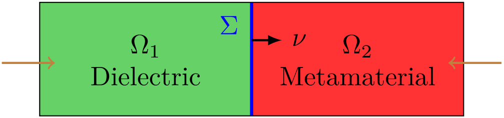
- Eigenvalue problem for E_z in 2D:

$$\boxed{\left|\begin{array}{l}\text{Find } v \in \mathrm{H}^1_0(\Omega) \setminus \{0\} \text{ such that:}\\ \operatorname{div}(\mu^{-1}\,\nabla v) + k^2 \varepsilon v = 0 \quad \text{ in } \Omega.\end{array}\right.}$$

Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain Ω of $\mathbb{R}^2$:

$$\begin{array}{l} \varepsilon_1 := \varepsilon|_{\Omega_1} > 0 \\ \mu_1 := \mu|_{\Omega_1} > 0 \end{array}$$



$$\begin{array}{l} \varepsilon_2 := \varepsilon|_{\Omega_2} < 0 \\ \mu_2 := \mu|_{\Omega_2} < 0 \end{array}$$

- Eigenvalue problem for E_z in 2D:

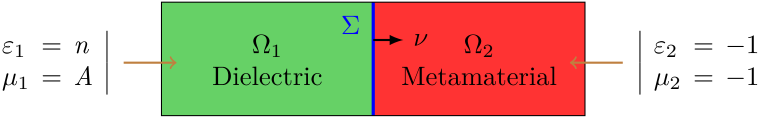
$$\left| \begin{array}{l} \text{Find } v \in \mathrm{H}_0^1(\Omega) \setminus \{0\} \text{ such that:} \\ \operatorname{div}(\mu^{-1}\,\nabla v) + k^2 \varepsilon v = 0 \quad \text{in } \Omega. \end{array} \right.$$

- k is a transmission eigenvalue if and only if there exists $v \in \mathrm{H}_0^1(\Omega)\setminus\{0\}$ such that, for all $v' \in \mathrm{H}_0^1(\Omega)$,

$$\int_{\Omega_1} \mu_1^{-1} \nabla v \cdot \overline{\nabla v'} - \int_{\Omega_2} |\mu_2|^{-1} \nabla v \cdot \overline{\nabla v'} = k^2 \left(\int_{\Omega_1} \varepsilon_1 v \overline{v'} - \int_{\Omega_2} |\varepsilon_2| v \overline{v'} \right).$$

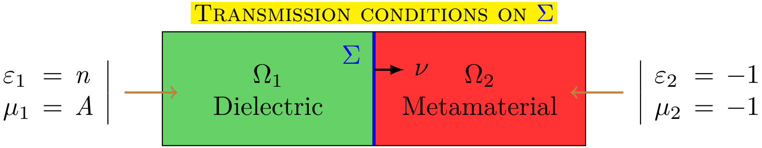
Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:



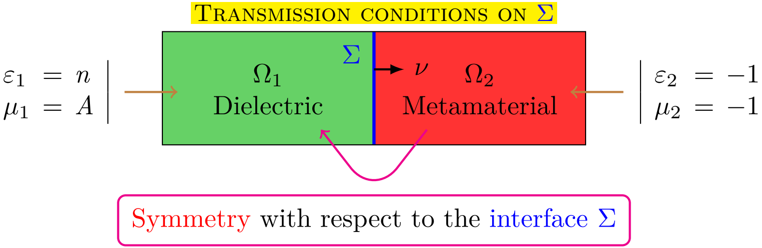
Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:



Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:



TRANSMISSION CONDITIONS ON Σ

$$\left.\begin{aligned}\varepsilon_1 &= n\\ \mu_1 &= A\end{aligned}\right|$$

Ω_1 Dielectric

Σ

ν

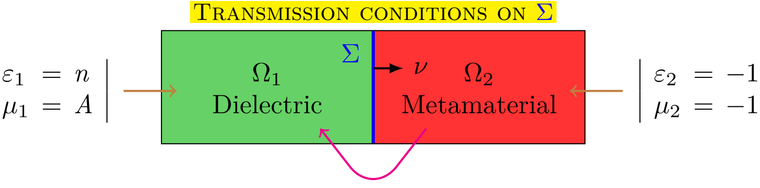
Ω_2 Metamaterial

$$\left|\begin{aligned}\varepsilon_2 &= -1\\ \mu_2 &= -1\end{aligned}\right.$$

Symmetry with respect to the interface Σ

Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:



TRANSMISSION CONDITIONS ON Σ

$$\left.\begin{array}{lcl}\varepsilon_1 & = & n\\ \mu_1 & = & A\end{array}\right|$$

Ω_1
Dielectric

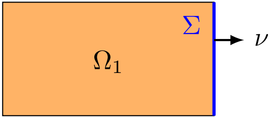
Σ ν

Ω_2
Metamaterial

$$\left|\begin{array}{lcl}\varepsilon_2 & = & -1\\ \mu_2 & = & -1\end{array}\right.$$

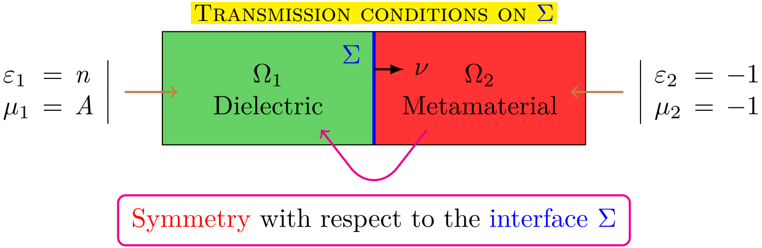
Symmetry with respect to the interface Σ

- We obtain a problem analogous to the ITEP in Ω_1:



Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:



TRANSMISSION CONDITIONS ON Σ

$$\begin{array}{rcl|} \varepsilon_1 & = & n \\ \mu_1 & = & A \end{array}$$

Σ ν

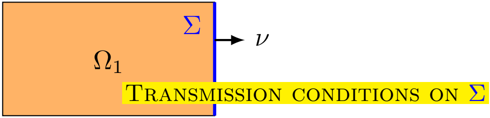
Ω_1 Dielectric

Ω_2 Metamaterial

$$\left| \begin{array}{rcl} \varepsilon_2 & = & -1 \\ \mu_2 & = & -1 \end{array} \right.$$

Symmetry with respect to the interface Σ

- We obtain a problem analogous to the ITEP in Ω_1:



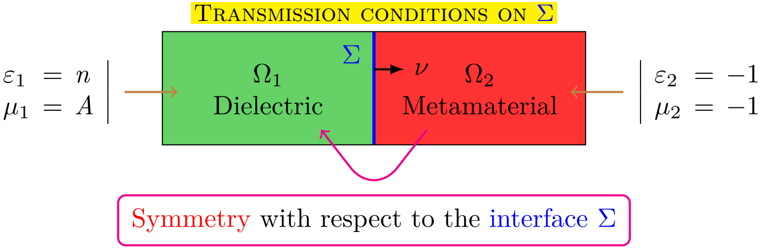
Σ ν

Ω_1

TRANSMISSION CONDITIONS ON Σ

Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:



TRANSMISSION CONDITIONS ON Σ

$\varepsilon_1 = n$
$\mu_1 = A$

Ω_1
Dielectric

Σ

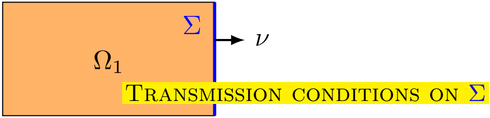
ν

Ω_2
Metamaterial

$\varepsilon_2 = -1$
$\mu_2 = -1$

Symmetry with respect to the interface Σ

- We obtain a problem analogous to the ITEP in Ω_1:



Σ

ν

Ω_1

TRANSMISSION CONDITIONS ON Σ

- The interface Σ in the DMTEP plays the role of the boundary $\partial\Omega$ in the ITEP.

Study of the ITEP

- Define on $\mathrm{X} \times \mathrm{X}$ the sesquilinear form

$$a((u,w),(u',w')) = \int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} - k^2(nu\overline{u'} - w\overline{w'}),$$

with $\mathrm{X} = \{(u,w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \,|\, u - w \in \mathrm{H}^1_0(\Omega)\}$.

Study of the ITEP

- Define on $\mathrm{X} \times \mathrm{X}$ the sesquilinear form

$$a((u,w),(u',w')) = \int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} - k^2(nu\overline{u'} - w\overline{w'}),$$

with $\mathrm{X} = \{(u,w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \,|\, u - w \in \mathrm{H}^1_0(\Omega)\}$.

- Introduce the isomorphism $\mathtt{T}(u,w) = (u - 2w, -w)$.

Study of the ITEP

- Define on $\mathrm{X} \times \mathrm{X}$ the sesquilinear form

$$a((u,w),(u',w')) = \int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} - k^2(nu\overline{u'} - w\overline{w'}),$$

with $\mathrm{X} = \{(u,w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \,|\, u - w \in \mathrm{H}^1_0(\Omega)\}$.

- Introduce the isomorphism $\mathtt{T}(u,w) = (u - 2w, -w)$.

- For $k \in \mathbb{R}i \backslash \{0\}$, $A > Id$ and $n > 1$, one finds

$$\Re e\, a((u,w), \mathtt{T}(u,w)) \geq C\,(\|u\|^2_{\mathrm{H}^1(\Omega)} + \|w\|^2_{\mathrm{H}^1(\Omega)}), \quad \forall (u,w) \in \mathrm{X}.$$

Study of the ITEP

- Define on $\mathrm{X}\times\mathrm{X}$ the sesquilinear form

$$a((u,w),(u',w')) = \int_\Omega A\nabla u\cdot\overline{\nabla u'} - \nabla w\cdot\overline{\nabla w'} - k^2(nu\overline{u'} - w\overline{w'}),$$

with $\mathrm{X} = \{(u,w)\in \mathrm{H}^1(\Omega)\times\mathrm{H}^1(\Omega)\,|\,u-w\in\mathrm{H}^1_0(\Omega)\}$.

- Introduce the isomorphism $\mathtt{T}(u,w) = (u-2w,-w)$.

- For $k\in\mathbb{R}i\backslash\{0\}$, $A > Id$ and $n>1$, one finds

$$\Re e\, a((u,w),\mathtt{T}(u,w)) \geq C\,(\|u\|^2_{\mathrm{H}^1(\Omega)} + \|w\|^2_{\mathrm{H}^1(\Omega)}),\quad \forall(u,w)\in\mathrm{X}.$$

- Using the analytic Fredholm theorem, one deduces the

PROPOSITION. Suppose that $A > Id$ and $n>1$. Then the set of transmission eigenvalues is discrete and countable.

Study of the ITEP

- Define on X × X the sesquilinear form

$$a((u,w),(u',w')) = \int_\Omega A\nabla u\cdot\overline{\nabla u'} - \nabla w\cdot\overline{\nabla w'} - k^2(nu\overline{u'} - w\overline{w'}),$$

with $\mathrm{X} = \{(u,w)\in \mathrm{H}^1(\Omega)\times \mathrm{H}^1(\Omega)\,|\,u-w\in \mathrm{H}^1_0(\Omega)\}$.

- Introduce the isomorphism $\mathtt{T}(u,w) = (u-2w,-w)$.

- For $k \in \mathbb{R}i\backslash\{0\}$, $A > Id$ and $n > 1$, one finds

$$\Re e\, a((u,w),\mathtt{T}(u,w)) \geq C\,(\|u\|^2_{\mathrm{H}^1(\Omega)} + \|w\|^2_{\mathrm{H}^1(\Omega)}),\quad \forall (u,w)\in \mathrm{X}.$$

- Using the analytic Fredholm theorem, one deduces the

PROPOSITION. Suppose that $A > Id$ and $n > 1$. Then the set of transmission eigenvalues is discrete and countable.

- This result can be extended to situations where $A - Id$ and $n-1$ change sign in Ω working with $\mathtt{T}(u,w) = (u-2\chi w,-w)$.