

Model reduction for the Dirichlet Laplacian in thin domains

Lucas Chesnel¹

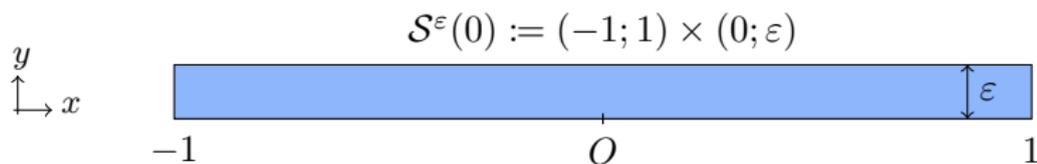
Collaboration with S.A. Nazarov².

¹Idefix team, Inria/Institut Polytechnique de Paris/EDF, France

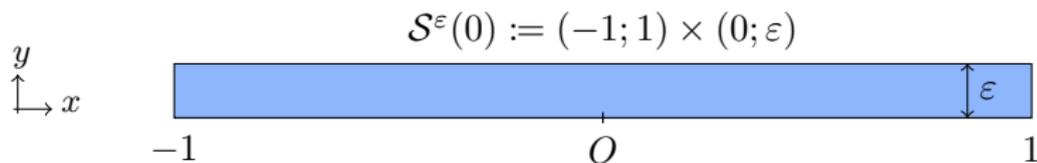
²FMM, St. Petersburg State University, Russia

The logo for Inria, featuring the word "Inria" in a stylized, red, cursive script.

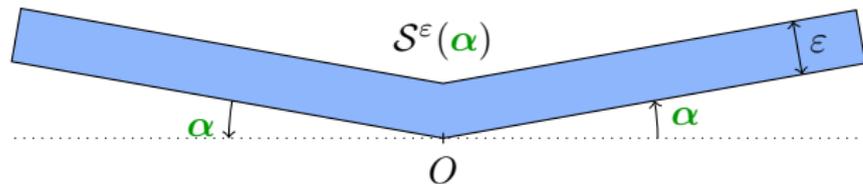
- ▶ Motivated by the study of **quantum waveguides**, we want to investigate the properties of the **Dirichlet Laplacian** in **thin** domains.
- ▶ For $\varepsilon > 0$ small, define the **thin straight** strip



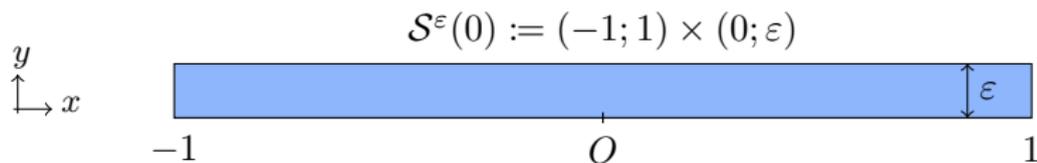
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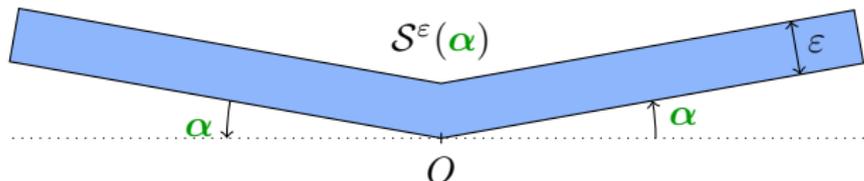
as well as the **thin broken** strip of angle α



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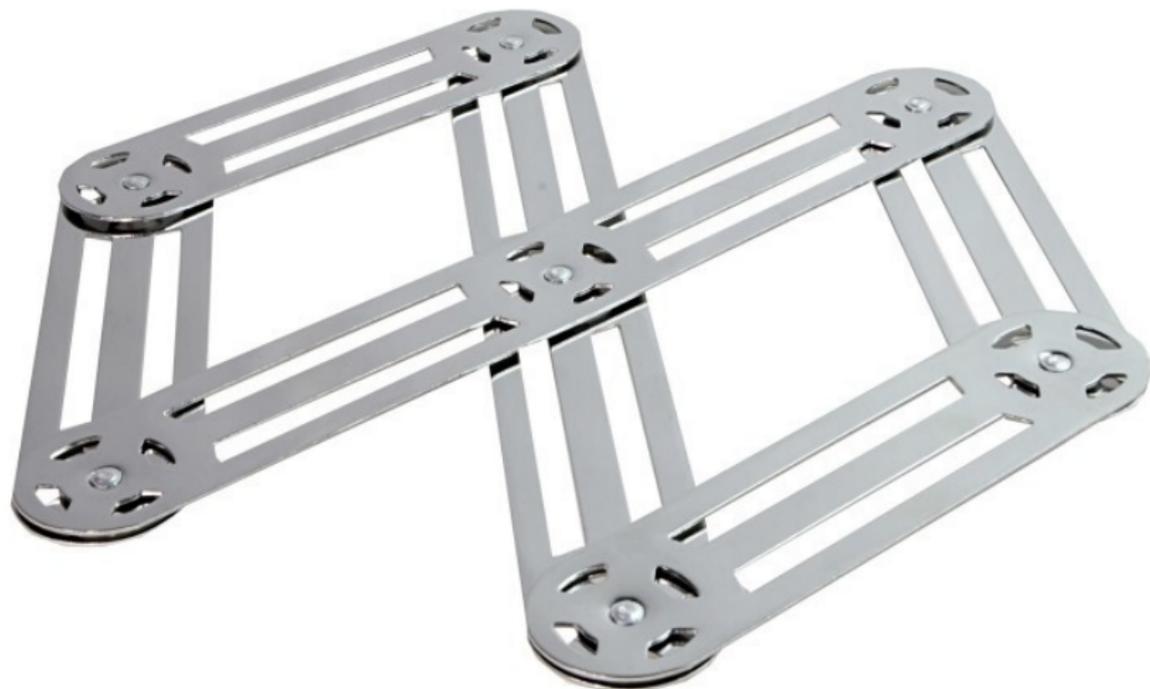
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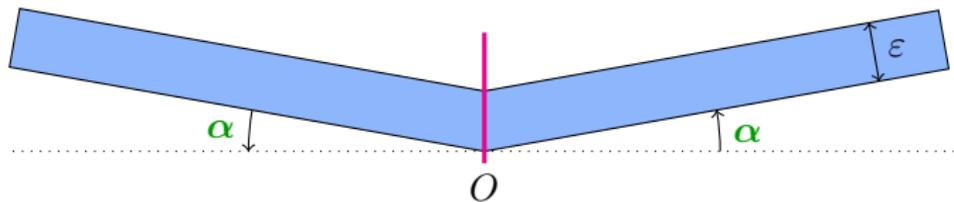
- ▶ Consider the **spectral** problem for the Dirichlet Laplacian

$$\begin{cases} -\Delta u = \lambda^\varepsilon u & \text{in } \mathcal{S}^\varepsilon(\alpha) \\ u = 0 & \text{on } \partial\mathcal{S}^\varepsilon(\alpha). \end{cases}$$

- ▶ Another **important** example of application:

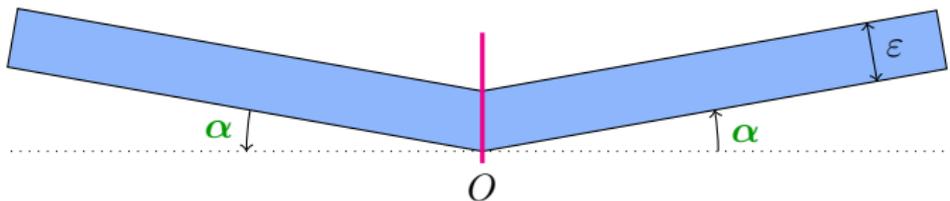


- ▶ Let us exploit **symmetry** wrt to the line $x = 0$



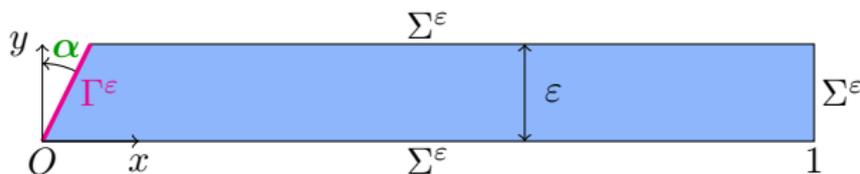
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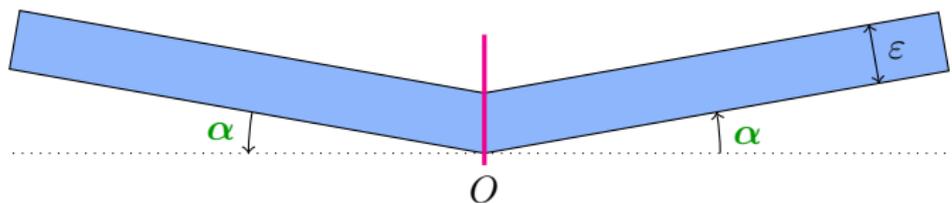
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the spectral problem with **mixed** BCs

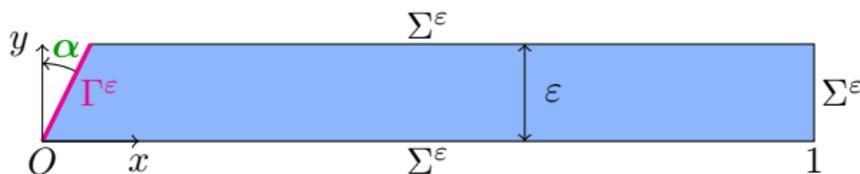
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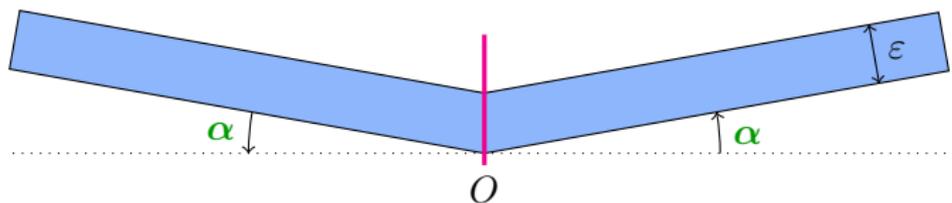
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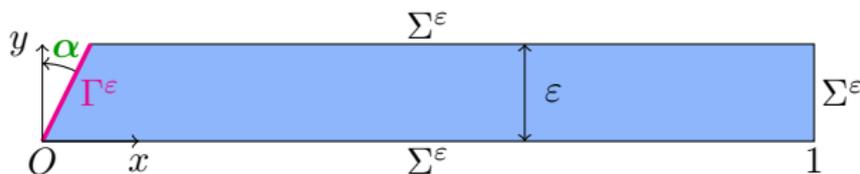
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- ▶ Its spectrum forms a sequence of **normal** eigenvalues

$$0 < \lambda_1^\epsilon < \lambda_2^\epsilon \leq \dots \leq \lambda_n^\epsilon \leq \dots \rightarrow +\infty.$$

► For $\alpha = 0$, T^ε is a rectangle \Rightarrow **explicit computations** can be done.

For the first eigenvalues, one obtains, for $n \in \mathbb{N}$,

$$\lambda_n^\varepsilon = \frac{\pi^2}{\varepsilon^2} + (n + 1/2)^2 \pi^2, \quad u_n^\varepsilon(x, y) = \cos(\pi(n + 1/2)x) \sin(\pi y/\varepsilon).$$



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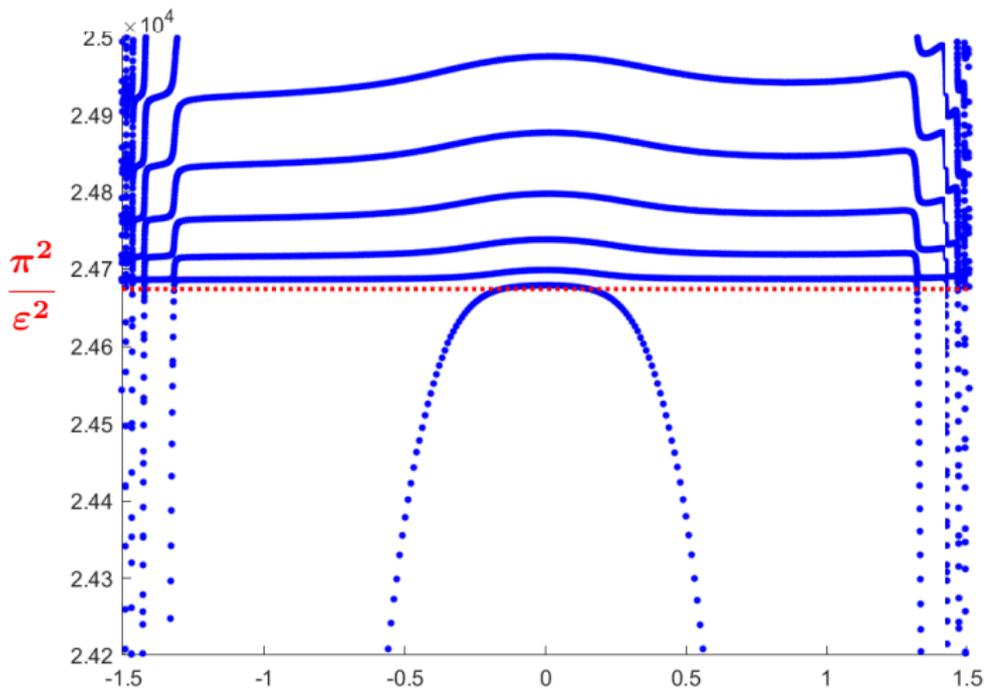
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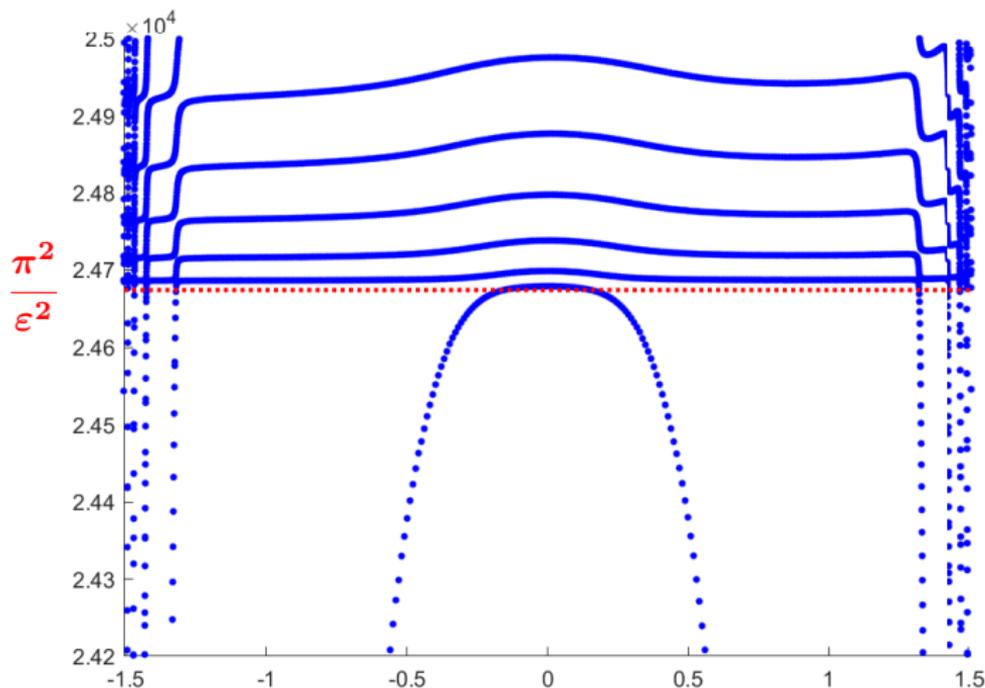
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- ▶ For $\alpha \neq 0$, let us do **numerics** with **Freefem++**.

- Spectrum of $(\mathcal{P}^\varepsilon)$ with respect to $\alpha \in (-\pi/2; \pi/2)$ for $\varepsilon = 0.02$:

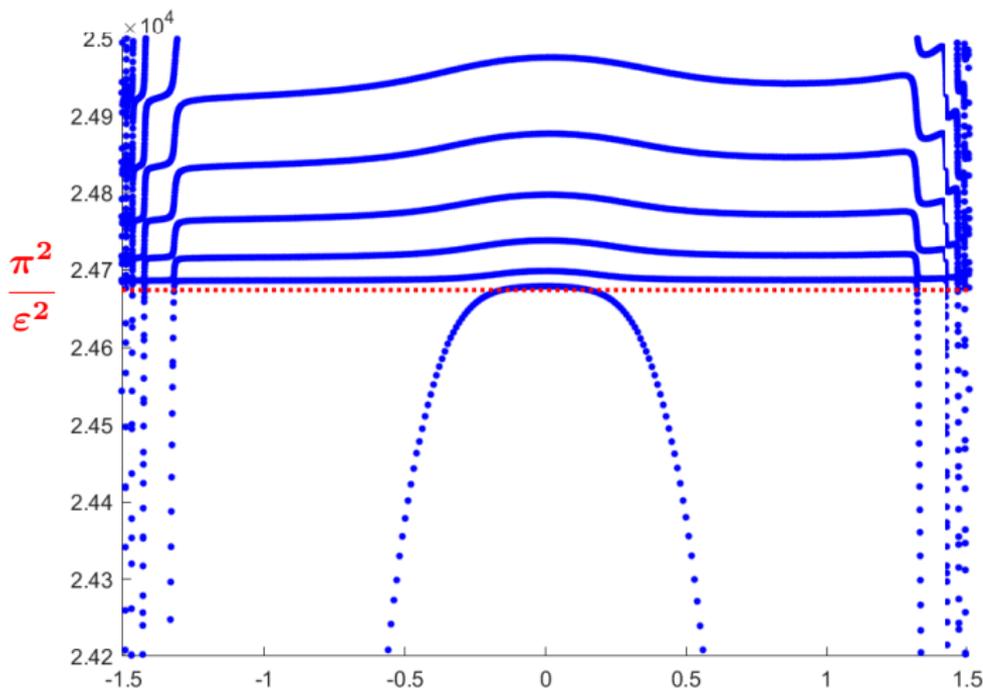


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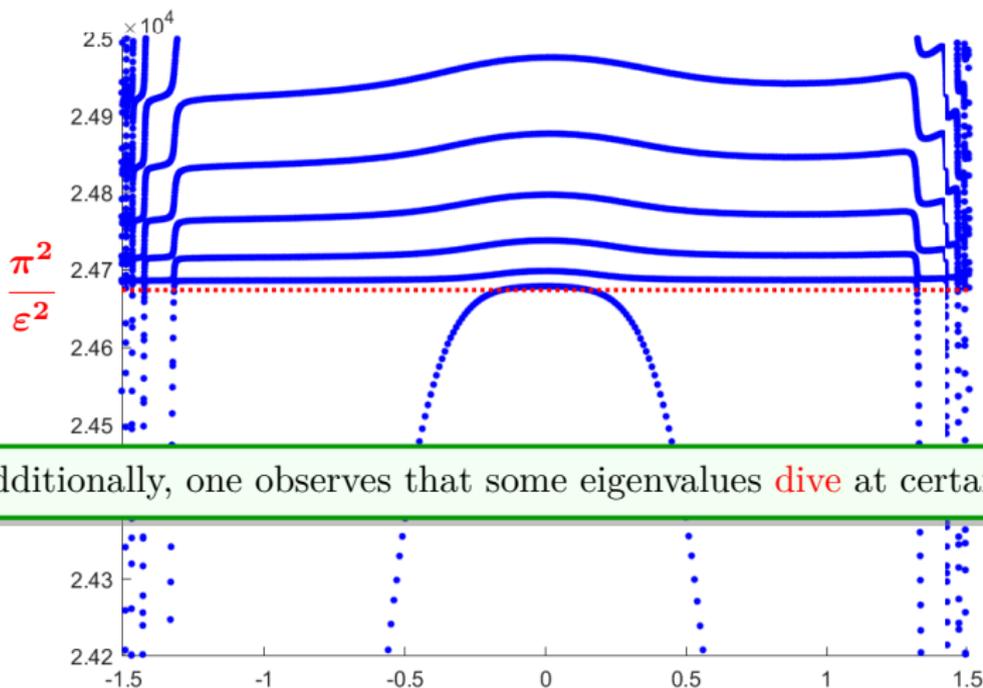
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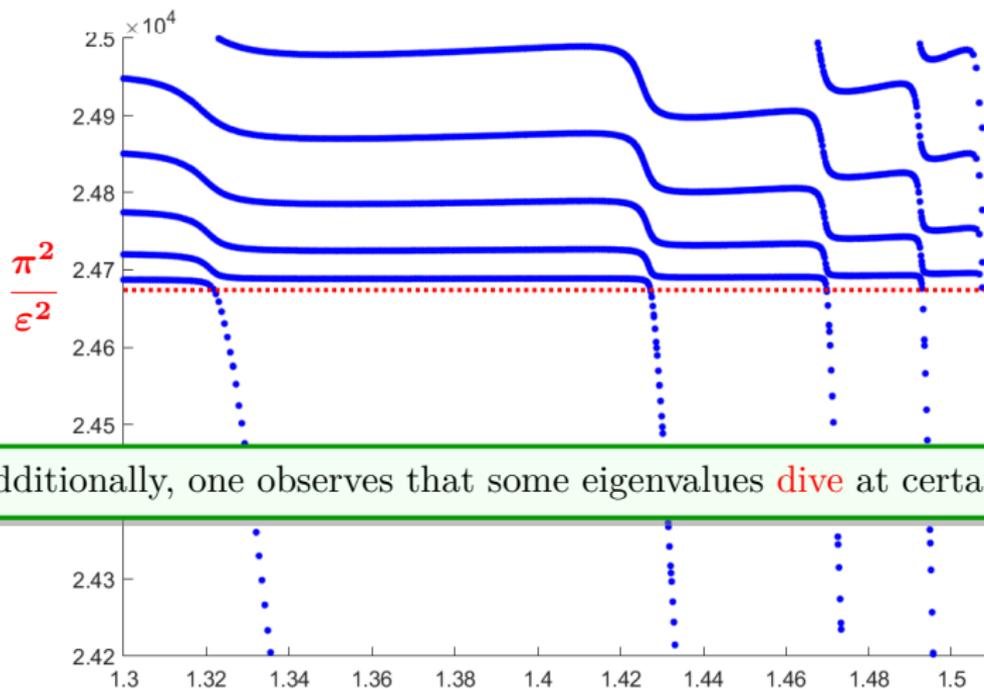
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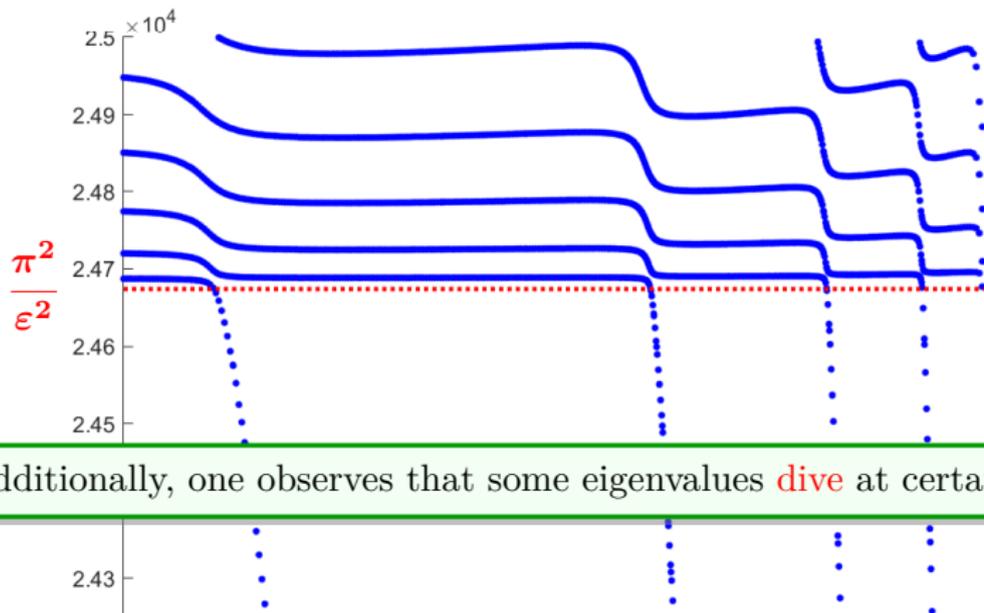
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Goal of the talk

We wish to explain this phenomenon of **diving eigenvalue** and **characterize the angles** at which it occurs.

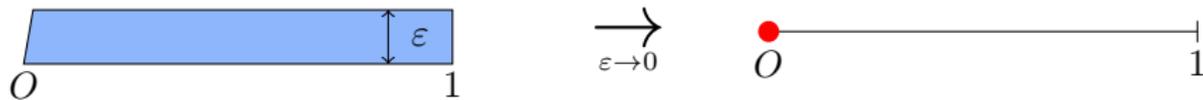
Outline of the talk

- 1 Preparatory work
- 2 Asymptotic analysis at a fixed α
- 3 Model problems at the critical angles
- 4 Spectral breathing in periodic waveguides

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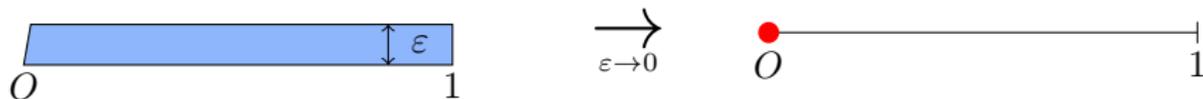
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- ▶ Roughly speaking, at the limit $\varepsilon \rightarrow 0$, we will obtain a **1D model problem**, with an **explicit dependence on α** , on the segment $I := (0; 1)$.



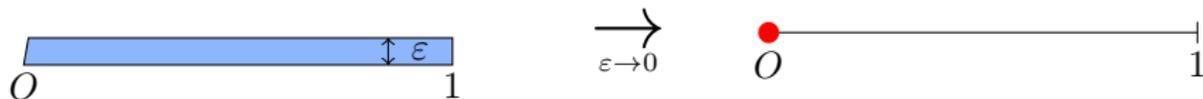
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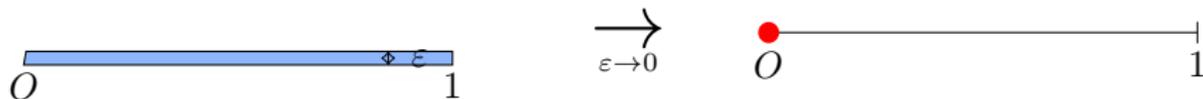
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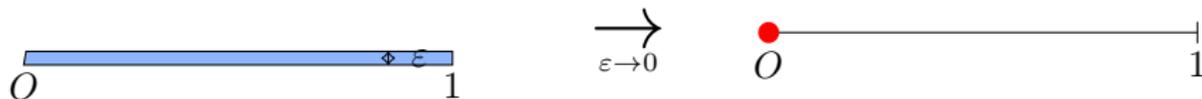
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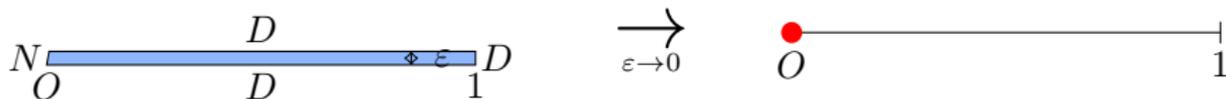
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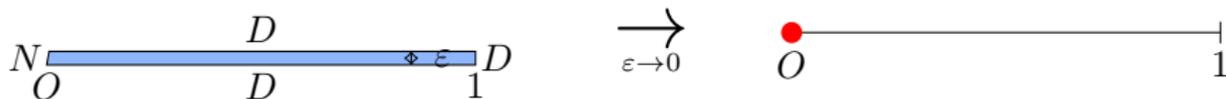


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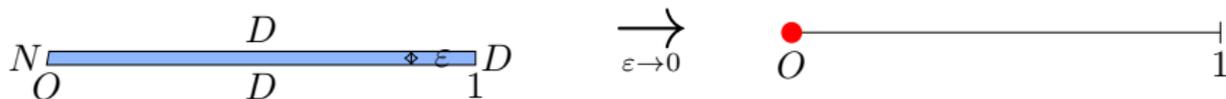
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Which condition should we impose **at O** ?

- ▶ To identify it, we will use techniques of **matched asymptotic expansions** (see **Post 05**, **Grieser 08**).
- ▶ Classically, we will consider different expansions **far** from O and **in a neighbourhood** of O that we will **match** in some intermediate regions.

- ▶ To capture the **rapid** variations of the eigenfunctions close to O , introduce the variables

$$(X, Y) = (x/\varepsilon, y/\varepsilon).$$

- ▶ As $\varepsilon \rightarrow 0$, in the variables X, Y , Ω^ε turns into the **near field geometry** Ω :



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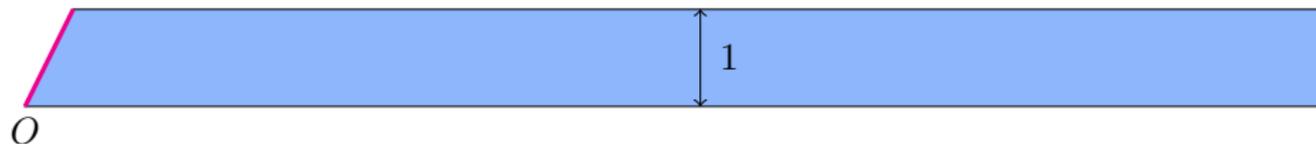
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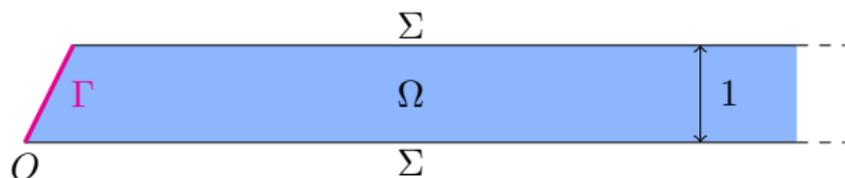
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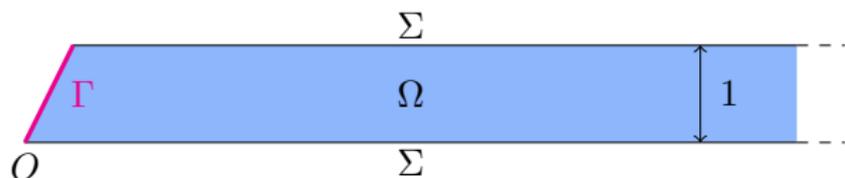
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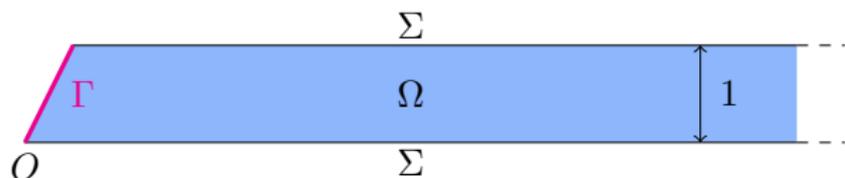
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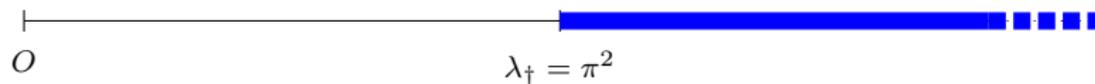
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Denote by A^Ω the unbounded operator of $L^2(\Omega)$ such that

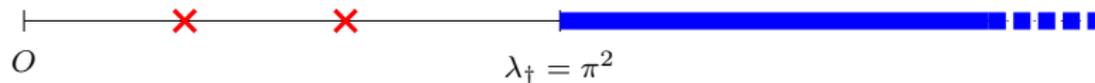
$$\left\{ \begin{array}{l} \mathcal{D}(A^\Omega) := \{v \in H^1(\Omega) \mid \Delta v \in L^2(\Omega), v = 0 \text{ on } \Sigma, \partial_\nu v = 0 \text{ on } \Gamma\} \\ A^\Omega v = -\Delta v. \end{array} \right.$$

Spectrum of A^Ω :



- The **continuous spectrum** of A^Ω occupies the ray $[\pi^2; +\infty)$.

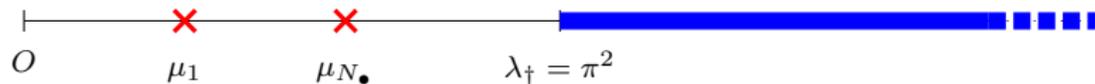
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- **Discrete spectrum** depends on α . There holds

$\sigma_d(A^\Omega) = \emptyset$ if $\alpha = 0$		$\sigma_d(A^\Omega) \neq \emptyset$ if $\alpha \neq 0$
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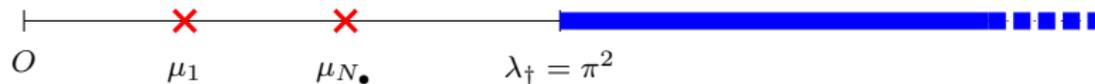
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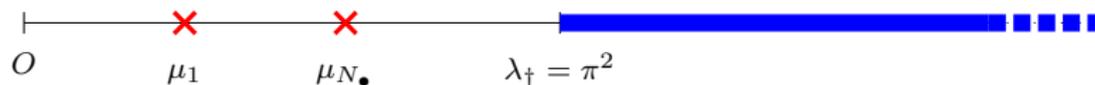
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*Eigenfunctions associated with two eigenvalues of the **discrete** spectrum:*



- In the sequel, the properties of the problem

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- ▶ Define the **wave-packets**, with a **non-zero flux of energy**,

$$W_+ = w_1 - iw_0, \quad W_- = w_1 + iw_0,$$

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i.e. at the **threshold of the continuous spectrum of A^Ω** , will play a key role.

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- For all $\alpha \in [0; \pi/2)$, we have $|\mathbb{S}| = |\mathbb{S}(\alpha)| = 1$. But in general $\mathbb{S}(\alpha) \neq -1$.



For **most angles** α , there holds $X_\dagger = \{0\}$.

- 1 Preparatory work
- 2 Asymptotic analysis at a fixed α
- 3 Model problems at the critical angles
- 4 Spectral breathing in periodic waveguides

First main results

Pick $\alpha \in [0; \pi/2)$. To simplify, assume absence of trapped modes for (\mathcal{P}_\dagger) .

THEOREM: There are constants C_n, δ_n such that as $\varepsilon \rightarrow 0$, we have:

For $n = 1, \dots, N_\bullet$:

$$\left| \lambda_n^\varepsilon - \varepsilon^{-2} \mu_n \right| \leq C_n e^{-(1+\delta_n)\sqrt{\pi^2 - \mu_n}/\varepsilon};$$

For $n = N_\bullet + m, m \in \mathbb{N}^*$:

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COMMENTS:

① The asymptotics of the first N_\bullet eigenvalues is dictated by $\sigma_d(A^\Omega)$.

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COMMENTS:

- 1 The asymptotics of the first N_\bullet eigenvalues is dictated by $\sigma_d(A^\Omega)$.
- 2 The behavior of the next eigenvalues depends on the existence or not of almost stabilizing solutions.

Elements of proof – first N_\bullet eigenvalues

For $1 \leq n \leq N_\bullet$, let $\mu_n \in (0; \pi^2)$, v be an eigenpair of the **discrete spectrum** of A^Ω :



Inserting $(\varepsilon^{-2}\mu_n, v(\cdot/\varepsilon))$ in $(\mathcal{P}^\varepsilon)$ only leaves a **small discrepancy** at $x = 1$ because v is **exponentially decaying at infinity**.

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► Let $\lambda_n^\varepsilon, u^\varepsilon$ be the n -th eigenpair of $(\mathcal{P}^\varepsilon)$. As $\varepsilon \rightarrow 0$, we can prove the asymptotic expansion

$$\lambda_n^\varepsilon = \varepsilon^{-2}\mu_n + \dots, \quad u^\varepsilon(z) = v(z/\varepsilon) + \dots$$

where $z := (x, y)$.

Elements of proof – higher eigenvalues

For $n > N_\bullet$, let u^ε be an eigenfunction associated with λ_n^ε .

FAR-FIELD EXPANSION. Far from O , consider the ansätze, as $\varepsilon \rightarrow 0$,

$$\lambda_n^\varepsilon = \varepsilon^{-2}\pi^2 + \kappa + \dots, \quad u^\varepsilon(z) = \gamma(x) \sin(\pi y/\varepsilon) + \dots \text{ for } x \in I.$$

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Take $U^0 = c^0 W, U' = c' W$ where $c^0, c' \in \mathbb{R}$ and W is the one above.

Elements of proof – higher eigenvalues

- Let us **match** the two representations of u^ε in some **intermediate region** where **both** $x \rightarrow 0$ and $x/\varepsilon \rightarrow +\infty$:

$$\sqrt{\varepsilon} < x < 2\sqrt{\varepsilon}$$


The diagram shows a horizontal line segment from 0 to 1. A small green shaded region is located near the origin, representing the intermediate region where both x and x/ε approach infinity.

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- ▶ We conclude that we must impose

$$\gamma(0) = c^0 (1 - \mathbb{S}), \quad 0 = c^0 (1 + \mathbb{S}) \quad \text{and} \quad \gamma'(0) = c' (1 + \mathbb{S}).$$

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- Now the study depends on the value of \mathbb{S} .

Case $\mathbb{S} \neq -1 \Leftrightarrow \mathbf{X}_\dagger = \{\mathbf{0}\}$.

We take $c^0 = 0$ and impose $\gamma(0) = 0$, *i.e.* **Dirichlet** at O in (\mathcal{P}_{1D}) .

Solving (\mathcal{P}_{1D}) , we get

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We impose $\gamma'(0) = 0$, *i.e.* **Neumann** at O in (\mathcal{P}_{1D}) , and take $c^0 = \gamma(0)/2$.

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$$\left| \begin{array}{l} \lambda_n^\varepsilon = \varepsilon^{-2}\pi^2 + (m - 1/2)^2\pi^2 + \dots \\ u^\varepsilon(z) = \cos(\pi(m - 1/2)x) \sin(\pi y/\varepsilon) + \dots \end{array} \right. \text{far-field expansion.}$$

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Are there α such that

$$\mathbb{S}(\alpha) = -1 \quad \Leftrightarrow \quad \dim X_{\dagger} = 1 \quad \Leftrightarrow \quad \left| \begin{array}{l} (\mathcal{P}_{\dagger}) \text{ admits a bounded solution} \\ \text{which does not decay at infinity} \end{array} \right. ?$$

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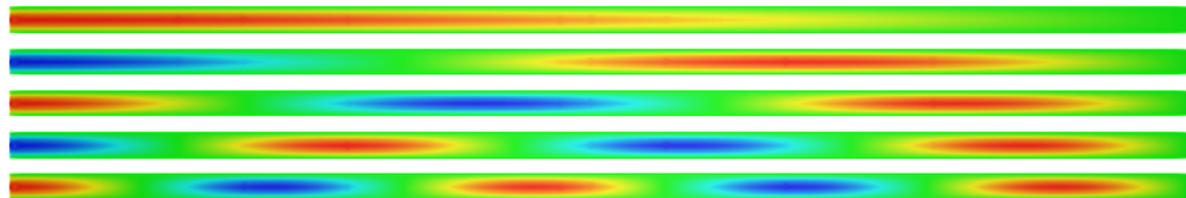
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Numerically, we find non zero α_k^* but proving their existence is an open pb.

Numerics

- ▶ Eigenfunctions of $(\mathcal{P}^\varepsilon)$ associated with the 5 first eigenvalues for $\varepsilon = 0.02$:

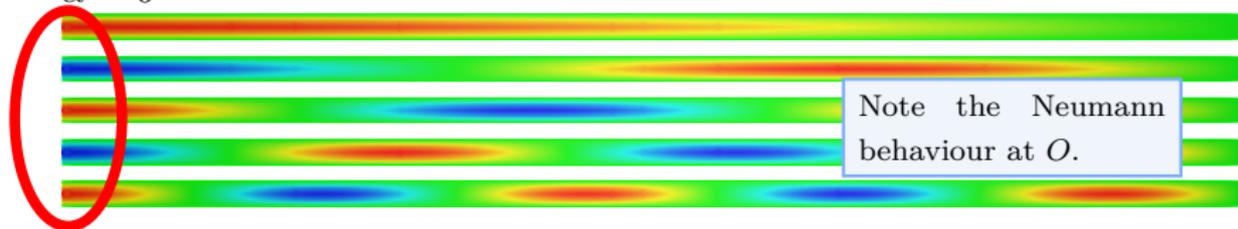
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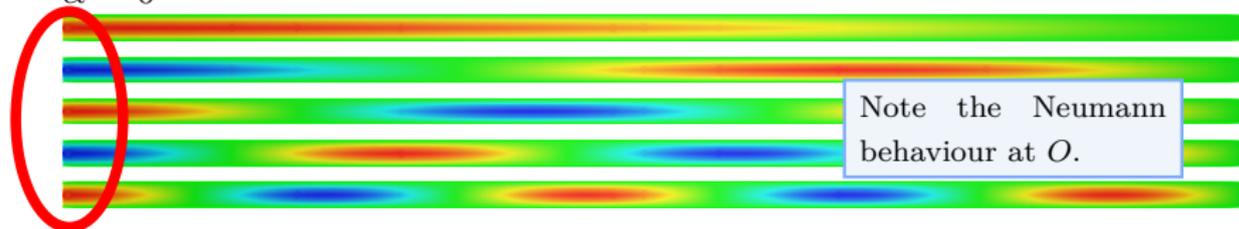
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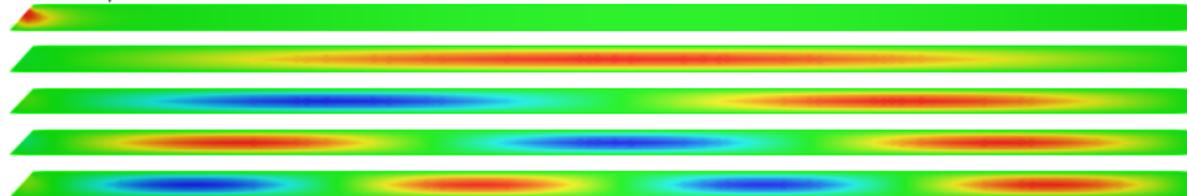
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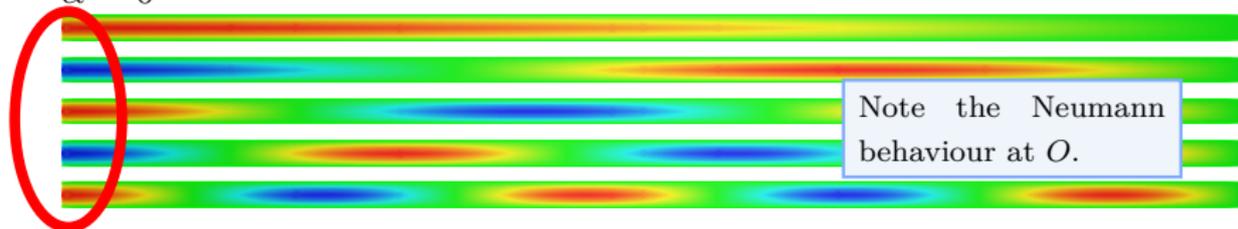
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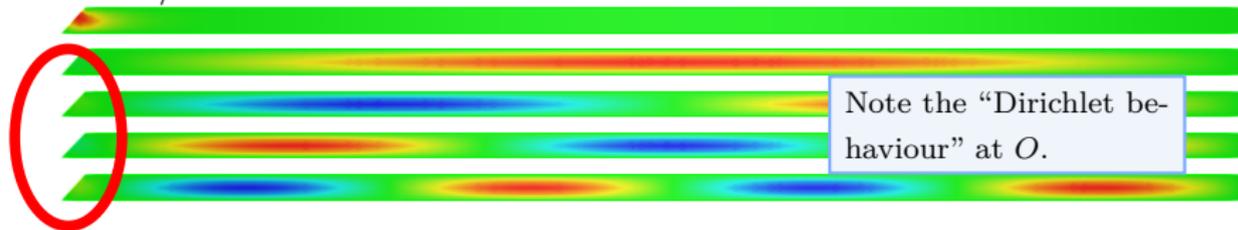
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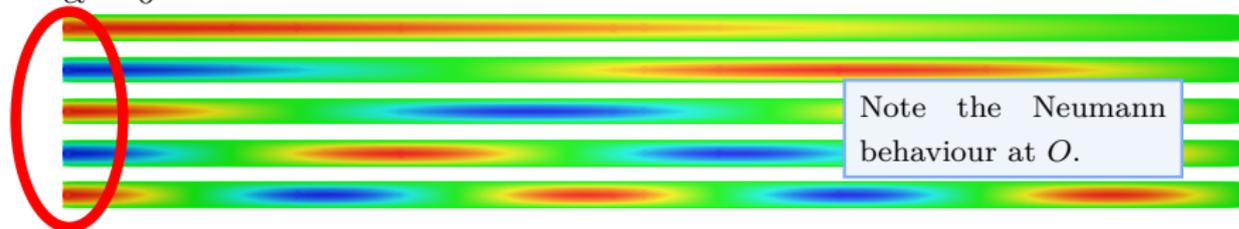
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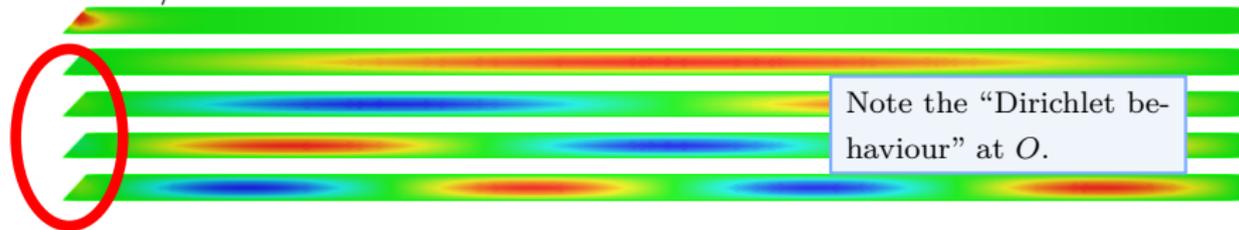
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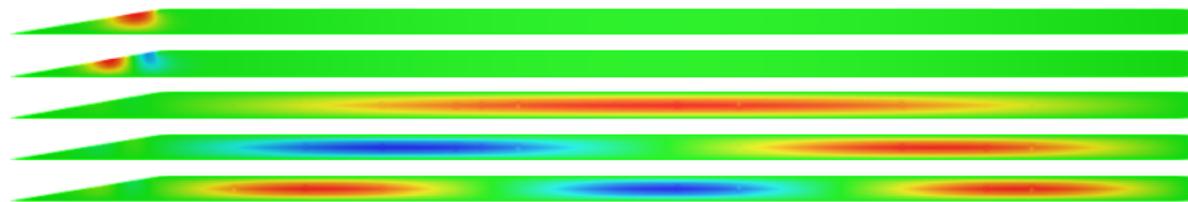
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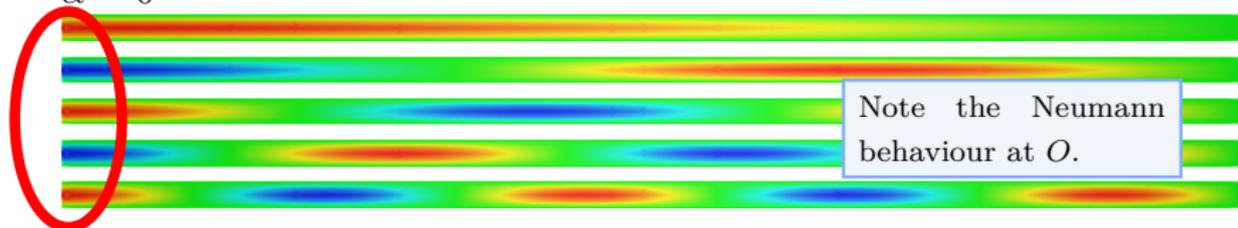
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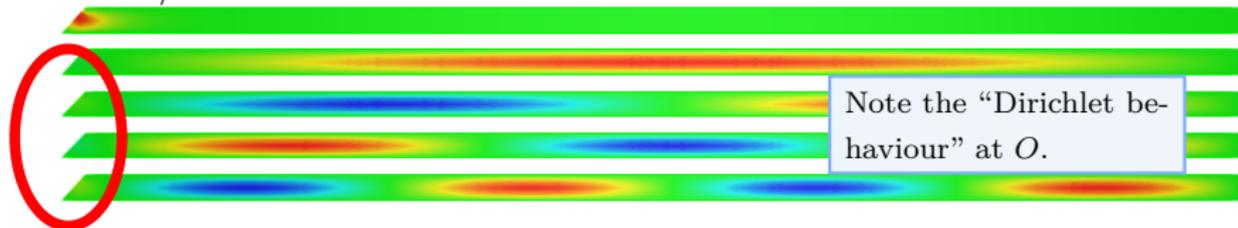
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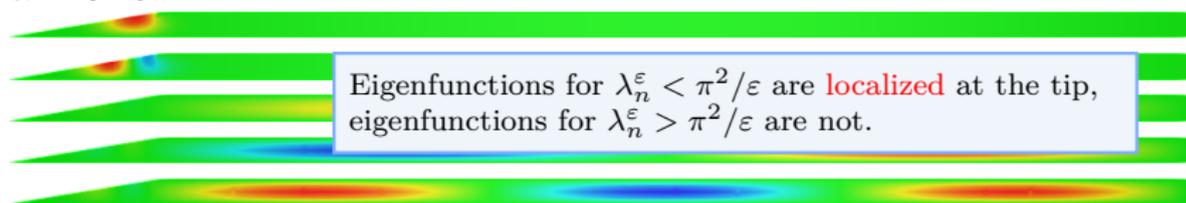
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- Numerically, to compute the **critical angles**, we need to approximate

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$$\left| \begin{array}{l} \Delta \hat{W} + \pi^2 \hat{W} = 0 \quad \text{in } \Omega_L := \{z \in \Omega \mid x < L\} \\ \hat{W} = 0 \quad \text{on } \Sigma \cap \partial\Omega_L \\ \partial_\nu \hat{W} = 0 \quad \text{on } \Gamma \end{array} \right. \quad \text{bounded domain}$$
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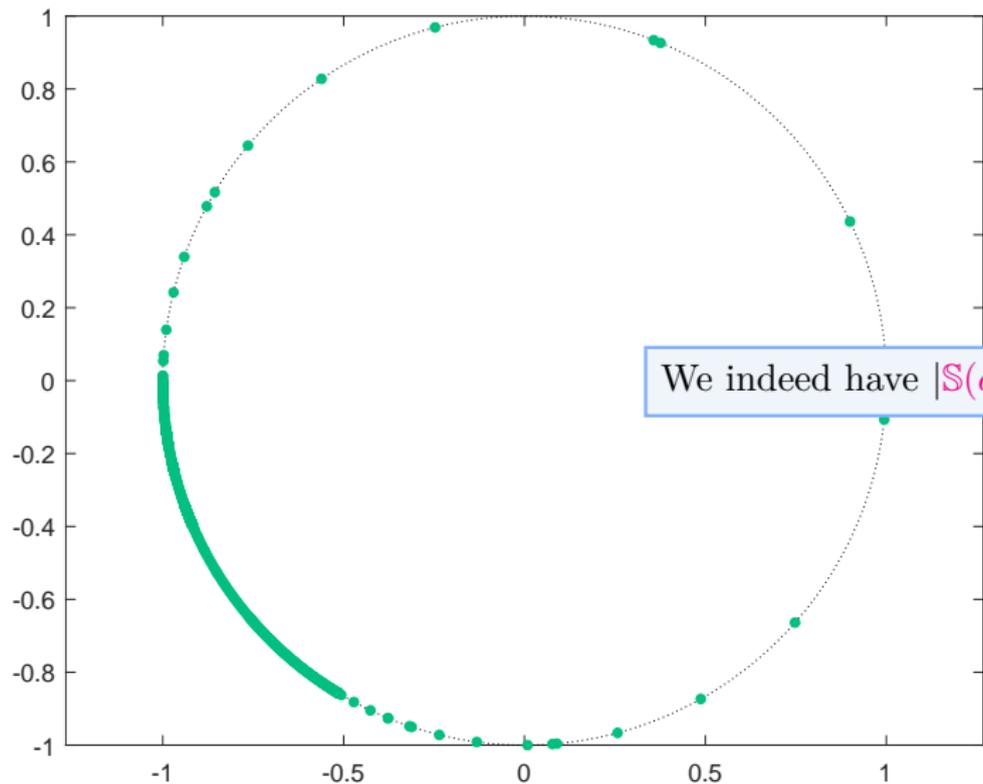
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- ▶ Finally, from \hat{W} we obtain an approximation of \mathbb{S} .

Numerics

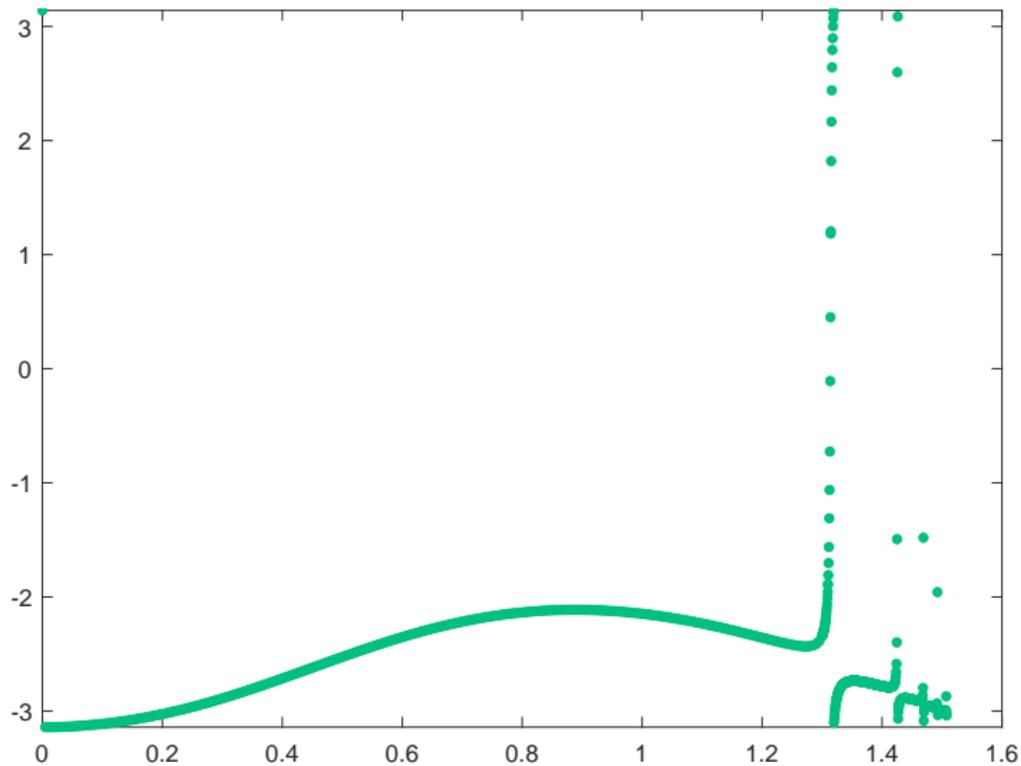
- ▶ $\mathbb{S}(\alpha)$ for 400 values of $\alpha \in [0; 0.48\pi)$ in the complex plane



We indeed have $|\mathbb{S}(\alpha)| = 1$.

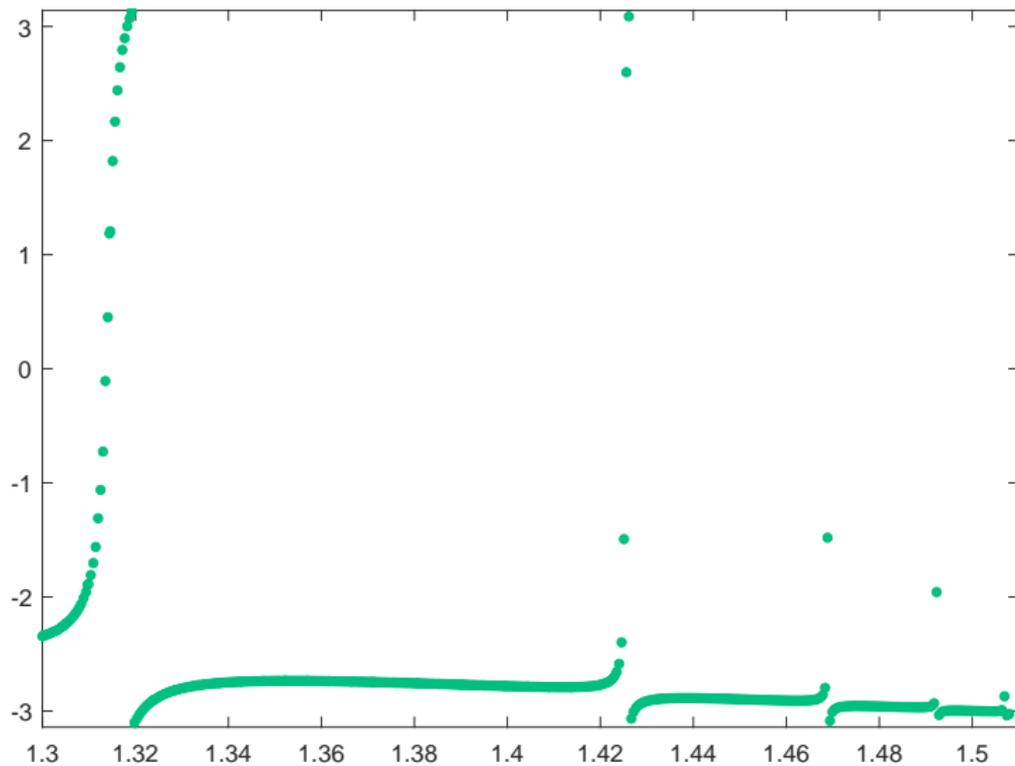
Numerics

- Phase of $\mathbb{S}(\alpha)$ for $\alpha \in [0; 0.48\pi)$



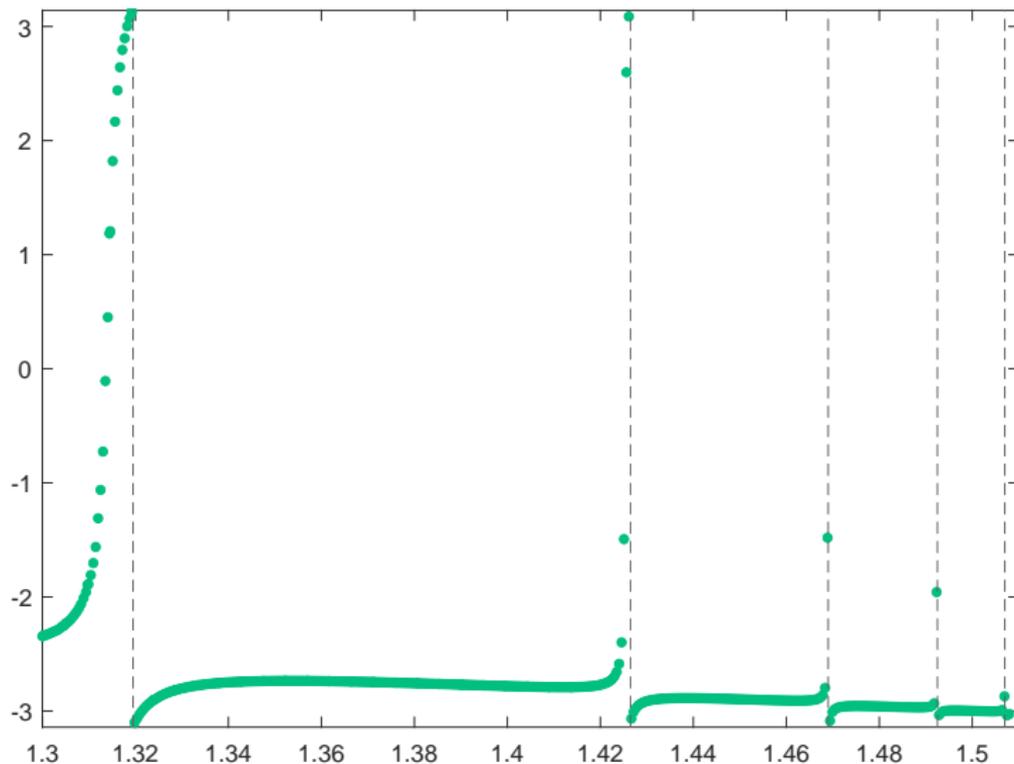
Numerics

- Phase of $\mathfrak{S}(\alpha)$ for $\alpha \in [1.3; 0.48\pi)$



Numerics

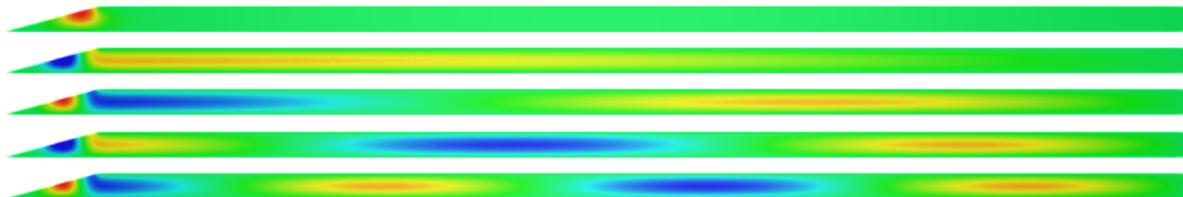
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Numerically, we obtain a sequence (α_k^*) which accumulates at $\pi/2$.

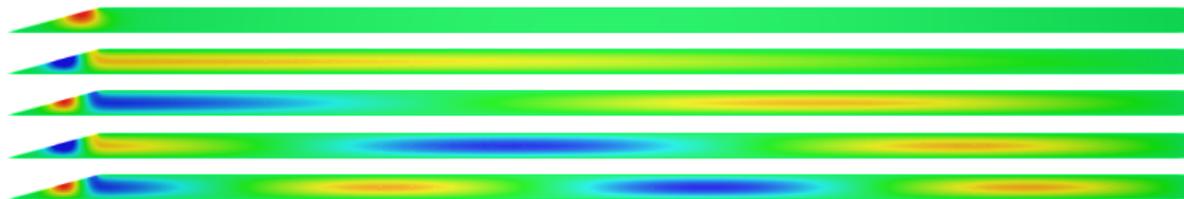
Numerics at a positive critical angle

- ▶ First eigenfunctions of $(\mathcal{P}^\varepsilon)$ for $\varepsilon = 0.02$ and $\alpha = \alpha_1^* \approx 1.321$:



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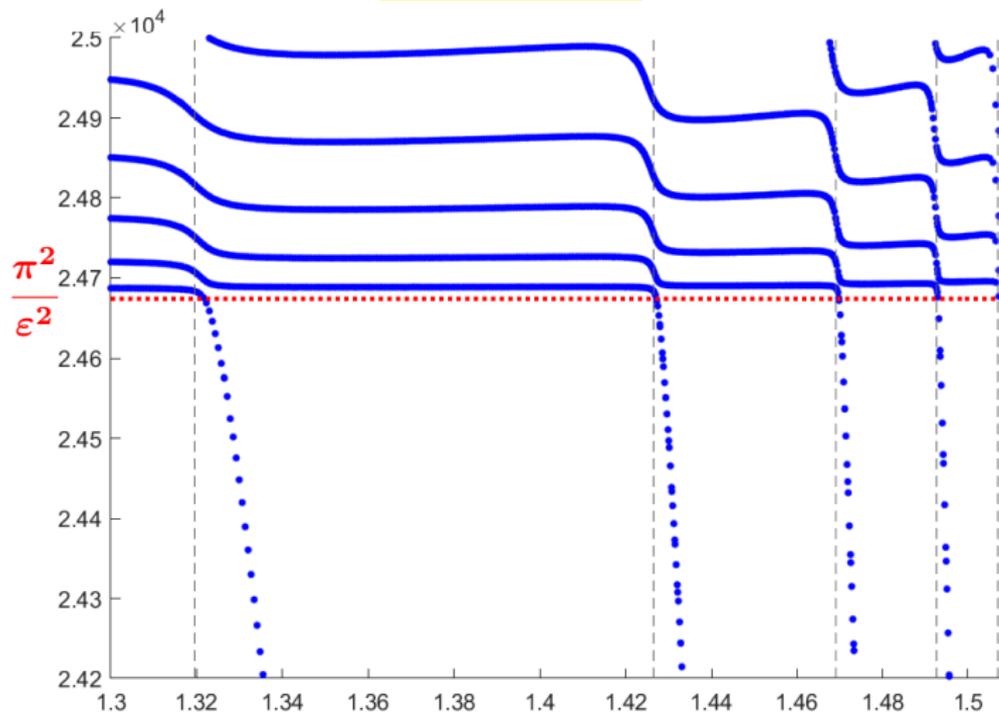
- ▶ First eigenfunctions of $(\mathcal{P}^\varepsilon)$ for $\varepsilon = 0.02$ and $\alpha = \alpha_1^* \approx 1.321$:



We indeed have a “**Neumann behaviour**” at O for the far-field of the eigenfunctions associated to the $\lambda_n^\varepsilon > \pi^2/\varepsilon^2$.

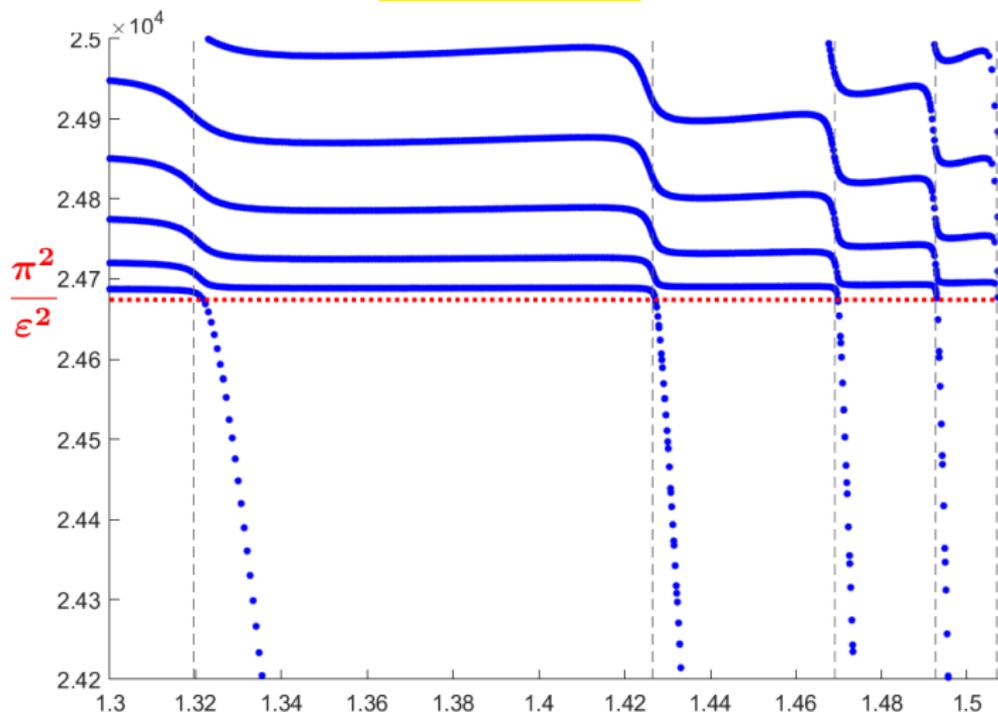
Initial picture

- ▶ Eigenvalues of $(\mathcal{P}^\varepsilon)$ wrt $\alpha \in (1.3; 0.48\pi)$



Initial picture

- Eigenvalues of $(\mathcal{P}^\varepsilon)$ wrt $\alpha \in (1.3; 0.48\pi)$



- ✓ We have identified the **critical angles** where an eigenvalue dives;
- ✗ We have **not explained** the phenomenon of **diving** eigenvalue.

- 1 Preparatory work
- 2 Asymptotic analysis at a fixed α
- 3 Model problems at the critical angles
- 4 Spectral breathing in periodic waveguides

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Goal now:

To describe the **transition** between the model with **Dirichlet** at O valid for $\alpha \neq \alpha_k^*$ and the one with **Neumann** at O valid for $\alpha = \alpha_k^*$.

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- For $\tau \in \mathbb{R}$ and $k \in \mathbb{N}$, define the trapezoid

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- ▶ We wish to obtain an asymptotic expansion of the $\lambda_{\tau,n}^\varepsilon$ as $\varepsilon \rightarrow 0$.

Asymptotic analysis

- For an eigenpair $(\lambda_\tau^\varepsilon, u_\tau^\varepsilon)$ close to π^2/ε^2 , consider the ansätze

$$\left| \begin{array}{ll} \lambda_\tau^\varepsilon & = \varepsilon^{-2}\pi^2 + \eta_\tau + \dots \\ u_\tau^\varepsilon(z) & = \gamma_\tau(x) \sin(\pi y/\varepsilon) + \dots & \text{(far-field expansion)} \\ u_\tau^\varepsilon(z) & = U_\tau^0(z/\varepsilon) + \varepsilon U_\tau^1(z/\varepsilon) + \dots & \text{(near-field expansion)} \end{array} \right.$$

where the constants η_τ and the functions γ_τ , U_τ^0 , U_τ^1 are **to be determined**.

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- ▶ Inserting it in $(\mathcal{P}^\varepsilon)$, we obtain

$$\begin{cases} \partial_x^2 \gamma_\tau + \eta_\tau \gamma_\tau &= 0 & \text{in } I \\ \gamma_\tau(1) &= 0 \end{cases} \quad \begin{cases} \Delta U_\tau^0 + \pi^2 U_\tau^0 &= 0 & \text{in } \Omega(\alpha_k^*) \\ U_\tau^0 &= 0 & \text{on } \Sigma(\alpha_k^*) \\ \partial_\nu U_\tau^0 &= 0 & \text{on } \Gamma(\alpha_k^*) \end{cases}$$

$$\begin{cases} \Delta U_\tau' + \pi^2 U_\tau' &= 0 & \text{in } \Omega(\alpha_k^*) \\ U_\tau' &= 0 & \text{on } \Sigma(\alpha_k^*) \\ \partial_\nu U_\tau' &= -\tau \partial_s U_\tau^0 - \tau s (\partial_s^2 U_\tau^0 + \pi^2 U_\tau^0) & \text{on } \Gamma(\alpha_k^*). \end{cases}$$

Asymptotic analysis

- From the Taylor series

$$\begin{aligned}\gamma_\tau(x) \sin(\pi y/\varepsilon) &= (\gamma_\tau(0) + \partial_x \gamma_\tau(0)x + \dots) \sin(\pi y/\varepsilon) \\ &= (\gamma_\tau(0) + \varepsilon \partial_x \gamma_\tau(0) \frac{x}{\varepsilon} + \dots) \sin(\pi y/\varepsilon),\end{aligned}$$

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$$U_\tau^0 = \gamma_\tau(0) W$$

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where C_0 (not unique), $C_1 \in \mathbb{R}$ and $\tilde{U}'_\tau \in H^1(\Omega(\alpha_k^*))$.

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- ▶ Matching **the terms in x at order ε** , we conclude that we must impose

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Model problem at positive critical angles

- Finally, for γ_τ , we obtain the **model problem** with **Robin BC** at O

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THEOREM: Consider a **positive** critical angle α_k^* and some $\tau \in \mathbb{R}$. For $n = N_\bullet + m$, $m \in \mathbb{N}^*$, there are constants $C_{n,\tau}$, $\delta_{n,\tau}$, s.t. as $\varepsilon \rightarrow 0$,

$$|\lambda_{n,\tau}^\varepsilon - (\varepsilon^{-2} \pi^2 + \eta_{m,\tau})| \leq C_n \varepsilon^{1+\delta_n},$$

where $\eta_{m,\tau}$ is the m -th eigenvalue of (\mathcal{P}_{1D}^τ) .

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THEOREM: Consider a **positive** critical angle α_k^* and some $\tau \in \mathbb{R}$. For $n = N_\bullet + m$, $m \in \mathbb{N}^*$, there are constants $C_{n,\tau}$, $\delta_{n,\tau}$, s.t. as $\varepsilon \rightarrow 0$,

$$|\lambda_{n,\tau}^\varepsilon - (\varepsilon^{-2} \pi^2 + \eta_{m,\tau})| \leq C_n \varepsilon^{1+\delta_n},$$

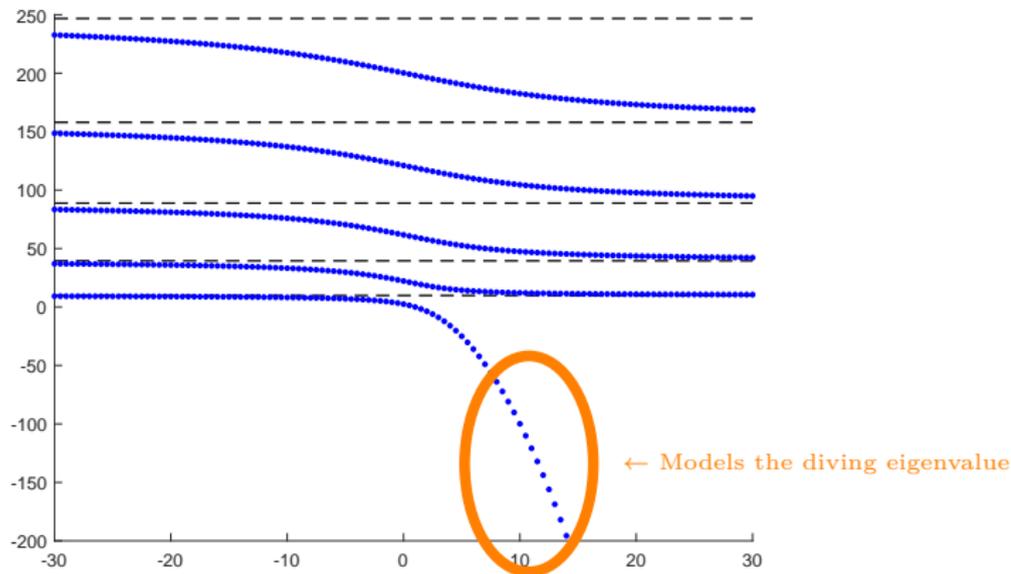
where $\eta_{m,\tau}$ is the m -th eigenvalue of (\mathcal{P}_{1D}^τ) .

Solving explicitly (\mathcal{P}_{1D}^τ) , we find

$$\left\{ \begin{array}{l} \lim_{\tau \rightarrow -\infty} \eta_{1,\tau} = \pi^2 \\ \eta_{1,\tau} \underset{\tau \rightarrow +\infty}{\sim} -4B^2\tau^2 \end{array} \right. \quad \text{and for } m \geq 2, \quad \left\{ \begin{array}{l} \lim_{\tau \rightarrow -\infty} \eta_{m,\tau} = m^2 \pi^2 \\ \lim_{\tau \rightarrow +\infty} \eta_{m,\tau} = (m-1)^2 \pi^2. \end{array} \right.$$

Model problem at positive critical angles

- Spectrum of (\mathcal{P}_{1D}^τ) wrt τ :



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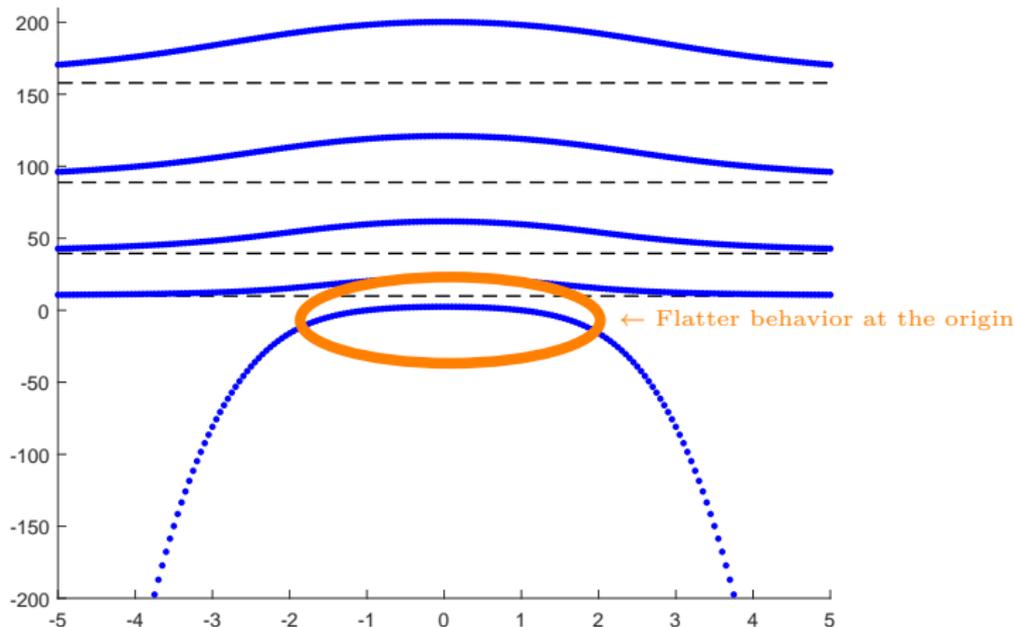
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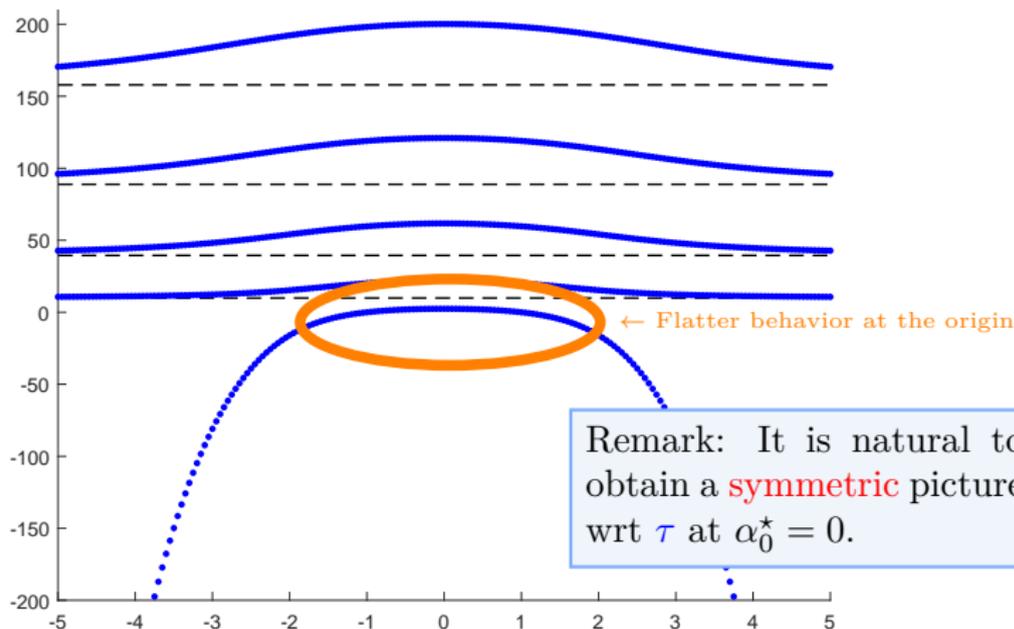


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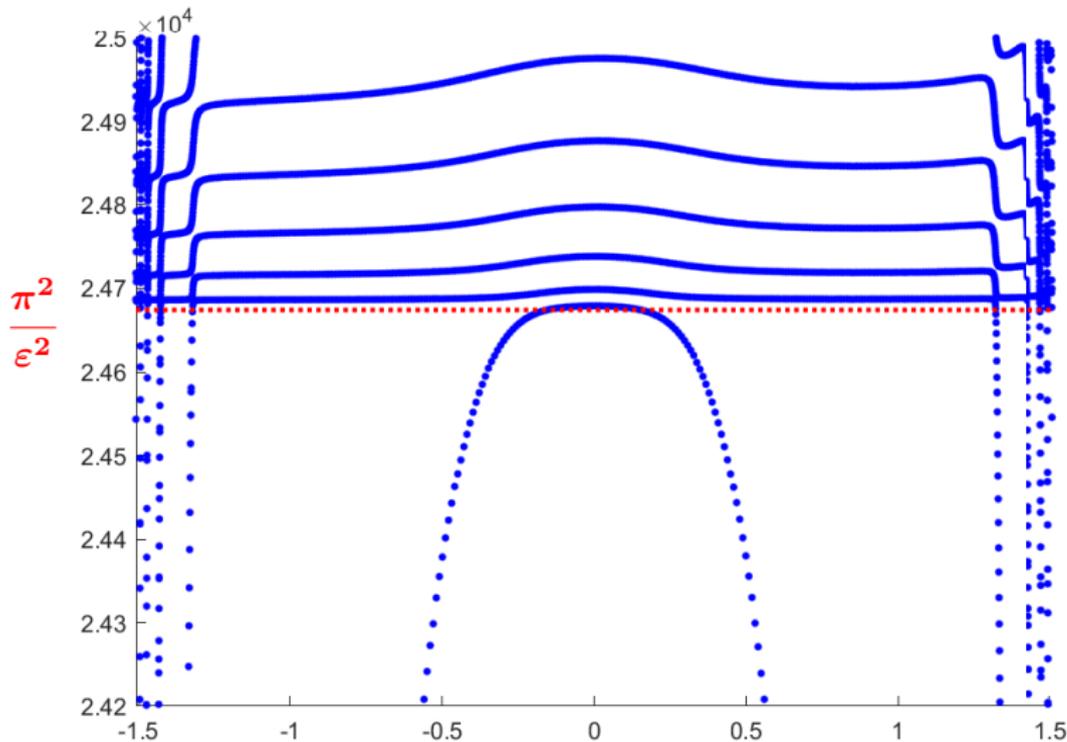


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Numerics

- Spectrum of $(\mathcal{P}^\varepsilon)$ with respect to $\alpha \in (-\pi/2; \pi/2)$ for $\varepsilon = 0.02$:



This explains the **mild** behavior of the **diving** eigenvalue at $\alpha = 0$.

Numerics in the broken strip

- ▶ Eigenfunctions for α varying around α_1^* and $\varepsilon = 0.02$:

Eigenfunction 2

Eigenfunction 4

Eigenfunction 3

And for the Neumann Laplacian?

- Consider the problem with **Neumann** BCs

$$\left| \begin{array}{l} -\Delta u = \lambda^\varepsilon u \quad \text{in } T^\varepsilon(\alpha) \\ \partial_\nu u = 0 \quad \text{on } \partial T^\varepsilon(\alpha). \end{array} \right.$$



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No large variation of the behavior of eigenvalues when varying α .

- 1 Preparatory work
- 2 Asymptotic analysis at a fixed α
- 3 Model problems at the critical angles
- 4 Spectral breathing in periodic waveguides



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- ♠ We studied the asymptotics of the spectrum of the **Dirichlet Laplacian** in **thin broken** strips.
 - All eigenvalues go to $+\infty$ as $O(\varepsilon^{-2})$;
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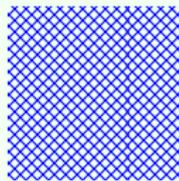
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Possible extensions and open questions

- 1) We can work similarly in **quasi 1D periodic waveguides**.
- 2) Can one find examples of Ω such that $\dim X_{\dagger} = 2$?
- 3) Can one work with **other models**, *i.e.* $\Delta\Delta u - k^4 u = 0$ + Dirichlet BC?

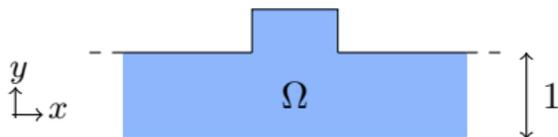


Thank you!

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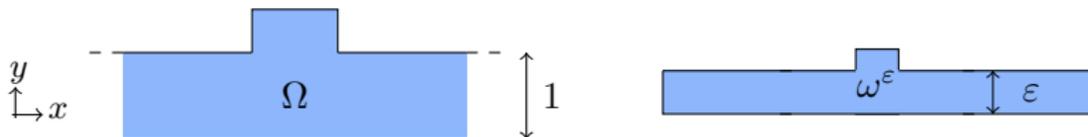
Setting

- ▶ We consider the **propagation of waves** in a 2D **thin periodic quantum waveguide** Π^ε .
- ▶ Start with some domain $\Omega \subset \mathbb{R}^2$ which coincides with the strip $\mathbb{R} \times (0; 1)$ outside of a bounded region (the **resonator**).



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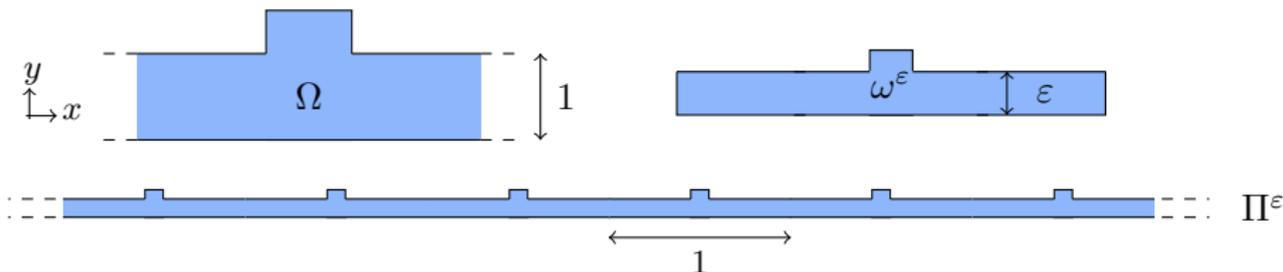
- ▶ **Shrink** Ω by a small factor $\varepsilon > 0$ to create the **unit cell**

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► Define the **periodic** waveguide

$$\Pi^\varepsilon := \{z \in \mathbb{R}^2 \mid (x - m, y) \in \omega^\varepsilon, m \in \mathbb{Z}\}.$$

- ▶ In Π^ε we consider the **spectral** problem for the **Dirichlet** Laplacian

$$(\mathcal{P}^\varepsilon) \left| \begin{array}{ll} -\Delta u^\varepsilon = \lambda^\varepsilon u^\varepsilon & \text{in } \Pi^\varepsilon \\ u^\varepsilon = 0 & \text{on } \partial\Pi^\varepsilon. \end{array} \right.$$

- ▶ Denote by A^ε the unbounded operator of $L^2(\Pi^\varepsilon)$ such that

$$\mathcal{D}(A^\varepsilon) := \{v \in H_0^1(\Pi^\varepsilon) \mid \Delta v \in L^2(\Pi^\varepsilon)\} \quad \text{and} \quad A^\varepsilon v = -\Delta v.$$

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Goal of the section

We wish to study the lower part of $\sigma(A^\varepsilon)$, the **spectrum** of A^ε , as $\varepsilon \rightarrow 0$.

- The Floquet Bloch transform

$$u^\varepsilon(z) \mapsto U^\varepsilon(z, \eta) = \frac{1}{(2\pi)^{1/2}} \sum_{j \in \mathbb{Z}} e^{i\eta j} u^\varepsilon(x + j, y), \quad \eta \in \mathbb{R},$$

converts $(\mathcal{P}^\varepsilon)$ into a **spectral** pb set in ω^ε with **quasi-periodic** BCs at $x = \pm 1/2$

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- For $\eta \in [0; 2\pi)$, the spectrum of $(\mathcal{P}^\varepsilon(\eta))$ is discrete, made of the unbounded sequence of real eigenvalues

$$0 < \lambda_1^\varepsilon(\eta) \leq \lambda_2^\varepsilon(\eta) \leq \dots \leq \lambda_p^\varepsilon(\eta) \leq \dots$$

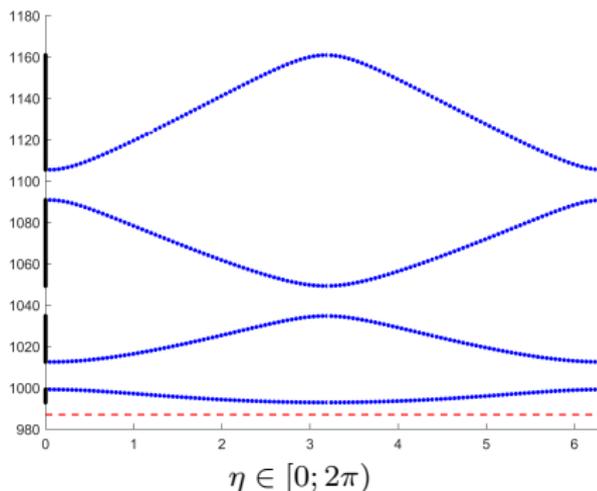
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are continuous so that the
spectral bands

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- ▶ Finally, we have

$$\sigma(A^\varepsilon) = \bigcup_{p \in \mathbb{N}^* := \{1, 2, \dots\}} \Upsilon_p^\varepsilon.$$

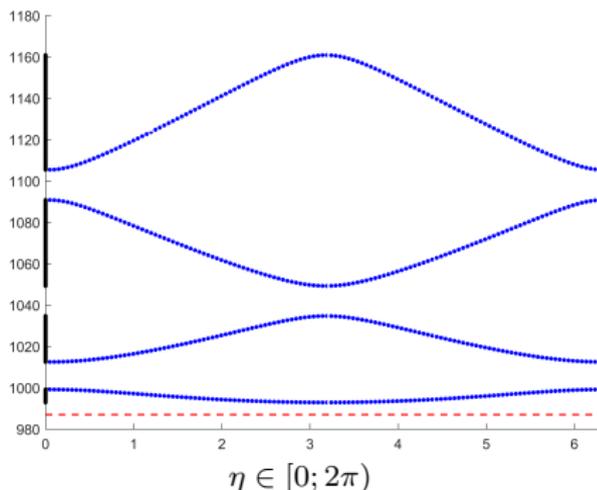
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To study the behaviour of $\sigma(A^\varepsilon)$ as $\varepsilon \rightarrow 0$, we have to consider the asymptotics of $\lambda_p^\varepsilon(\eta)$ as $\varepsilon \rightarrow 0$.

Asymptotic analysis

- ▶ For $p \in \mathbb{N}^*$, denote by $a_{p-}^\varepsilon \leq a_{p+}^\varepsilon$ the bounds of the **spectral band** Υ_p^ε :

$$\Upsilon_p^\varepsilon = [a_{p-}^\varepsilon; a_{p+}^\varepsilon].$$

- ▶ Working as in the first section, we can establish the

THEOREM: There are constants $c_{p-} < c_{p+}$, $C_p > 0$, $\delta_p > 0$ s.t. as $\varepsilon \rightarrow 0$,

For $p = 1, \dots, N_\bullet$:

$$\left| a_{p\pm}^\varepsilon - \left(\varepsilon^{-2} \mu_p + \varepsilon^{-2} e^{-\sqrt{\pi^2 - \mu_p}/\varepsilon} c_{p\pm} \right) \right| \leq C_p e^{-(1+\delta_p)\sqrt{\pi^2 - \mu_p}/\varepsilon};$$

For $p = N_\bullet + m$, $m \in \mathbb{N}^*$:

$$i) \text{ if } \mathbf{X}_\dagger = \{0\}, \quad \left| a_{p\pm}^\varepsilon - \left(\varepsilon^{-2} \pi^2 + m^2 \pi^2 + \varepsilon c_{p\pm} \right) \right| \leq C_p \varepsilon^{1+\delta_p};$$

$$ii) \text{ if } \dim \mathbf{X}_\dagger = 1, \quad \left| a_{p\pm}^\varepsilon - \left(\varepsilon^{-2} \pi^2 + c_{p\pm} \right) \right| \leq C_p \varepsilon^{1+\delta_p};$$

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When **$\dim X_\dagger \neq 1$** , the Υ_p^ε are of length $O(\varepsilon)$. Moreover, between Υ_p^ε and $\Upsilon_{p+1}^\varepsilon$, there is a gap whose length tends to $(2m+1)\pi^2$.



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Generically, the propagation of waves in Π^ε is **hampered** and occurs only for very **narrow** intervals of frequencies.

When **$\dim X_\dagger = 1$** , the situation is **very different** because asymptotically the Υ_p^ε are of length $c_{p+} - c_{p-}$, with in general $c_{p+} > c_{p-}$.



For **particular Ω** , waves can propagate in Π^ε for **much larger intervals** of frequencies than above.

Spectral breathing

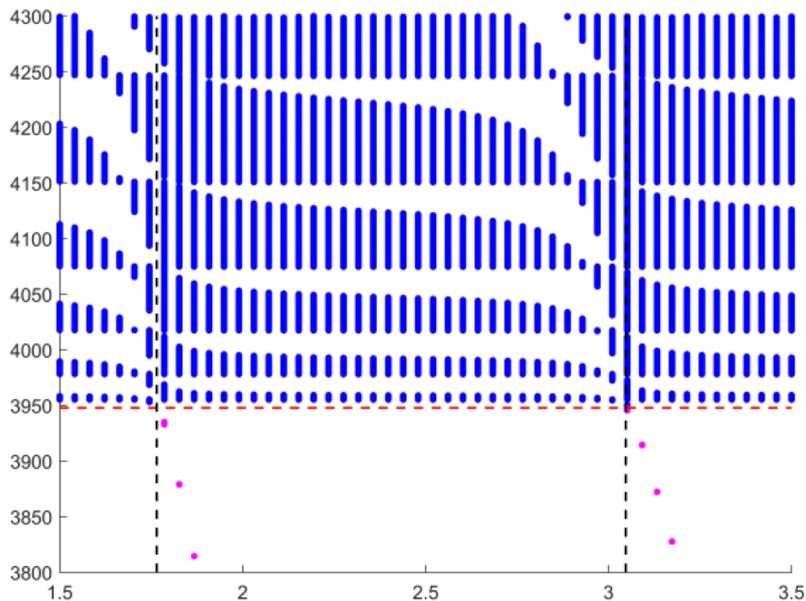
Then we can describe the change of $\sigma(A^\varepsilon)$ when **perturbing the inner field geometry** around a Ω_\star where $\dim X_\dagger = 1$.

→ Note that we make a **periodic perturbation** of Π^ε .

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Then we can describe the change of $\sigma(A^\varepsilon)$ when perturbing the inner field geometry around a Ω_\star where $\dim X_\dagger = 1$.

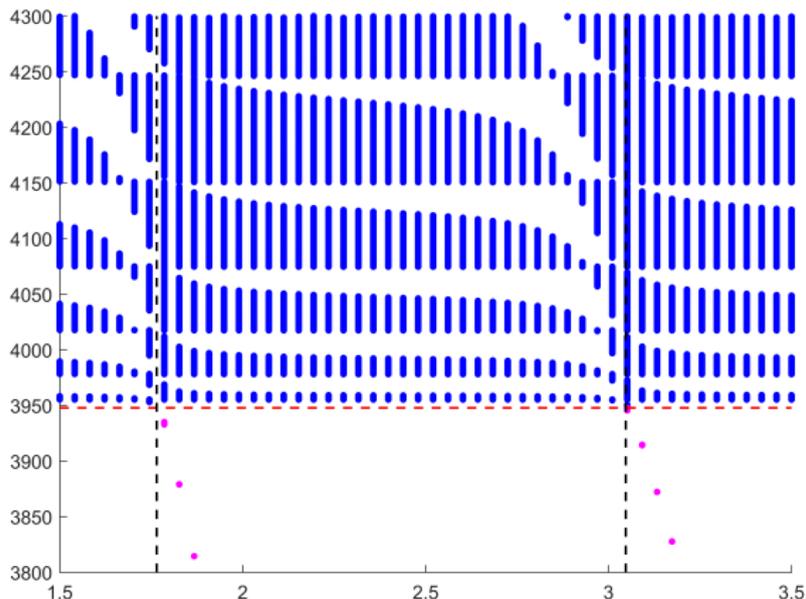
→ Note that we make a periodic perturbation of Π^ε .



Spectral breathing

Then we can describe the change of $\sigma(A^\varepsilon)$ when **perturbing the inner field geometry** around a Ω_\star where $\dim X_\dagger = 1$.

→ Note that we make a **periodic perturbation** of Π^ε .

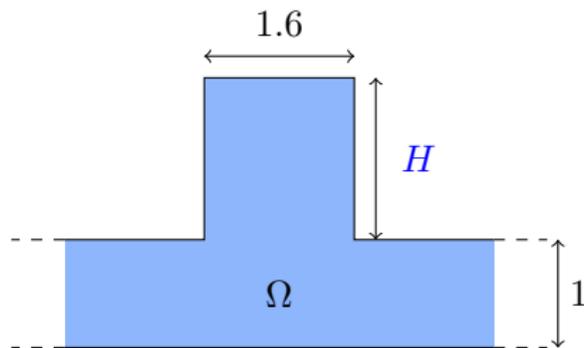


► By deriving **model problems** as in the previous section, one shows:

- When **perturbing** the near field geometry around Ω_\star , the spectral bands **expand and shrink** \Rightarrow **breathing phenomenon** of $\sigma(A^\varepsilon)$.
- In the process, a band **dives** below π^2/ε^2 , **stops breathing** and becomes **extremely short**.

Numerical illustration

- ▶ We start from the inner field geometry



that we shrink by a factor ε , cut at $x = \pm 1/2$ and periodize.

- ▶ We use `Freefem++` to compute the spectrum of $(\mathcal{P}^\varepsilon(\eta))$ in the corresponding unit cell for different values of H .

Numerical illustration
