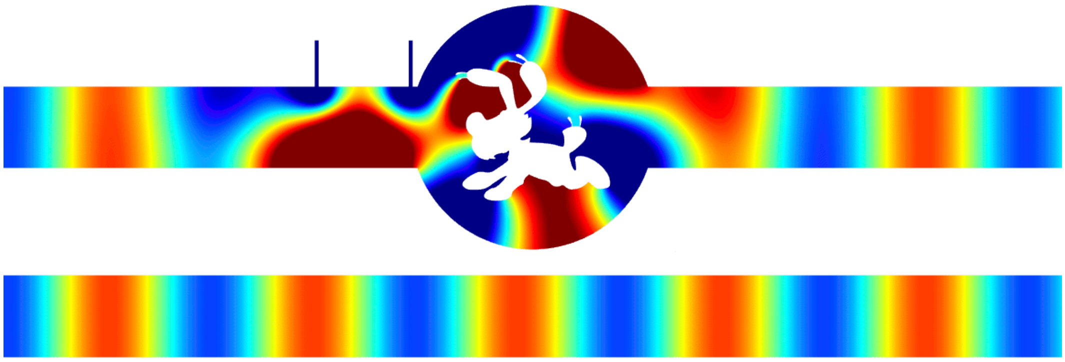
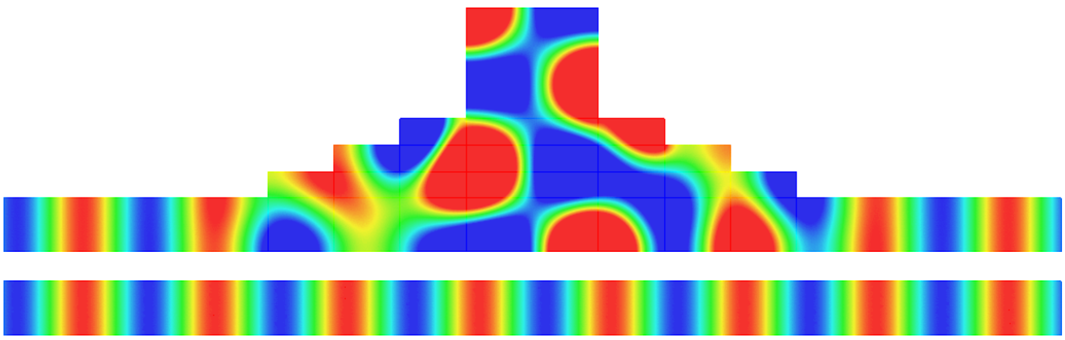


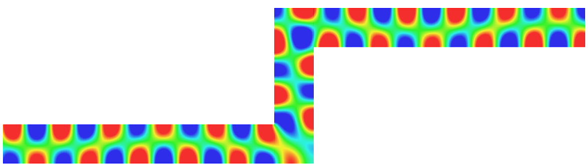
A few techniques to achieve invisibility in waveguides

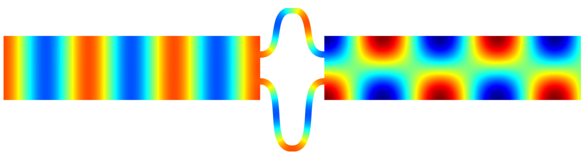
Lucas Chesnel

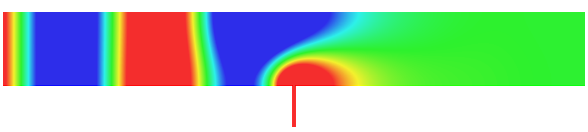
Version June 27, 2025

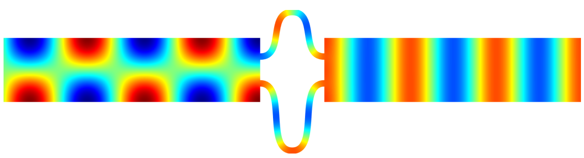
Inria, Ensta Paris, Institut Polytechnique de Paris
E-mail: lucas.chesnel@inria.fr











Introduction

Figure 1: Examples of waveguides (source Wikipedia).

The aim of these lecture notes is to consider a concrete problem, namely the identification of situations of invisibility in waveguides, to present techniques and tools of applied mathematics that can be useful in other contexts. We will be interested in the propagation of scalar waves in guides which are unbounded in one direction. Such problems arise in many fields of physics. For example air ducts and horns in acoustics carry sound waves in musical instruments as well as in loudspeakers. Conductive metal pipes are exploited to propagate high frequency radio waves while optical fibers serve as waveguides for light in electromagnetics. Waveguides problems also appear in water waves theory, in classical mechanics or in quantum mechanics. In general, the diffraction of an incident wave in such structures in presence of an obstacle generates a reflection and a transmission characterized by some scattering coefficients. Generally speaking, our goal is to play with the geometry, the frequency and/or the index material to control these scattering coefficients.

This document is divided in four chapters. In the first one, we present classical results concerning waveguide theory. This is a rather long story and the aim here is not to be exhaustive but instead to present the main ideas and ingredients that will be useful to address the invisibility problematic. In Chapter II, we develop perturbative techniques based in particular on the use of shape derivatives to design invisible defects of the reference geometry. With these approaches, in principle we construct small amplitude invisible obstacles. In Chapter III, we exploit resonant phenomena to provide examples of larger invisible obstacles. There, we also propose a method to hid given objects by perturbing (in a singular way) the boundaries of the waveguide. Finally, in Chapter IV we change the point of view, assume that the obstacle is given, and construct a non self-adjoint operator whose eigenvalues coincide with frequencies such that there are incident fields which produce zero reflection.

Our approaches mainly rely on techniques of asymptotic analysis as well as spectral theory for self-adjoint and non self-adjoint operators. Wherever possible, we will illustrate the results by numerical experiments.

The first chapter contains classical material. In the next three, some more recent results are presented. They have been obtained with different colleagues among them, Antoine Bera, Anne-Sophie Bonnet-Ben Dhia, Jérémy Heleine, Sergei Nazarov, Vincent Pagneux. I thank them warmly.

This is the first version of this document and it probably contains typos. I would be grateful to those finding them and sending them to lucas.chesnel@inria.fr. Remarks, suggestions are also welcome.

Key words. Waveguides, scattering, invisibility, asymptotic analysis, spectral theory, complex resonances, spectral theory, shape derivative.

Contents

Chapter I

Waveguide problems

1 Setting

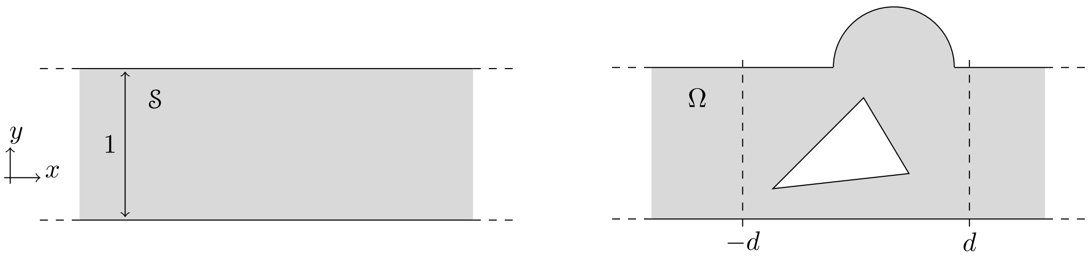


Figure I.1: Left: reference strip $\mathcal{S}$. Right: perturbed waveguide Ω.

In this chapter, we present general results concerning waveguide theory. To make it simple, we stick to a 2D scalar problem.

Set $I := (0;1)$ and consider $\Omega \subset \mathbb{R}^2$ a waveguide which coincides with the reference strip $\mathcal{S} := \{(x,y) \in \mathbb{R} \times I\}$ outside of a compact region located in the zone $\{(x,y) \in \mathbb{R}^2 \,|\, |x| < d\}$ for some $d > 0$ (see Figure I.1). We assume that the domain Ω is connected with Lipschitz boundary. Let us study the wave equation, for $t \geq 0$,

$$\left| \begin{array}{rcll} \dfrac{1}{c^2}\dfrac{\partial^2 U}{\partial t^2} - \Delta U &=& F & \text{in } \Omega \\ U &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{I.1}$$

with some initial conditions. Here $c > 0$, the celerity of waves in the medium filling Ω, is assumed to be constant. The homogeneous Dirichlet Boundary Conditions (BCs) are relevant in certain circumstances in electromagnetics when the Maxwell's problem has some invariance with respect to one spatial variable. Assume that the excitation F is time harmonic, *i.e.* of the form

$$F(x,y,t) = f(x,y)e^{-i\omega t},$$

for some pulsation $\omega > 0$ corresponding to a temporal period $T := 2\pi/w$. Then it is natural to look for solutions of (I.1) which are also harmonic for long times[1]. More precisely, we are led to search for U solving (I.1) of the form

$$U(x,y,t) = u(x,y)e^{-i\omega t}. \tag{I.2}$$

Inserting (I.2) in (I.1), we find that u satisfies the problem

$$\left| \begin{array}{rcll} \Delta u + k^2 u &=& f & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array} \right. \tag{I.3}$$

[1]The is the limiting amplitude principle, which can be violated due to trapped modes that we will meet later.

where $k := \omega/c > 0$ denotes the wavenumber.

We wish to endow (I.3) with a well-suited functional framework. This is not straightforward for two reasons. First, the form associated with (I.3) is not coercive except when k is small. Second, the domain Ω is unbounded so that the term involving k cannot be seen as a compact perturbation of the principal part.

Before proceeding further, we introduce a few spaces that will be useful in the analysis. Denote by $\mathrm{L}^2(\Omega)$ the usual Lebesgue space of square-integrable functions. It is a Hilbert space for the inner product

$$(u,v)_{\mathrm{L}^2(\Omega)} = \int_\Omega uv\,dxdy, \qquad \forall u,v \in \mathrm{L}^2(\Omega). \tag{I.4}$$

We will also work with the Sobolev spaces

$$\begin{aligned}\mathrm{H}^1(\Omega) &:= \{u \in \mathrm{L}^2(\Omega)\,|\,\nabla u \in (\mathrm{L}^2(\Omega))^2\}\\ \mathrm{H}^1_0(\Omega) &:= \{u \in \mathrm{H}^1(\Omega)\,|\,u = 0 \text{ on } \partial\Omega\}\end{aligned}$$

that we endow with the inner product

$$(u,v)_{\mathrm{H}^1(\Omega)} = \int_\Omega \nabla u\cdot\nabla v + uv\,dxdy. \tag{I.5}$$

They also are Hilbert spaces. We define the norms

$$\|\cdot\|_{\mathrm{L}^2(\Omega)} = (\cdot,\cdot)^{1/2}_{\mathrm{L}^2(\Omega)}, \qquad \|\cdot\|_{\mathrm{H}^1(\Omega)} = (\cdot,\cdot)^{1/2}_{\mathrm{H}^1(\Omega)}.$$

For a non-empty set $\mathcal{O}$, $\mathscr{C}^\infty_0(\mathcal{O})$ refer to the space of infinitely differentiable functions whose support is bounded and in $\mathcal{O}$.

2 Dirichlet problem for $0 < k < \pi$

Assume that f in (I.3) belongs to $\mathrm{L}^2(\Omega)$. The natural variational formulation of that problem writes

$$\left|\begin{array}{l}\text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that}\\ a(u,v) = \ell(v), \qquad \forall v \in \mathrm{H}^1_0(\Omega),\end{array}\right. \tag{I.6}$$

with

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v - k^2uv\,dxdy, \qquad\qquad \ell(v) = \int_\Omega fv\,dxdy.$$

The bilinear form $a(\cdot,\cdot)$ is continuous in $\mathrm{H}^1_0(\Omega)$. Therefore, with the Riesz representation theorem, we can introduce the linear bounded operator $A(k) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ such that

$$(A(k)u,v)_{\mathrm{H}^1(\Omega)} = a(u,v), \qquad \forall u,v \in \mathrm{H}^1_0(\Omega). \tag{I.7}$$

In this section, we establish the following statement.

Theorem I.1. *Pick $k \in (0;\pi)$. The operator $A(k)$ decomposes as*

$$A(k) = B + K$$

where $B : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isomorphism and $K : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is compact (B and K are allowed to depend on k).

Proof. Define the bilinear form $b(\cdot,\cdot)$ such that

$$b(u,v) = \int_\Omega \nabla u \cdot \nabla v + ((1+k^2)\mathbb{1}_{\Omega_d} - k^2)uv\, dxdy, \qquad \forall u,v \in \mathrm{H}^1_0(\Omega),$$

where $\mathbb{1}_{\Omega_d}$ stands for the indicator function of the set $\Omega_d := \{(x,y) \in \Omega \,|\, |x| < d\}$. Since $b(\cdot,\cdot)$ is continuous in $\mathrm{H}^1_0(\Omega)$, we can define the bounded operator $B : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ such that

$$(Bu,v)_{\mathrm{H}^1(\Omega)} = b(u,v), \qquad \forall u,v \in \mathrm{H}^1_0(\Omega).$$

From the Lax-Milgram theorem, to show that B is an isomorphism, it suffices to prove that $b(\cdot,\cdot)$ is coercive in $\mathrm{H}^1_0(\Omega)$. Below we establish the 1D Poincaré inequality

$$\pi^2 \int_I \varphi^2\, dt \le \int_I (\partial_t \varphi)^2\, dt, \qquad \forall \varphi \in \mathrm{H}^1_0(I) := \{\psi \in \mathrm{H}^1(I) \,|\, \psi(0) = \psi(1) = 0\}, \tag{I.8}$$

where we recall that $I = (0;1)$. Integrating this estimate with respect to $x \in (-\infty;-d) \cup (d;+\infty)$ for $u \in \mathscr{C}^\infty_0(\Omega)$ (the space of infinitely differentiable functions supported in Ω) and using the density of $\mathscr{C}^\infty_0(\Omega)$ in $\mathrm{H}^1_0(\Omega)$, we obtain

$$\pi^2 \int_{\Omega\setminus\overline{\Omega_d}} u^2\, dxdy \le \int_{\Omega\setminus\overline{\Omega_d}} |\nabla u|^2\, dxdy, \qquad \forall u \in \mathrm{H}^1_0(\Omega).$$

Therefore we can write

$$\begin{array}{rcl} b(u,u) & \ge & \left(1 - \dfrac{k^2}{\pi^2}\right) \displaystyle\int_{\Omega\setminus\overline{\Omega_d}} |\nabla u|^2\, dxdy + \|u\|^2_{\mathrm{H}^1(\Omega_d)} \\ & \ge & \left(1 - \dfrac{k^2}{\pi^2}\right) \displaystyle\int_{\Omega\setminus\overline{\Omega_d}} |\nabla u|^2\, dxdy + \|u\|^2_{\mathrm{H}^1(\Omega_d)} \\ & \ge & (1+\pi^2)^{-1}\left(1 - \dfrac{k^2}{\pi^2}\right) \|u\|^2_{\mathrm{H}^1(\Omega\setminus\overline{\Omega_d})} + \|u\|^2_{\mathrm{H}^1(\Omega_d)} \ge \alpha\, \|u\|^2_{\mathrm{H}^1(\Omega)} \end{array}$$

with $\alpha = (1+\pi^2)^{-1}(1-k^2/\pi^2)$. This shows that $b(\cdot,\cdot)$ is coercive in $\mathrm{H}^1_0(\Omega)$.

Now set $K = A(k) - B$. We have

$$(Ku,v)_{\mathrm{H}^1(\Omega)} = -(1+k^2) \int_\Omega uv\, dxdy, \qquad \forall u,v \in \mathrm{H}^1_0(\Omega), \tag{I.9}$$

To establish that $K : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is compact, we have to prove that from any bounded sequence (u_n) of functions of $\mathrm{H}^1_0(\Omega)$, we can extract a subsequence such that (Ku_n) converges in $\mathrm{H}^1_0(\Omega)$. By taking $v = Ku$ in (I.39), we obtain, for all $u \in \mathrm{H}^1_0(\Omega)$,

$$\|Ku\|_{\mathrm{H}^1(\Omega)} \le (1+k^2)\, \|u\|_{\mathrm{L}^2(\Omega_d)}.$$

In particular, for $u_{mn} := u_m - u_n$, we obtain

$$\|Ku_{mn}\|_{\mathrm{H}^1(\Omega)} \le (1+k^2)\, \|u_{mn}\|_{\mathrm{L}^2(\Omega_d)}. \tag{I.10}$$

Since Ω_d is bounded, the Rellich theorem ensures that the embedding of $\mathrm{H}^1(\Omega_d)$ in $\mathrm{L}^2(\Omega_d)$ is compact. We deduce that we can extract from (u_n), which is bounded in $\mathrm{H}^1(\Omega)$ and so in $\mathrm{H}^1(\Omega_d)$, a subsequence, still denoted (u_n) such that (u_n) converges in $\mathrm{L}^2(\Omega_d)$. Thus (u_n) is a Cauchy sequence in $\mathrm{L}^2(\Omega_d)$. From (I.10), we infer that (Ku_n) is a Cauchy sequence in $\mathrm{H}^1_0(\Omega)$. Since this space is complete, we infer that (Ku_n) indeed converges in $\mathrm{H}^1_0(\Omega)$. $\square$

This shows that $A(k)$ satisfies the Fredholm alternative. Either $A(k)$ is injective and in this case it is an isomorphism of $\mathrm{H}^1_0(\Omega)$. Or $A(k)$ has a kernel of finite dimension $\mathrm{span}(u_1, \dots, u_P)$ and in that case the equation

$$A(k)u = F \qquad \text{in } \mathrm{H}^1_0(\Omega) \tag{I.11}$$

has a solution (defined up to $\mathrm{span}(u_1, \dots, u_P)$) if and only if F satisfies the compatibility conditions

$$(F, u_p)_{\mathrm{H}^1(\Omega)} = 0, \qquad p = 1, \dots, P. \tag{I.12}$$

Let us emphasize that by multiplying (I.11) by u_p and using the symmetry of $A(k)$, one easy finds that the conditions (I.12) are necessary for the existence of a solution.

We prove now the Poincaré inequality needed in (I.8).

Lemma I.2. *We have*

$$\inf_{\varphi \in \mathrm{H}^1_0(I)\setminus\{0\}} \frac{\|\partial_t \varphi\|^2_{\mathrm{L}^2(I)}}{\|\varphi\|^2_{\mathrm{L}^2(I)}} = \pi^2 \tag{I.13}$$

so that there holds

$$\pi^2 \|\varphi\|^2_{\mathrm{L}^2(I)} \le \|\partial_t \varphi\|^2_{\mathrm{L}^2(I)}, \qquad \forall \varphi \in \mathrm{H}^1_0(I). \tag{I.14}$$

Proof. To obtain 1D Poincaré inequalities as (I.14), a classical approach consists in working with explicit representations. More precisely, for $\varphi \in \mathscr{C}^\infty_0(I)$, we can write, for $s \in (0;1)$,

$$\varphi(s) = \int_0^s \partial_t \varphi(t)\, dt.$$

According to the Cauchy-Schwarz inequality in L^2, this implies, for all $s \in (0; 1/2)$,

$$\varphi^2(s) \le \int_0^s 1\, dt \int_0^s (\partial_t \varphi(t))^2\, dt \le s\, \|\partial_t \varphi\|^2_{\mathrm{L}^2(0;1/2)}.$$

Integrating this identity between 0 and $1/2$, we obtain

$$\|\varphi\|^2_{\mathrm{L}^2(0;1/2)} \le \frac{1}{8} \|\partial_t \varphi\|^2_{\mathrm{L}^2(0;1/2)}.$$

By establishing a similar estimate on $(1/2; 1)$ (note that $\varphi(1) = 0$) and using the density of $\mathscr{C}^\infty_0(I)$ in $\mathrm{H}^1_0(I)$, we find

$$\|\varphi\|^2_{\mathrm{L}^2(I)} \le \frac{1}{8} \|\partial_t \varphi\|^2_{\mathrm{L}^2(I)} \qquad \Leftrightarrow \qquad 8\, \|\varphi\|^2_{\mathrm{L}^2(I)} \le \|\partial_t \varphi\|^2_{\mathrm{L}^2(I)}, \qquad \forall \varphi \in \mathrm{H}^1_0(I).$$

This is a nice Poincaré inequality but it is not optimal (observe that (I.14) is better because $8 < \pi^2$). Looking for the best Poincaré inequality leads us to consider the minimization problem

$$\inf_{\varphi \in \mathrm{H}^1_0(I)\setminus\{0\}} \frac{\|\partial_t \varphi\|^2_{\mathrm{L}^2(I)}}{\|\varphi\|^2_{\mathrm{L}^2(I)}}. \tag{I.15}$$

Below we prove that this infimum, equal to some $\lambda > 0$, is actually a minimum because it is attained at some functions $u \in \mathrm{H}^1_0(I) \setminus \{0\}$. Moreover we establish that these quantities satisfy

$$-\partial^2_{tt} u = \lambda u \qquad \text{in } I. \tag{I.16}$$

In other words, λ is an eigenvalue (the smallest) of the Dirichlet Laplacian and u is a corresponding eigenfunction. Since $I = (0; 1)$, a direct computation gives $\lambda = \pi^2$ with $u(t) = \sin(\pi t)$ (up to a multiplicative constant which does not change the ratio in (I.15)). Thus we obtain (I.13) and so (I.14).

To prove that the infimum in (I.15) is reached, let us first remark that solving (I.15) is equivalent to solve the constrained minimization problem

$$\inf_{\varphi \in K} \left\{ J(\varphi) = \int_I \partial_t \varphi^2 \, dt \right\}$$

with $\mathscr{B} := \{\varphi \in \mathrm{H}^1_0(I) \mid \int_I \varphi^2 \, dt = 1\}$. The functional J is positive in $\mathscr{B}$, therefore we have $\inf_{\mathscr{B}} J \geq 0$. Consider $(u_n) \in \mathscr{B}^{\mathbb{N}}$ a minimizing sequence for J, *i.e.* a sequence of functions of $\mathscr{B}$ such that

$$\lim_{n \to +\infty} J(u_n) = \inf_{\mathscr{B}} J.$$

The sequence $(J(u_n))$ is bounded and so (u_n) is bounded in $\mathrm{H}^1_0(I)$. Then we know that we can extract a subsequence, still denoted by (u_n), such that there is some $u \in \mathrm{H}^1_0(I)$ such that

$$u_n \rightharpoonup u \text{ weakly in } \mathrm{H}^1_0(I), \qquad u_n \to u \text{ strongly in } \mathrm{L}^2(I).$$

Now, by writing

$$0 \leq (\partial_t(u - u_n), \partial_t(u - u_n))_{\mathrm{L}^2(\Omega)} = \|\partial_t u\|^2_{\mathrm{L}^2(I)} + \|\partial_t u_n\|^2_{\mathrm{L}^2(I)} - 2(\partial_t u, \partial_t u_n)_{\mathrm{L}^2(I)},$$

we obtain $2(\partial_t u, \partial_t u_n)_{\mathrm{L}^2(I)} - \|\partial_t u\|^2_{\mathrm{L}^2(I)} \leq \|\partial_t u_n\|^2_{\mathrm{L}^2(I)}$. By passing to the inferior limit, we deduce

$$\|\partial_t u\|^2_{\mathrm{L}^2(I)} \leq \liminf_{n \to +\infty} \|\partial_t u_n\|^2_{\mathrm{L}^2(I)}$$

and so

$$J(u) \leq \liminf_{n \to +\infty} J(u_n) = \inf_{\mathscr{B}} J.$$

But, for all $n \in \mathbb{N}$, we have $\|u_n\|_{\mathrm{L}^2(I)} = 1$. Since (u_n) converges strongly to u in $\mathrm{L}^2(I)$, we deduce $\|u\|_{\mathrm{L}^2(I)} = 1$, which shows that u belongs to $\mathscr{B}$. Thus J attains its infimum in $\mathscr{B}$ at u.

Set $\lambda = J(u)$. For all $v \in \mathrm{H}^1_0(I) \setminus \{0\}$, we have $v/\|v\|_{\mathrm{L}^2(I)} \in \mathscr{B}$ and so

$$J(v/\|v\|_{\mathrm{L}^2(I)}) \geq \lambda.$$

This gives, for all $v \in \mathrm{H}^1_0(I) \setminus \{0\}$,

$$\int_I (\partial_t v)^2 - \lambda v^2 \, dt \geq 0.$$

Then we are led to consider the following minimization problem without constraint

$$\min_{v \in \mathrm{H}^1_0(I)} \left\{ \tilde{J}(v) = \int_I (\partial_t v)^2 - \lambda v^2 \, dt \right\}.$$

According to what precedes, the functional $\tilde{J}$ is non negative in $\mathrm{H}^1_0(I)$. Moreover, we have $\tilde{J}(u) = 0$ and $u \in \mathrm{H}^1_0(I)$. Therefore u is a minimizer of $\tilde{J}$ in $\mathrm{H}^1_0(I)$. Thus we must have, for all $v \in \mathrm{H}^1_0(I)$, $s \in \mathbb{R}$,

$$0 = \tilde{J}(u) \leq \tilde{J}(u + sv).$$

This is equivalent to

$$0 \leq s \left(\int_I \partial_t u \, \partial_t v - \lambda uv \, dt \right) + s^2 \tilde{J}(v), \qquad \forall v \in \mathrm{H}^1_0(I),\ s \in \mathbb{R},$$

which holds if and only if

$$\int_I \partial_t u \, \partial_t v \, dt = \lambda \int_I uv \, dt, \qquad \forall v \in \mathrm{H}^1_0(I).$$

We deduce that, in the sense of distributions, we must have the equation (I.16). □

In Theorem I.1, we proved that $A(k)$ satisfies the Fredholm alternative for $k \in (0;\pi)$. We give now a result of injectivity in certain geometries.

Proposition I.3. *Assume that $k \in (0;\pi)$ and $\Omega \subset \mathcal{S} = \mathbb{R} \times I$. Then the operator $A(k)$ defined in (I.7) is injective, and so is an isomorphism of $\mathrm{H}^1_0(\Omega)$. In that case, Problem (I.6) admits a unique solution for all $f \in \mathrm{L}^2(\Omega)$.*

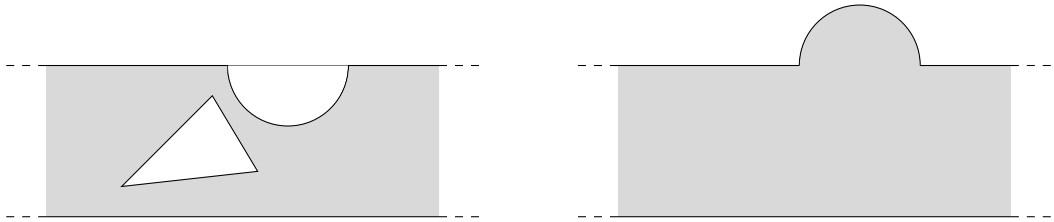


Figure I.2: Left: example of domain where $A(k)$ is an isomorphism for all $k \in (0;\pi)$. Right: example of domain where $A(k)$ is not an isomorphism for all $k \in (0;\pi)$.

Remark I.4. *In Figure I.2 left, we give an example of domain Ω satisfying the assumption $\Omega \subset \mathcal{S}$. We emphasize that the injectivity of $A(k)$ for all $k \in (0;\pi)$ does not always hold and in waveguides with exterior bumps (with respect to the reference strip $\mathcal{S}$) as in Figure I.2 right, $A(k)$ can have a non zero kernel for certain $k \in (0;\pi)$.*

Proof. When $\Omega \subset \mathcal{S}$, if a function belongs to $\mathrm{H}^1_0(\Omega)$, then its extension by zero to $\mathcal{S}$ is an element of $\mathrm{H}^1_0(\mathcal{S})$. Therefore we have

$$\inf_{u \in \mathrm{H}^1_0(\mathcal{S})\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dxdy}{\displaystyle\int_\Omega u^2\,dxdy} \le \inf_{u \in \mathrm{H}^1_0(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dxdy}{\displaystyle\int_\Omega u^2\,dxdy}. \tag{I.17}$$

Now integrating the 1D Poincaré inequality (I.14) with respect to $x \in \mathbb{R}$ for $u \in \mathscr{C}^\infty_0(\mathcal{S})$ and using the density of $\mathscr{C}^\infty_0(\mathcal{S})$ in $\mathrm{H}^1_0(\mathcal{S})$, we obtain

$$\pi^2 \int_{\mathcal{S}} u^2\,dxdy \le \int_{\mathcal{S}} |\nabla u|^2\,dxdy, \qquad \forall u \in \mathrm{H}^1_0(\mathcal{S}).$$

From (I.17), this gives

$$\pi^2 \int_{\Omega} u^2\,dxdy \le \int_{\Omega} |\nabla u|^2\,dxdy, \qquad \forall u \in \mathrm{H}^1_0(\Omega).$$

Therefore, if $u \in \mathrm{H}^1_0(\Omega)$ is such that $A(k)u = 0$, then we have

$$\begin{aligned} 0 = a(u,u) &= \int_\Omega |\nabla u|^2 - k^2u^2\,dxdy \\ &\ge (\pi^2 - k^2)\int_\Omega u^2\,dxdy, \end{aligned}$$

which ensures that $u \equiv 0$ in Ω. This shows that $A(k)$ is injective and Theorem I.1 together with the Fredholm alternative guarantee that $A(k)$ is an isomorphism of $\mathrm{H}^1_0(\Omega)$. $\square$

3 Dirichlet problem for $k > \pi$

In the previous paragraph, we studied Problem (I.3) for $k < \pi$. Now we wish to understand what happens for $k \ge \pi$. To proceed, we first compute what one usually calls the "modes" of (I.3). They play a key role in the physical phenomena and so in the mathematical properties of (I.3).

3.1 Computation of modes

The modes are defined as the solutions with separate variables, *i.e.* of the form

$$u(x,y) = \alpha(x)\varphi(y), \tag{I.18}$$

which solve Problem (I.3) in the reference strip $\mathcal{S}$ for $f \equiv 0$. Inserting (I.18) in the equation $\Delta u + k^2 u = 0$ in $\mathcal{S}$, this yields

$$\alpha''(x)\varphi(y) + \alpha(x)\varphi''(y) + k^2\alpha(x)\varphi(y) = 0.$$

Dividing this identity by $\alpha(x)\varphi(y)$, we find that we must have

$$\left|\begin{array}{ll} -\varphi''(y) = \lambda\,\varphi(y) & \text{in } I \\ \varphi(0) = \varphi(1) = 0 & \end{array}\right. \tag{I.19}$$

and

$$-\alpha''(x) = (k^2 - \lambda)\,\alpha(x) \qquad \text{in } \mathbb{R} \tag{I.20}$$

for some constant λ to be determined. Problem (I.19) is a spectral problem: we wish to find the values of λ such that (I.19) admits a non zero solution φ. A direct calculation shows that the eigenpairs of (I.19) are given by

$$\lambda_n = n\pi, \qquad \varphi_n(y) = \sqrt{2}\sin(n\pi y), \qquad n \in \mathbb{N}^* := \{1, 2, \dots\}. \tag{I.21}$$

Note that the φ_n have been chosen such that they satisfy the orthonormality conditions

$$(\varphi_m, \varphi_n)_{\mathrm{L}^2(I)} = \delta_{m,n}$$

where $\delta_{m,n}$ stands for the Kronecker symbol. Then solving the second order ODE (I.20), finally we find that when $k \neq n\pi$ for all $n \in \mathbb{N}^*$, the modes coincide with the family $\{w_n^\pm\}_{n\in\mathbb{N}^*}$ where

$$w_n^\pm(x,y) = e^{\pm i\beta_n x}\varphi_n(y), \qquad \beta_n := \sqrt{k^2 - n^2\pi^2}. \tag{I.22}$$

Here and below, the complex square root is chosen (this is a convention) such that if $z = re^{i\theta}$ with $r \geq 0$ and $\theta \in [0; 2\pi)$, then $\sqrt{z} = \sqrt{r}e^{i\theta/2}$. As a consequence, for any $z \in \mathbb{C}$, there holds $\Im m\,\sqrt{z} \geq 0$.

Let us make a few observations concerning these modes. To set ideas, introduce $N \in \mathbb{N}$ such that $k \in (N\pi; (N+1)\pi)$.

$\star$ For $n = 1, ..., N$ (ignore this case if $N = 0$), we have

$$w_n^\pm(x,y) = e^{\pm i\sqrt{k^2-n^2\pi^2}x}\varphi_n(y).$$

Since $\sqrt{k^2 - n^2\pi^2} > 0$ for $n = 1, ..., N$, these modes do not decay at infinity. They are called propagating modes. For a fixed $k > 0$, there is always a finite number of propagating modes. Moreover they do not exist when $k \in (0; \pi)$ (the situation studied in the previous paragraph). On the contrary, for all $k > \pi$, the modes $w_1^\pm$ are propagating. Going back to time-domain, we observe that these modes lead to consider solutions of (I.1) of the form

$$W_n^\pm(x,y,t) = e^{i(\pm\sqrt{k^2-n^2\pi^2}x - \omega t)}\varphi_n(y).$$

The waves W_n^+ propagate to the right while the W_n^- propagate to the left. For this reason, we will say that the w_n^+ are rightgoing modes while the w_n^- are leftgoing.

$\star$ For $n = N+1, N+2, ...$, we have

$$w_n^\pm(x,y) = e^{\mp\sqrt{n^2\pi^2-k^2}x}\varphi_n(y).$$

Since $\sqrt{n^2\pi^2 - k^2} > 0$ for $n = N+1, N+2, \dots$, these modes are exponentially decaying as $x \to \pm\infty$ and exponentially growing as $x \to \mp\infty$. There are an infinite number of them.

Though these modes have been computed for the problem in the reference strip $\mathcal{S}$, we will also use them in the analysis of Problem (I.3) in the perturbed domain Ω.

3.2 Ill-posedness in $\mathrm{H}_0^1(\Omega)$

In this paragraph, our goal is to show that the existence of propagating modes for $k > \pi$ is responsible for the ill-posedness in the Fredholm sense of the operator $A(k) : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ defined in (I.7).

Definition I.5. *Let* X *and* Y *be two Banach spaces, and let* $T : \mathrm{X} \to \mathrm{Y}$ *be a continuous linear map. The operator* T *is said to be a Fredholm operator if and only if the following two conditions are fulfilled*

i) $\dim(\ker T) < +\infty$ *and* $\mathrm{range}\, T$ *is closed;*

ii) $\dim(\mathrm{coker}\, T) < +\infty$ *where* $\mathrm{coker}\, T := (\mathrm{Y}/\mathrm{range}\, T)$.

Besides, the index of a Fredholm operator T *is defined by* $\mathrm{ind}\, T = \dim(\ker T) - \dim(\mathrm{coker}\, T)$.

To prove that the range of $A(k)$ is not closed, we start by recalling a lemma due to J. Peetre [33] (see also Theorem 12.12 in [41]).

Lemma I.6. *Let* X, Y, Z *be three reflexive Banach spaces, such that* X *is compactly embedded into* Z. *Let* $T : \mathrm{X} \to \mathrm{Y}$ *be a continuous linear map. Then the assertions below are equivalent:*

i) $\dim(\ker T) < +\infty$ *and* $\mathrm{range}\, T$ *is closed in* Y*;*

ii) there exists $C > 0$ *such that* $\|u\|_{\mathrm{X}} \le C\,(\|Tu\|_{\mathrm{Y}} + \|u\|_{\mathrm{Z}})$, $\forall u \in \mathrm{X}$.

Proposition I.7. *For* $k > \pi$, *the operator* $A(k) : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ *defined in (I.7) is not Fredholm.*

Remark I.8. *We exclude the case* $k = \pi$ *because the computations are a bit different. However in that situation too one can prove that* $A(k) : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ *is not Fredholm.*

Proof. Set again $\Omega_d := \{(x,y) \in \Omega \,|\, |x| < d\}$ where d appears before (I.3). Our goal is to show that we cannot have the existence of $C > 0$ such that there holds

$$\|u\|_{\mathrm{H}^1(\Omega)} \le C\,(\|A(k)u\|_{\mathrm{H}^1(\Omega)} + \|u\|_{\mathrm{L}^2(\Omega_d)}), \qquad \forall u \in \mathrm{H}_0^1(\Omega). \tag{I.23}$$

To proceed, consider some functions $\psi_+, \psi_- \in \mathscr{C}^\infty(\mathbb{R})$ such that

$$\psi_+(x) = \left|\begin{array}{ll} 1 & \text{for } x > d+1 \\ 0 & \text{for } x < d \end{array}\right. \qquad\qquad \psi_-(x) = \left|\begin{array}{ll} 1 & \text{for } x < 0 \\ 0 & \text{for } x > 1. \end{array}\right.$$

Then for $m \in \mathbb{N}$, set $\psi_m(x) = \psi_+(x)\psi_-(x-m)$ and

$$u_m(x,y) = \psi_m(x) w_1^+(x,y) = \psi_m(x) e^{i\beta_1 x} \varphi_1(y),$$

where w_1^+ is the mode appearing in (I.22) which is propagating for $k > \pi$ (because then $\beta_1 = \sqrt{k^2 - \pi^2} \in \mathbb{R}$).

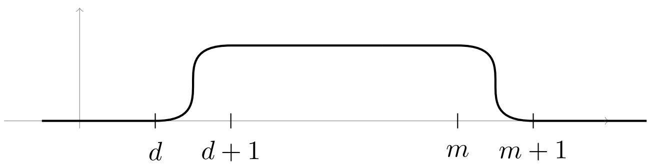


Figure I.3: Graphs of the cut-off function ψ_m.

Exploiting that the support of u_m becomes unbounded as $m \to +\infty$, it is straightforward to show that

$$\lim_{m \to +\infty} \|u_m\|_{\mathrm{H}^1(\Omega)} = +\infty.$$

On the other hand, clearly $(\|u_m\|_{\mathrm{L}^2(\Omega_d)})$ remains bounded as $m \to +\infty$. Now, for $v \in \mathrm{H}^1_0(\Omega)$, we have

$$(A(k)u_m, v)_{\mathrm{H}^1(\Omega)} = \int_\Omega \nabla u_m \cdot \nabla v - k^2 u_m v \, dxdy = -\int_\Omega (\Delta u_m + k^2 u_m) v \, dxdy. \tag{I.24}$$

But there holds

$$\begin{array}{rcl} \Delta u_m + k^2 u_m & = & \psi_m(\Delta w_1^+ + k^2 w_1^+) + 2\nabla\psi_m \cdot \nabla w_1^+ + w_1^+ \Delta\psi_m \\ & = & 2\nabla\psi_m \cdot \nabla w_1^+ + w_1^+ \Delta\psi_m. \end{array}$$

By observing that ψ_m, $\Delta\psi_m$ are non zero only in $(d; d+1) \times (0;1) \cup (m; m+1) \times (0;1)$ and that their norms in $\mathrm{L}^\infty(\Omega)$ remain bounded independently of $m \in \mathbb{N}$, we find that there exists $C > 0$ independent of m such that we have

$$\|\Delta u_m + k^2 u_m\|_{\mathrm{L}^2(\Omega)} \le C.$$

By taking $v = A(k)u_m$ in (I.24), we conclude that $(A(k)u_m)$ remains bounded in $\mathrm{H}^1_0(\Omega)$ as $m \to +\infty$. This shows that Estimate (I.23) does not hold.
Finally, since Ω_d is bounded, the embedding of $\mathrm{H}^1_0(\Omega)$ in $\mathrm{L}^2(\Omega_d)$ is compact. From Lemma I.6, we deduce that $A(k) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is not Fredholm when $k > \pi$. □

By working in weighted Sobolev spaces, one can show that $A(k)$ has a kernel of finite dimension for all $k > \pi$. Therefore, the loss of Fredholmness is due to the fact that the range of $A(k)$ is not closed then $k > \pi$. Thus, even by removing the kernel if there exists one, we cannot create an operator in $\mathrm{H}^1_0(\Omega)$ which admits a continuous inverse when propagating modes exist. This leads us to think that we have to take them into account in the functional framework.

To proceed, we will apply some strategy which is classical in applied mathematics related to physics: we will add a bit of dissipation in the medium characterized by some parameter $\eta > 0$ and then take the limit as $\eta \to 0$. More precisely, with dissipation the definition of the physical solution, the one in $\mathrm{H}^1_0(\Omega)$, becomes obvious. Then we will define the solution without dissipation as the limit as $\eta \to 0$ of the solution with dissipation. This is called the limiting absorption in scattering theory. In fluid mechanics, dissipation is more often refereed to as viscosity but the idea is the same.

3.3 Problem with dissipation

To model dissipation, let us work on the problem

$$\left|\begin{array}{rcll} \Delta u_\eta + (k^2 + ik\eta)u_\eta & = & f & \text{in } \Omega \\ u_\eta & = & 0 & \text{on } \partial\Omega \end{array}\right. \tag{I.25}$$

with $\eta > 0$. To get an idea of why this is a relevant way to model dissipation, let us come back to time domain. With the time harmonic convention $U_\eta(x,y,t) = u_\eta(x,y)e^{-i\omega t}$, Problem (I.25) originates from the study of the wave equation, for $t \ge 0$,

$$\left|\begin{array}{rcll} \dfrac{\partial^2 U_\eta}{\partial t^2} + \eta \dfrac{\partial U_\eta}{\partial t} - \dfrac{1}{c^2}\Delta U_\eta & = & F & \text{in } \Omega \\ U_\eta & = & 0 & \text{on } \partial\Omega, \end{array}\right. \tag{I.26}$$

with some initial conditions. Assume that the forcing term F is null. Then multiplying (I.26) by $\partial_t \overline{U}$ and integrating in Ω, we obtain

$$\frac{\partial}{\partial t}\frac{1}{2}\int_\Omega \left|\frac{\partial U_\eta}{\partial t}\right|^2 + \frac{1}{c^2}|\nabla U_\eta|^2 \, dxdy = -\eta \int_\Omega \left|\frac{\partial U_\eta}{\partial t}\right|^2 dxdy.$$

Therefore, the energy

$$E(t) = \frac{1}{2}\int_{\Omega}\left|\frac{\partial U_\eta}{\partial t}\right|^2 + \frac{1}{c^2}|\nabla U_\eta|^2\,dxdy$$

indeed decreases, due to the term $\eta\partial_t U_\eta$, when $\eta > 0$.

The variational formulation associated with (I.25) writes

$$\left|\begin{array}{l}\text{Find } u_\eta \in \mathrm{H}^1_0(\Omega) \text{ such that}\\ a_\eta(u_\eta, v) = \ell(v), \qquad \forall v \in \mathrm{H}^1_0(\Omega),\end{array}\right. \tag{I.27}$$

where the sesquilinear (resp. antilinear) forms $a_\eta(\cdot,\cdot)$ (resp. $\ell(\cdot)$) are such that

$$a_\eta(w,v) = \int_\Omega \nabla w\cdot\nabla\overline{v} - (k^2 + ik\eta)w\overline{v}\,dxdy, \qquad\qquad \ell(v) = \int_\Omega f\overline{v}\,dxdy.$$

Note that the functions are now assumed to be complex valued and the inner products introduced in (I.4), (I.5) are changed accordingly.

Theorem I.9. *For all $k > 0$, for all $\eta > 0$, Problem (I.27) admits a unique solution $u_\eta \in \mathrm{H}^1_0(\Omega)$.*

Proof. For $v \in \mathrm{H}^1_0(\Omega)$, we have

$$\Re e\, a_\eta(v,v) = \int_\Omega |\nabla v|^2 - k^2|v|^2\,dxdy, \qquad \Im m\, a_\eta(v,v) = -k\eta\int_\Omega |v|^2\,dxdy,$$

Therefore, we obtain, for $\gamma > 0$,

$$\Re e\,((1+i\gamma)a_\eta(v,v)) = \Re e\, a_\eta(v,v) - \gamma\,\Im m\, a_\eta(v,v) = \int_\Omega |\nabla v|^2 + (\gamma k\eta - k^2)\,|v|^2\,dxdy.$$

Thus for any $\eta > 0$, for $\gamma > 0$ large enough, the form $(1+i\gamma)a_\eta(\cdot,\cdot)$ is coercive in $\mathrm{H}^1_0(\Omega)$. With the complex version of the Lax-Milgram theorem, this is enough to conclude that Problem (I.27) admits a unique solution in that case. □

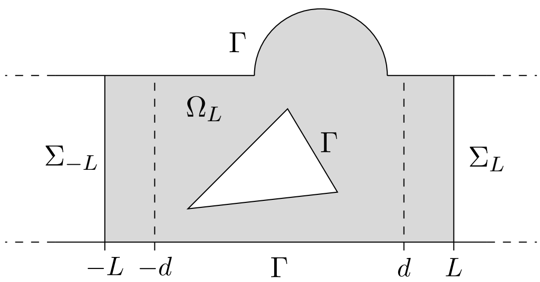


Figure I.4: Domain Ω_L.

Now assume that f in (I.25) is given in $\mathrm{L}^2(\Omega)$ and supported in Ω_d. Introduce some $L > d$ and define the bounded domain

$$\Omega_L := \{(x,y) \in \Omega\,|\,|x| < L\}$$

(see Figure I.4). To take the limit η tends to zero in (I.27), we first derive a problem set in Ω_L whose solution coincides with $u_\eta|_{\Omega_L}$. To proceed, we must impose *ad hoc* transparent conditions on the artificial boundaries

$$\Sigma_\pm := \{\pm L\} \times I$$

that do not create spurious reflections. Set $\mathcal{S}_+ := (L;+\infty)\times I$ and define the Dirichlet-to-Neumann operator such that

$$\begin{array}{rccc}\Lambda^\eta_+: & \mathrm{H}^{1/2}_{00}(\Sigma_L) & \to & \mathrm{H}^{-1/2}(\Sigma_L)\\ & \varphi & \mapsto & \dfrac{\partial v_\varphi}{\partial \nu},\end{array}$$

where $\partial_\nu = \partial_x$ on Σ_L and $v_\varphi \in \mathrm{H}^1(\mathcal{S}_+)$ is the function such that

$$\left|\begin{array}{rcll} \Delta v_\varphi + (k^2 + ik\eta) v_\varphi & = & 0 & \text{in } \mathcal{S}_+ \\ v_\varphi & = & 0 & \text{on } \partial\Omega \cap \partial\mathcal{S}_+ \\ v_\varphi & = & \varphi & \text{on } \Sigma_L. \end{array}\right. \tag{I.28}$$

Here $\mathrm{H}^{1/2}_{00}(\Sigma_L)$ stands for the space of traces on Σ_L of elements of $\mathrm{H}^1_0(\Omega_L)$. It coincides with the functions which belong to $\mathrm{H}^{1/2}(\{L\}\times\mathbb{R})$ when extended by zero. Moreover $\mathrm{H}^{-1/2}(\Sigma_L)$ denotes the dual space of $\mathrm{H}^{1/2}_{00}(\Sigma_L)$. Classically, one shows that the linear operator $\Lambda^\eta_+ : \mathrm{H}^{1/2}_{00}(\Sigma_L) \to \mathrm{H}^{-1/2}(\Sigma_L)$ is continuous. If u_η solves (I.27), then it satisfies

$$\frac{\partial u_\eta}{\partial \nu} = \Lambda^\eta_+(u_\eta) \qquad \text{on } \Sigma_L, \tag{I.29}$$

where again $\partial_\nu = \partial_x$ on Σ_L.

In the following, it will be useful to have an explicit representation of the action of Λ^η_+. We will obtain it by working with the modes of (I.28). Assume that k is not equal to one of the $n\pi$, $n \in \mathbb{N}^*$. In every transverse section $\{x\}\times I$ of $\mathcal{S}_+$, we have the decomposition in Fourier series

$$v_\varphi(x,y) = \sum_{n=1}^{+\infty} \alpha_n(x)\, \varphi_n(y) \tag{I.30}$$

where the φ_n are the ones introduced in (I.21) and the α_n are to be determined. Inserting (I.30) into (I.25), similarly to (I.20), we find that the α_n must be of the form

$$\alpha_n(x) = A_n\, e^{i\beta^\eta_n x} + B_n\, e^{-i\beta^\eta_n x}$$

for some constants A_n, $B_n \in \mathbb{C}$ with

$$\beta^\eta_n = \sqrt{k^2 + i\eta - n^2\pi^2}.$$

But according to our convention for the complex square root after (I.22), for $\eta > 0$, the imaginary part of β^η_n is positive for all $n \in \mathbb{N}^*$. As a consequence, $x \mapsto e^{i\beta^\eta_n x}$ is exponentially decaying at $+\infty$ while $x \mapsto e^{-i\beta^\eta_n x}$ is exponentially growing. Since v_φ belongs to $\mathrm{H}^1(\mathcal{S}_+)$, we must impose $B_n = 0$ for all $n \in \mathbb{N}^*$. Thus for $x > L$, we have the expansion

$$v_\varphi(x,y) = \sum_{n=1}^{+\infty} (\varphi, \varphi_n)_{\mathrm{L}^2(\Sigma_L)}\, e^{i\beta^\eta_n (x-L)}\, \varphi_n(y)$$

so that there holds

$$\Lambda^\eta_+(\varphi) = \sum_{n=1}^{+\infty} i\beta^\eta_n\, (\varphi, \varphi_n)_{\mathrm{L}^2(\Sigma_L)}\, e^{i\beta^\eta_n (x-L)}\, \varphi_n(y).$$

We work completely similarly in $\mathcal{S}_- := (-\infty; -L)\times I$ and define the Dirichlet-to-Neumann operator such that

$$\begin{array}{rccc} \Lambda^\eta_- : & \mathrm{H}^{1/2}_{00}(\Sigma_{-L}) & \to & \mathrm{H}^{-1/2}(\Sigma_{-L}) \\ & \varphi & \mapsto & \dfrac{\partial v_\varphi}{\partial \nu}, \end{array}$$

where this time $\partial_\nu = -\partial_x$ on Σ_{-L} and $v_\varphi \in \mathrm{H}^1(\mathcal{S}_-)$ is the function such that

$$\left|\begin{array}{rcll} \Delta v_\varphi + (k^2 + ik\eta) v_\varphi & = & 0 & \text{in } \mathcal{S}_- \\ v_\varphi & = & 0 & \text{on } \partial\Omega \cap \partial\mathcal{S}_- \\ v_\varphi & = & \varphi & \text{on } \Sigma_{-L}. \end{array}\right.$$

The operator $\Lambda_-^\eta : \mathrm{H}^{1/2}_{00}(\Sigma_{-L}) \to \mathrm{H}^{-1/2}(\Sigma_{-L})$ is continuous and if u_η solves (I.27), then it satisfies

$$\frac{\partial u_\eta}{\partial \nu} = \Lambda_-^\eta(u_\eta) \quad \text{ on } \Sigma_{-L}, \tag{I.31}$$

with $\partial_\nu = -\partial_x$ on Σ_{-L}. Besides, we have the representation

$$\Lambda_-^\eta(\varphi) = \sum_{n=1}^{+\infty} i\beta_n^\eta\,(\varphi, \varphi_n)_{\mathrm{L}^2(\Sigma_{-L})}\, e^{-i\beta_n^\eta(x+L)}\,\varphi_n(y).$$

Now we have everything to write our problem in Ω_L. Set $\Gamma := \partial\Omega \cap \partial\Omega_L$ and $\mathrm{H}^1_0(\Omega_L;\Gamma) := \{v \in \mathrm{H}^1(\Omega_L)\,|\, v = 0 \text{ on } \Gamma\}$.

Proposition I.10. *If $u_\eta \in \mathrm{H}^1_0(\Omega)$ solves (I.25), then its restriction to Ω_L satisfies*

$$\left|\begin{array}{lcll} \text{Find } u_\eta \in \mathrm{H}^1_0(\Omega_L;\Gamma) \text{ such that} & & & \\ \Delta u_\eta + (k^2 + ik\eta)u_\eta & = & f & \text{in } \Omega_L \\ \dfrac{\partial u_\eta}{\partial \nu} & = & \Lambda_\pm^\eta(u_\eta) & \text{on } \Sigma_{\pm L} \end{array}\right. \tag{I.32}$$

where $\partial_\nu = \pm\partial_x$ at $x = \pm L$. Conversely, if $u_\eta \in \mathrm{H}^1_0(\Omega)$ satisfies (I.32), it can be extended as a solution in $\mathrm{H}^1_0(\Omega)$ of (I.25).

Proof. The first part of the statement comes from (I.29), (I.31). Now if $u_\eta \in \mathrm{H}^1_0(\Omega)$ solves (I.32), define $\hat{u}_\eta$ such that

$$\hat{u}_\eta(x,y) = \left|\begin{array}{ll} u_\eta(x,y) & \text{in } \Omega_L \\ \displaystyle\sum_{n=1}^{+\infty} (u_\eta, \varphi_n)_{\mathrm{L}^2(\Sigma_{\pm L})}\, e^{\pm i\beta_n^\eta(x\mp L)}\,\varphi_n(y) & \text{in } \mathcal{S}_\pm. \end{array}\right.$$

The function $\hat{u}_\eta$ satisfies $\Delta\hat{u}_\eta + (k^2 + ik\eta)\hat{u}_\eta = f$ in $\Omega_L \cup \mathcal{S}_+ \cup \mathcal{S}_-$. Moreover we have $[\hat{u}_\eta]|_{\Sigma_{\pm L}} = 0$ as well as $[\partial_x\hat{u}_\eta]|_{\Sigma_{\pm L}} = 0$ where $[\cdot]|_{\Sigma_{\pm L}}$ stands for the jump at $x = \pm L$. This is enough to conclude that $\hat{u}_\eta$ is the solution of (I.32) in $\mathrm{H}^1_0(\Omega)$. $\square$

3.4 Problem without dissipation

Taking the limit η tends to zero in (I.32), we are led to consider the problem

$$\left|\begin{array}{lcll} \text{Find } u \in \mathrm{H}^1_0(\Omega_L;\Gamma) \text{ such that} & & & \\ \Delta u + k^2 u & = & f & \text{in } \Omega_L \\ \dfrac{\partial u}{\partial \nu} & = & \Lambda_\pm(u) & \text{on } \Sigma_{\pm L} \end{array}\right. \tag{I.33}$$

where $\Lambda_\pm$ are the operators such that

$$\begin{array}{rccl} \Lambda_\pm(\varphi): & \mathrm{H}^{1/2}_{00}(\Sigma_{\pm L}) & \to & \mathrm{H}^{-1/2}(\Sigma_{\pm L}) \\ & \varphi & \mapsto & \displaystyle\sum_{n=1}^{+\infty} i\beta_n\,(\varphi, \varphi_n)_{\mathrm{L}^2(\Sigma_{\pm L})}\, e^{\pm i\beta_n(x\mp L)}\,\varphi_n(y). \end{array} \tag{I.34}$$

The study of Problem (I.33) leads to consider the following variational problem

$$\left|\begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega_L;\Gamma) \text{ such that} \\ a^{\mathrm{out}}(u,v) = \ell(v), \qquad \forall v \in \mathrm{H}^1_0(\Omega_L;\Gamma), \end{array}\right. \tag{I.35}$$

with

$$a^{\text{out}}(u,v) = \int_{\Omega_L} \nabla u\cdot\nabla\overline{v} - k^2 u\overline{v}\,dxdy - \langle \Lambda_+(u), v\rangle_{\Sigma_L} - \langle \Lambda_-(u), v\rangle_{\Sigma_{-L}},\quad \ell(v) = \int_{\Omega_L} f\overline{v}\,dxdy. \tag{I.36}$$

It is important to understand that the Dirichlet-to-Neumann operators $\Lambda_\pm$ in (I.33) on $\Sigma_{\pm L}$ have a double action. First, they allow to look for a solution which decompose on propagating modes which are not in $\mathrm{H}^1_0(\Omega)$. But we cannot allow for all propagating modes in the functional space because otherwise by combining them, we could create non zero functions satisfying the homogeneous problem, *i.e* we would obtain for all frequencies a non zero kernel. But by working with the $\Lambda_\pm$, we also impose radiation conditions and select outgoing behaviors (this explains the choice of the index $^{\text{out}}$ for "outgoing"). At the end, for $\pm x > L$ the solution, which is the physical one, is searched as a superposition of outgoing propagating modes and evanescent modes.

Besides, in (I.36) $\langle\cdot,\cdot\rangle_{\Sigma_{\pm L}}$ stands for the antilinear duality pairing between $\mathrm{H}^{-1/2}(\Sigma_{\pm L})$ and $\mathrm{H}^{1/2}_{00}(\Sigma_{\pm L})$. By exploiting that the operators $\Lambda_\pm(\varphi) : \mathrm{H}^{1/2}_{00}(\Sigma_{\pm L}) \to \mathrm{H}^{-1/2}(\Sigma_{\pm L})$ are continuous and that the trace mappings from $\mathrm{H}^1_0(\Omega_L)$ to $\mathrm{H}^{1/2}_{00}(\Sigma_{\pm L})$ are also continuous, we deduce that the sesquilinear form $a^{\text{out}}(\cdot,\cdot)$ is continuous in $\mathrm{H}^1_0(\Omega_L)$. Therefore, with the Riesz representation theorem, we can introduce the linear operator $A^{\text{out}}(k) : \mathrm{H}^1_0(\Omega_L;\Gamma) \to \mathrm{H}^1_0(\Omega_L;\Gamma)$ such that

$$(A^{\text{out}}(k)u, v)_{\mathrm{H}^1(\Omega_L)} = a^{\text{out}}(u,v), \qquad \forall u, v \in \mathrm{H}^1_0(\Omega_L;\Gamma). \tag{I.37}$$

One has the following statement.

Theorem I.11. *For $k \in (\pi; +\infty)\setminus\{\mathbb{N}\pi\}$, the operator $A^{\text{out}}(k)$ decomposes as*

$$A^{\text{out}}(k) = B^{\text{out}} + K$$

where $B^{\text{out}} : \mathrm{H}^1_0(\Omega_L;\Gamma) \to \mathrm{H}^1_0(\Omega_L;\Gamma)$ is an isomorphism and $K : \mathrm{H}^1_0(\Omega_L;\Gamma) \to \mathrm{H}^1_0(\Omega_L;\Gamma)$ is compact (B^{out} and K are allowed to depend on k).

Proof. Define the continuous operator $B^{\text{out}} : \mathrm{H}^1_0(\Omega_L;\Gamma) \to \mathrm{H}^1_0(\Omega_L;\Gamma)$ such that

$$(B^{\text{out}}u, v)_{\mathrm{H}^1(\Omega_L)} = a^{\text{out}}(u,v), \qquad \forall u, v \in \mathrm{H}^1_0(\Omega_L;\Gamma),$$

with

$$b^{\text{out}}(u,v) = \int_{\Omega_L} \nabla u\cdot\nabla\overline{v} + u\overline{v}\,dxdy - \langle \Lambda_+(u), v\rangle_{\Sigma_L} - \langle \Lambda_-(u), v\rangle_{\Sigma_{-L}}.$$

For $u \in \mathrm{H}^1_0(\Omega_L;\Gamma)$, we have

$$\Re e\, b^{\text{out}}(u,u) = \|u\|^2_{\mathrm{H}^1(\Omega_L)} - \Re e\,\langle \Lambda_+(u), u\rangle_{\Sigma_L} - \Re e\,\langle \Lambda_-(u), u\rangle_{\Sigma_{-L}}. \tag{I.38}$$

But (I.34) provides

$$\langle \Lambda_\pm(u), u\rangle_{\Sigma_{\pm L}} = \sum_{n=1}^{+\infty} i\beta_n\, |(u,\varphi_n)_{\mathrm{L}^2(\Sigma_{\pm L})}|^2.$$

Introduce $N \in \mathbb{N}^*$ such that $k \in (N\pi; (N+1)\pi)$. The $i\beta_n$ for $n = 1,\dots,N$ are purely imaginary. On the other hand, the $i\beta_n$ for $n > N$ are real negative. Thus we have

$$\Re e\,\langle \Lambda_\pm(u), u\rangle_{\Sigma_{\pm L}} = \sum_{n=N+1}^{+\infty} i\beta_n\, |(u,\varphi_n)_{\mathrm{L}^2(\Sigma_{\pm L})}|^2 = \sum_{n=N+1}^{+\infty} -\sqrt{n^2\pi^2 - k^2}\, |(u,\varphi_n)_{\mathrm{L}^2(\Sigma_{\pm L})}|^2 < 0.$$

With (I.38), this gives

$$\Re e\, b^{\text{out}}(u,u) \geq \|u\|^2_{\mathrm{H}^1(\Omega_L)}.$$

From the complex version of the Lax-Milgram theorem, we deduce that $B^{\text{out}} : \mathrm{H}^1_0(\Omega_L;\Gamma) \to \mathrm{H}^1_0(\Omega_L;\Gamma)$ is an isomorphism.

Now set $K = A^{\text{out}}(k) - B^{\text{out}}$. We have

$$(Ku, v)_{\mathrm{H}^1(\Omega_L)} = -(1+k^2)\int_{\Omega_L} u\overline{v}\,dxdy, \qquad \forall u, v \in \mathrm{H}^1_0(\Omega_L;\Gamma). \tag{I.39}$$

Since Ω_L is bounded, the embedding of $\mathrm{H}^1_0(\Omega_L;\Gamma)$ in $\mathrm{L}^2(\Omega_L)$ is compact and we can show that $K : \mathrm{H}^1_0(\Omega_L;\Gamma) \to \mathrm{H}^1_0(\Omega_L;\Gamma)$ is compact by working as for the operator K appearing in the proof of Theorem I.1. $\square$

This shows that $A^{\text{out}}(k)$ satisfies the Fredholm alternative. Either $A^{\text{out}}(k)$ is injective and in this case it is an isomorphism of $\mathrm{H}^1_0(\Omega_L;\Gamma)$. Or $A^{\text{out}}(k)$ has a kernel of finite dimension $\mathrm{span}(u_1, \dots, u_P)$ and in that case the equation

$$A^{\text{out}}(k)u = F \quad \text{ in } \mathrm{H}^1_0(\Omega_L;\Gamma)$$

has a solution (defined up to $\mathrm{span}(u_1, \dots, u_P)$) if and only if F satisfies the compatibility conditions

$$(F, u_p)_{\mathrm{H}^1(\Omega_L)} = 0, \qquad p = 1, \dots, P. \tag{I.40}$$

For a given geometry, one can show that the set of $k \in (\pi; +\infty) \setminus \{\mathbb{N}\pi\}$ such that $A^{\text{out}}(k)$ is not injective is discrete and accumulates only at $+\infty$.

If u solves (I.33), then by defining $\hat{u}$ such that

$$\hat{u}(x,y) = \left| \begin{array}{ll} u(x,y) & \text{in } \Omega_L \\ \displaystyle\sum_{n=1}^{+\infty} (u, \varphi_n)_{\mathrm{L}^2(\Sigma_{\pm L})}\, e^{\pm i\beta_n(x\mp L)}\, \varphi_n(y) & \text{for } \pm x > L, \end{array} \right. \tag{I.41}$$

we obtain a solution of (I.3). In general, this $\hat{u}$ does not belong to $\mathrm{H}^1_0(\Omega)$ because it involves propagating modes.

In the (rare) cases where $A^{\text{out}}(k)$ is not injective, we show now that the element u of its kernel do not decompose on the propagating modes. As a consequence, the corresponding extensions $\hat{u}$ are exponentially decaying as $x \to \pm\infty$ and so are localized in a neighborhood of the perturbation in the geometry. For this reason one usually call them trapped modes.

Proposition I.12. *Pick $k \in (\pi; +\infty) \setminus \{\mathbb{N}\pi\}$ and consider some u in $\ker A^{\text{out}}(k)$. Then its corresponding extension $\hat{u}$ defined via (I.41) decays as $O(e^{-\sqrt{(N+1)^2\pi^2 - k^2}|x|})$ as $x \to \pm\infty$ and so belongs to $\mathrm{H}^1_0(\Omega)$.*

Proof. Introduce again $N \in \mathbb{N}^*$ such that $k \in (N\pi; (N+1)\pi)$. If u is an element of $\ker A^{\text{out}}(k)$, we have $a^{\text{out}}(u,u) = 0$ and so

$$0 = \Im m\, a^{\text{out}}(u,u) = -\sum_{n=1}^{N} \beta_n \left(|(u, \varphi_n)_{\mathrm{L}^2(\Sigma_{+L})}|^2 + |(u, \varphi_n)_{\mathrm{L}^2(\Sigma_{-L})}|^2\right).$$

Since the $\beta_1, \dots, \beta_N$ are all positive, this proves that the corresponding $\hat{u}$ in (I.41) decomposes only on the evanescent modes. $\square$

3.5 Limiting absorption principle

In this paragraph, we prove that the dissipative solution converges to the solution of Problem (I.33) without dissipation as η tends to zero.

Theorem I.13. *Fix $k \in (\pi; +\infty) \setminus \{\mathbb{N}\pi\}$ and assume that the operator $A^{\text{out}}(k)$ defined in (I.37) is injective. Then there is η_0 such that we have*

$$\|u_\eta - u\|_{\mathrm{H}^1(\Omega_L)} \leq C\eta\, \|f\|_{\mathrm{L}^2(\Omega_L)}, \qquad \forall \eta \in (0; \eta_0],$$

with a constant $C > 0$ which is independent of η. Here u, u_η are the functions solving respectively (I.33), (I.32).

Proof. Let F be the element of $\mathrm{H}^1_0(\Omega_L; \Gamma)$ such that

$$(F, v)_{\mathrm{H}^1(\Omega_L)} = \ell(v), \qquad \forall v \in \mathrm{H}^1_0(\Omega_L; \Gamma).$$

Denote also by $A^\eta : \mathrm{H}^1_0(\Omega_L; \Gamma) \to \mathrm{H}^1_0(\Omega_L; \Gamma)$ the operator such that

$$(A^\eta w, v)_{\mathrm{H}^1(\Omega_L)} = a_\eta(w, v), \qquad \forall w, v \in \mathrm{H}^1_0(\Omega_L; \Gamma).$$

We have, for all $\eta > 0$,

$$A^\eta u_\eta = F = A^{\text{out}} u.$$

This gives $A^{\text{out}}(u - u_\eta) = (A^\eta - A^{\text{out}})u_\eta$. Since A^{out} is assumed to be injective, Theorem I.11 together with the Fredholm alternative guarantee that A^{out} is an isomorphism of $\mathrm{H}^1_0(\Omega_L; \Gamma)$. Thus we can write

$$u - u_\eta = (A^{\text{out}})^{-1}(A^\eta - A^{\text{out}})u_\eta. \tag{I.42}$$

Now from the definition of A^{out}, A^η, one establishes, for η small,

$$\|A^\eta - A^{\text{out}}\| \leq C\eta, \tag{I.43}$$

where $C > 0$ is a constant which may change from one line to another below, but remains independent of η. Gathering (I.42) and (I.43), we find

$$\|u - u_\eta\|_{\mathrm{H}^1(\Omega_L)} \leq C\eta\, \|u_\eta\|_{\mathrm{H}^1(\Omega_L)}. \tag{I.44}$$

By using the inequality $\|u_\eta\|_{\mathrm{H}^1(\Omega_L)} \leq \|u - u_\eta\|_{\mathrm{H}^1(\Omega_L)} + \|u\|_{\mathrm{H}^1(\Omega_L)}$ in (I.44), we obtain, for η small enough,

$$\|u - u_\eta\|_{\mathrm{H}^1(\Omega_L)} \leq C\eta\, \|u\|_{\mathrm{H}^1(\Omega_L)} \leq C\eta\, \|(A^{\text{out}})^{-1} f\|_{\mathrm{H}^1(\Omega_L)} \leq C\eta\, \|f\|_{\mathrm{L}^2(\Omega_L)}.$$

□

3.6 Scattering problem

In (I.33), we considered a problem with a source term f. In the following, we will be mostly interested in scattering problems for incident waves. To keep things as simple as possible, we assume all through this paragraph that $k \in (\pi; 2\pi)$ so that only the modes $w_1^\pm$ can propagate. To make short, we denote them by $w_\pm$ so that

$$w_\pm(x, y) = e^{\pm i\beta_1 x}\varphi_1(y) = e^{\pm i\sqrt{k^2 - \pi^2}x}\varphi_1(y). \tag{I.45}$$

The scattering of the rightgoing wave w_+ coming from the left branch of the waveguide leads us to consider the problem

$$\left|\begin{array}{rcll} \multicolumn{4}{l}{\text{Find } u_+ \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega) \text{ such that } u_+ - w_+ \text{ is outgoing and}} \\ \Delta u_+ + k^2 u_+ &=& 0 & \text{in } \Omega \\ u_+ &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{I.46}$$

Here $\mathrm{H}^1_{0,\mathrm{loc}}(\Omega)$ denotes the set of measurable functions v such that ζv belongs to $\mathrm{H}^1_0(\Omega)$ for all $\zeta \in \mathscr{C}^\infty_0(\mathbb{R}^2)$. Moreover, the sentence $u_+ - w_+$ is outgoing means that we impose the decomposition

$$u_+ - w_+ = \begin{cases} \displaystyle\sum_{n=1}^{+\infty} \alpha_n^+ e^{i\beta_n x}\,\varphi_n(y) & \text{for } x > L \\ \displaystyle\sum_{n=1}^{+\infty} \alpha_n^- e^{-i\beta_n x}\,\varphi_n(y) & \text{for } x < -L, \end{cases}$$

for some $\alpha_n^\pm \in \mathbb{C}$. In this context, u_+ is usually called the total field associated with the incident field w_+ while the quantity $u_+^s := u_+ - w_+$ is the scattered field.

Proposition I.14. *For all $k \in (\pi; 2\pi)$, (I.46) admits a solution. It is unique if trapped modes do not exist.*

Proof. Introduce some cut-off functions $\zeta \in \mathscr{C}^\infty(\mathbb{R})$ such that

$$\zeta(x) = \left|\begin{array}{ll} 1 & \text{for } x \le -L \\ 0 & \text{for } x \ge -d, \end{array}\right.$$

If u_+ satisfies (I.46), then $u_+^s = u_+ - \zeta w_+$ solves Problem (I.33) with

$$f = -\Delta(\zeta w_+) - k^2(\zeta w_+) \in \mathrm{L}^2(\Omega_L). \tag{I.47}$$

Conversely, if u_+^s satisfies (I.33) with the above f, then $u_+ := u_+^s + \zeta w_+$ is a solution of (I.46). Therefore it is sufficient to focus our attention on the study of (I.33) with f defined in (I.47).

If trapped modes do not exist, Theorem I.11 together with the Fredholm alternative ensure that (I.33) admits a unique solution.
Now if trapped modes span$(u_1, \dots, u_P)$, $P \ge 1$, exist, let us prove that for the particular f considered in (I.47) related to the incident mode, Problem (I.33) still have a solution. To proceed, we have to show that f satisfies the compatibility conditions appearing in (I.40). For $p = 1, \dots, P$, integrating twice by parts in Ω_L and using that $\Delta u_p + k^2 u_p = 0$ in Ω, we obtain

$$(f, u_p)_{\mathrm{L}^2(\Omega_L)} = -\int_{\Omega_L} \Big(\Delta(\zeta w_+) + k^2(\zeta w_+)\Big)\, u_p\, dxdy = -\int_{\Sigma_{-L}} \frac{\partial w_+}{\partial \nu} u_p - w_+ \frac{\partial u_p}{\partial \nu}\, dy.$$

Above we used the fact that $\zeta = 1$ on Σ_{-L}. Now we observe that w_+ is propagating while Proposition I.12 ensures that u_p decomposes only on the evanescent modes. From the orthogonality of the family $(\varphi_n)_{n\in\mathbb{N}^*}$ in $\mathrm{L}^2(I)$, we conclude that $(f, u_p)_{\mathrm{L}^2(\Omega_L)} = 0$. $\square$

Set $R_+ := \alpha_n^-$, $T_+ := 1 + \alpha_n^+$ so that for u_+ we have the representation

$$u_+ = \left|\begin{array}{rl} w_+ + R_+ w_- + \tilde{u}_+ & \text{for } x < -L \\ T_+ w_+ + \tilde{u}_+ & \text{for } x > L, \end{array}\right. \tag{I.48}$$

with $\tilde{u}_+ \in \mathrm{H}^1_0(\Omega)$. The quantities R_+, T_+ are usually called the reflection and transmission coefficients. Let us emphasize that they are uniquely defined, even if trapped modes exist a certain k. Indeed since trapped modes decay as $O(e^{-\sqrt{4\pi^2-k^2}|x|})$ as $x \to \pm\infty$ according to Proposition I.12, the scattering coefficients R_+, T_+ are insensitive to their existence.

In the same way, one shows that Problem (I.3) admits a solution $u_- \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega)$ with the decomposition

$$u_- = \left|\begin{array}{rl} T_- w_- + \tilde{u}_- & \text{for } x < -L \\ w_- + R_- w_+ + \tilde{u}_- & \text{for } x > L, \end{array}\right. \tag{I.49}$$

where R_-, $T_- \in \mathbb{C}$, $\tilde{u}_- \in \mathrm{H}^1_0(\Omega)$. It corresponds to the scattering of the leftgoing wave w_- coming from the right branch of the waveguide.

With the coefficients $R_\pm$, $T_\pm$ appearing in the decompositions (I.48), (I.49), we form the scattering matrix

$$\mathbb{S} := \left(\begin{array}{cc} R_+ & T_+ \\ T_- & R_- \end{array}\right) \in \mathbb{C}^{2\times 2}.$$

Due to physics, the matrix $\mathbb{S}$ has a very rigid structure. More precisely, we have the following statement:

Proposition I.15. *The scattering matrix $\mathbb{S}$ is symmetric ($T_+ = T_-$) and unitary ($\mathbb{S}\overline{\mathbb{S}}^\top = \mathrm{Id}_{2\times 2}$).*

Proof. For $l > 0$, set $\Omega_l := \{(x,y) \in \Omega \,|\, |x| < l\}$, $\Sigma_{\pm l} := \{\pm l\} \times I$. We have

$$0 = \int_{\Omega_l} (\Delta u_+ + k^2 u_+)u_- - u_+(\Delta u_- + k^2 u_-)\,dxdy = \int_{\Sigma_l \cup \Sigma_{-l}} \frac{\partial u_+}{\partial \nu} u_- - u_+ \frac{\partial u_-}{\partial \nu}\,dy.$$

Using decompositions (I.48), (I.49) in the above identity and taking the limit $l \to +\infty$, we obtain $0 = 2i\beta_1(T_+ - T_-)$, which gives $T_+ = T_-$. Then working similarly from the identities

$$0 = \int_{\Omega_l} (\Delta u_\pm + k^2 u_\pm)\overline{u_\pm} - u_\pm(\Delta \overline{u_\pm} + k^2 \overline{u_\pm})\,dxdy = \int_{\Sigma_l \cup \Sigma_{-l}} \frac{\partial u_\pm}{\partial \nu} \overline{u_\pm} - u_\pm \frac{\partial \overline{u_\pm}}{\partial \nu}\,dy,$$

one establishes the relations of conservation of energy

$$|R_\pm|^2 + |T_\pm|^2 = 1. \qquad \text{(I.50)}$$

Finally, by using that

$$0 = \int_{\Omega_l} (\Delta u_\pm + k^2 u_\pm)\overline{u_\mp} - u_\pm(\Delta \overline{u_\mp} + k^2 \overline{u_\mp})\,dxdy = \int_{\Sigma_l \cup \Sigma_{-l}} \frac{\partial u_\pm}{\partial \nu} \overline{u_\mp} - u_\pm \frac{\partial \overline{u_\mp}}{\partial \nu}\,dy,$$

one gets $R_\pm \overline{T_\mp} + T_\pm \overline{R_\mp} = 0$. $\square$

In the following, we simply set $T := T_+ = T_-$ so that we have

$$\mathbb{S} = \left(\begin{array}{cc} R_+ & T \\ T & R_- \end{array}\right).$$

In our study, we will need some formulas expressing the values of the scattering coefficients $R_\pm$, T with respect to $u_\pm$.

Proposition I.16. *For $k \in (\pi; 2\pi)$, the coefficients $R_\pm$, T appearing in (I.48), (I.49) in the decompositions of u_+, u_- satisfy*

$$R_\pm = \frac{1}{2i\beta_1} \int_{\Sigma_L \cup \Sigma_{-L}} \frac{\partial u_\pm}{\partial \nu} \overline{w_\mp} - u_\pm \frac{\partial \overline{w_\mp}}{\partial \nu}\,dy \quad \text{and} \quad T - 1 = \frac{1}{2i\beta_1} \int_{\Sigma_L \cup \Sigma_{-L}} \frac{\partial u_\pm}{\partial \nu} \overline{w_\pm} - u_\pm \frac{\partial \overline{w_\pm}}{\partial \nu}\,dy.$$

Proof. This is a consequence of the decompositions (I.48), (I.49) and of the fact that the family $(\varphi_n)_{n\in\mathbb{N}^*}$ is orthonormal in $\mathrm{L}^2(I)$. $\square$

3.7 Numerical approximation

Below we will have to compute numerical approximations of the quantities $u_{\pm}$, in particular to obtain $\mathbb{S}$. There are "three infinities" which are unpleasant to solve (I.46) with a computer. First, the domain Ω is unbounded. To face this difficulty, we will exploit the above analysis and consider a formulation set in Ω_L involving the Dirichlet-to-Neumann operators similar to (I.35). Second, $\mathrm{H}^1_0(\Omega_L;\Gamma)$ is a space of infinite dimension. We will work with a finite element method and solve a variational formulation in a space of finite dimension. Third, the radiation condition involves an infinite number of terms. Quite naturally, we will truncate the series appearing in the Dirichlet-to-Neumann operators at rank $M > N$ where N is the number of propagating modes.

To set ideas, assume that $k \in (\pi; 2\pi)$ and consider the approximation of u_+. Since $u_+ - w_+$ is outgoing, we have the conditions

$$\frac{\partial(u_+ - w_+)}{\partial\nu} = \Lambda_{\pm}(u_+ - w_+) \quad \text{ on } \Sigma_{\pm L}.$$

This gives

$$\frac{\partial u_+}{\partial\nu} = \Lambda_{\pm}(u_+) + \frac{\partial w_+}{\partial\nu} - \Lambda_{\pm}(w_+) \quad \text{ on } \Sigma_{\pm L}.$$

Since w_+ is rightgoing, using the definitions (I.34) of $\Lambda_{\pm}$, we obtain

$$\partial_\nu w_+ - \Lambda_+(w_+) = 0 \quad \text{ on } \Sigma_L, \qquad\qquad \partial_\nu w_+ - \Lambda_-(w_+) = -2i\beta_1 w_+ \quad \text{ on } \Sigma_{-L}.$$

Thus u_+ solves the problem

$$\left|\begin{array}{l} \text{Find } u_+ \in \mathrm{H}^1_0(\Omega_L;\Gamma) \text{ such that for all } v \in \mathrm{H}^1_0(\Omega_L;\Gamma),\\ \displaystyle\int_{\Omega_L} \nabla u_+ \cdot \nabla \overline{v} - k^2 u\overline{v}\,dxdy - \langle \Lambda_+(u_+), v\rangle_{\Sigma_L} - \langle \Lambda_-(u_+), v\rangle_{\Sigma_{-L}} = -2i\beta_1 \int_{\Sigma_{-L}} w_+\overline{v}\,dy. \end{array}\right.$$

Note that (I.34) yields

$$\langle \Lambda_{\pm}(u), v\rangle_{\Sigma_{\pm L}} = \sum_{n=1}^{+\infty} i\beta_n\,(u, \varphi_n)_{\mathrm{L}^2(\Sigma_{\pm L})}\,\overline{(v, \varphi_n)_{\mathrm{L}^2(\Sigma_{\pm L})}}.$$

Now introduce $(\mathcal{T}_h)_h$ a shape regular family of triangulations of $\overline{\Omega_L}$ (in other words, we mesh the domain $\overline{\Omega_L}$ with triangles). Define the family of Lagrange finite element spaces

$$\mathrm{V}_h := \left\{ v \in \mathrm{H}^1_0(\Omega_L;\Gamma) \text{ such that } v|_\tau \in \mathbb{P}_q(\tau) \text{ for all } \tau \in \mathcal{T}_h \right\},$$

where $\mathbb{P}_q(\tau)$ is the space of polynomials of degree at most q on the triangle τ. Finally, the problem we solve writes

$$\left|\begin{array}{l} \text{Find } u^h_+ \in \mathrm{V}_h \text{ such that for all } v^h \in \mathrm{V}_h,\\ \displaystyle\int_{\Omega_L} \nabla u^h_+ \cdot \nabla\overline{v^h} - k^2 u^h_+\overline{v^h}\,dxdy - \sum_{n=1}^{M} i\beta_n\,(u^h_+, \varphi_n)_{\mathrm{L}^2(\Sigma_L)}\,\overline{(v^h, \varphi_n)_{\mathrm{L}^2(\Sigma_L)}} \\ \displaystyle\qquad\qquad - \sum_{n=1}^{M} i\beta_n\,(u^h_+, \varphi_n)_{\mathrm{L}^2(\Sigma_{-L})}\,\overline{(v^h, \varphi_n)_{\mathrm{L}^2(\Sigma_{-L})}} = -2i\beta_1 \int_{\Sigma_{-L}} w_+\overline{v^h}\,dy. \end{array}\right. \tag{I.51}$$

One can show that for h small enough, L large enough (actually one has exponential convergence with respect to L), u^h_+ yields a good approximation of u_+. Then replacing u_+ by u^h_+ in the exact formulas

$$R_+ = \int_{\Sigma_{-L}} (u_+ - w_+)\,w_+\,dy, \qquad T = \int_{\Sigma_L} u_+ w_-\,dy,$$

we obtain good approximations of the scattering coefficients.

4 Neumann problem

In acoustics, we are led to study the same Helmholtz equation as in (I.3) but with homogeneous Neumann BCs which model sound hard walls. This yields the problem

$$\left|\begin{array}{rcll} \Delta u + k^2 u & = & 0 & \text{in } \Omega \\ \partial_\nu u & = & 0 & \text{on } \partial\Omega, \end{array}\right. \tag{I.52}$$

where u stands for the pressure of the fluid in Ω and ∂_ν refers to the outward normal derivative on $\partial\Omega$. As seen in the study of the Dirichlet case, the modes play a key role in the analysis. This time, we need to compute the solutions of (I.52) with separate variables in the reference strip $\mathcal{S}$. Reproducing what has been done in §3.1, we find that they coincide with the family $\{w_n^{\pm}\}_{n\in\mathbb{N}}$ where

$$w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\varphi_n(y), \qquad \beta_n := \sqrt{k^2 - n^2\pi^2}, \quad \varphi_n(y) = \left|\begin{array}{ll} 1 & \text{if } n = 0 \\ \sqrt{2}\cos(n\pi y) & \text{for } n > 0. \end{array}\right. \tag{I.53}$$

In particular for all $k > 0$, we have $w_0^{\pm}(x,y) = e^{\pm ikx}$, which shows that unlike the Dirichlet case, propagating modes always exist (this is also related to the fact that we have no Poincaré inequality as (I.14) in $\mathrm{H}^1(I)$). As a consequence, for all $k > 0$, $\mathrm{H}^1(\Omega)$ is not an adapted functional framework to study (I.53). The solution must decompose on the propagating modes and radiation conditions must be imposed to select the outgoing behaviour. The method, based in particular on the limiting absorption principle, is completely similar to what has been done in §3.3, 3.4, 3.5. We do not detail it and instead simply present the main results concerning the corresponding scattering problem.

To stick to the simplest setting, we assume that k belongs to $(0;\pi)$ so that only the modes $w_0^{\pm}$ can propagate. We denote them by $w_{\pm}$, so that

$$w_{\pm}(x,y) = e^{\pm ikx}. \tag{I.54}$$

Note that $w_{\pm}$ are plane waves, they do not depend on the variable y (which was not the case for the Dirichlet problem). The scattering of the rightgoing wave w_+ in Ω leads to consider the solution $u_+ \in \mathrm{H}^1_{\text{loc}}(\Omega)$ of (I.52) admitting the expansion

$$u_+ = \left|\begin{array}{rl} w_+ + R_+ w_- + \tilde{u}_+ & \text{for } x < -L \\ T_+ w_+ + \tilde{u}_+ & \text{for } x > L, \end{array}\right. \tag{I.55}$$

with $R_+, T_+ \in \mathbb{C}$ and $\tilde{u}_+ \in \mathrm{H}^1(\Omega)$. More precisely, by adapting what has been done in (3.6), one can show that (I.52) always admits a solution with the expansion (I.55). This solution is uniquely defined if trapped modes (solutions of (I.52) in $\mathrm{H}^1(\Omega)$) do not exist. Besides, by working as in Proposition I.12, one shows that trapped modes, if they exist, decay as $O(e^{-\sqrt{\pi^2-k^2}|x|})$ as $x \to \pm\infty$ so that the scattering coefficients R_+, T_+ in (I.55) are always uniquely defined.

Similarly, the scattering of the leftgoing plane wave w_- in Ω leads to consider the solution $u_- \in \mathrm{H}^1_{\text{loc}}(\Omega)$ of (I.52) admitting the expansion

$$u_- = \left|\begin{array}{rl} T_- w_- + \tilde{u}_- & \text{for } x < -L \\ w_- + R_- w_+ + \tilde{u}_- & \text{for } x > L, \end{array}\right. \tag{I.56}$$

with $R_-, T_- \in \mathbb{C}$ and $\tilde{u}_- \in \mathrm{H}^1(\Omega)$.

As in Proposition I.15, one proves that $T_+ = T_-$ and we set $T := T_+ = T_-$. The scattering matrix

$$\mathbb{S} = \begin{pmatrix} R_+ & T \\ T & R_- \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

is symmetric and unitary ($\mathbb{S}\overline{\mathbb{S}}^\top = \mathrm{Id}_{2\times 2}$). In particular, we have the relations of conservation of energy

$$|R_\pm|^2 + |T|^2 = 1. \tag{I.57}$$

As in the proof of Proposition I.16, one establishes the following statement:

Proposition I.17. *For $k \in (0;\pi)$, the coefficients $R_\pm$, T appearing in (I.55), (I.56) in the decompositions of u_+, u_- satisfy*

$$R_\pm = \frac{1}{2ik}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u_\pm}{\partial\nu}\overline{w_\mp} - u_\pm\frac{\partial\overline{w_\mp}}{\partial\nu}\,dy \quad \textit{and} \quad T = \frac{1}{2ik}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u_\pm}{\partial\nu}\overline{w_\pm} - u_\pm\frac{\partial\overline{w_\pm}}{\partial\nu}\,dy.$$

For the numerics, we work with a formulation very similar to (I.51) where the modes are replaced by the ones obtained in (I.53).

5 Invisibility questions

In the following, our general goal will be to find situations, by playing with the geometry, the wavenumber k, ... where we have some sort of invisibility.

The weakest invisibility that one can look for is $R_\pm = 0$. In the sequel, we shall say that one has zero reflection or that the defect is non reflecting. In such a situation, an observer generating an incident plane wave, located a bit far from the defect in the geometry and measuring the resulting backscattering field will only measure the evanescent component. Due to noise, one will get something similar to the field in the reference strip and therefore will be unable to detect the presence of the obstacle. Note that due to conservation of energy ((I.50) or (I.57)), the fact that $R_\pm = 0$ implies $|T| = 1$ and so $T = e^{i\theta}$ for a certain $\theta \in \mathbb{R}$. As a consequence in general there is a phase shift in the transmitted field which can reveal the presence of the defect if one probes the field in that part of the guide.

A more demanding definition of invisibility is to have $T = 1$. In that situation, we shall say that the defect is perfectly invisible or that we have perfect transmission without phase shift.

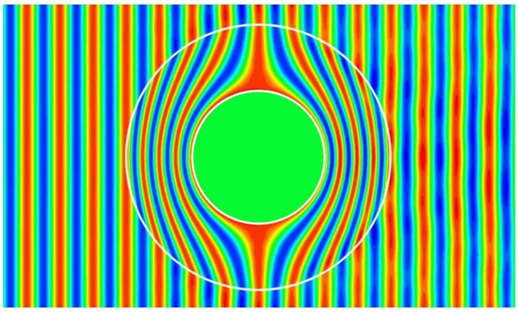


Figure I.5: Cloaking by transformation optics.

The problem of cloaking an object has a large number of applications and has been the subject of intense studies over the last decade in the theory of waves propagation. Let us mention that different notions of invisibility exist in literature. In particular, our approach is different from the cloaking via transformation optics pictured in Figure I.5 (see [34, 28]). Let us describe this latter device. Imagine that one wish to hide a given obstacle. One technique consists in surrounding it by a well chosen penetrable material, localized in the annulus around the green region on the picture, so that an incoming wave leaves the whole device as if there was nothing. Said differently, on the picture, an observer probing the field outside of the larger disk marked by the white thin circle obtains the same measurements as in free space and so cannot detect the presence of the obstacle. Mathematically, this idea is quite simple to implement, it boils down to a change of variable. However this change of variable is singular and for this reason the physical parameters of the *ad hoc* material in the cloaking device should take infinite values. For this reason, for the moment designing such structures, even working with so-called metamaterials, is still unreachable, in particular due to the presence of important losses.

What we propose is less ambitious because we only wish to control the scattering coefficients and not the inner field. In short, what we aim for is only cloaking at infinity. For this reason, it is more easily doable. Actually, we will not even need to play with penetrable materials. We will see that by working with a homogeneous material and acting only on the geometry is sufficient. On the other hand, though our setting is less ambitious, it is still relevant in numerous applications. Indeed, the evanescent part of the field that we neglect is exponentially decaying at infinity and therefore is really difficult to distinguish from noise a few wavelengths far from the obstacle. Finally, though we simply wish to control a finite number of complex coefficients, this problem is not trivial because the link between the variation of the parameters (geometry, k, ...) and the variation of the scattering coefficients is non linear and not explicit. Additionally, let us emphasize that due to the fact that there is no coercivity in the problem, optimization methods fail due to the presence of local minima.

In the next three chapters, we present several ideas to reach invisibility.

Chapter II

Invisible perturbations of the reference geometry

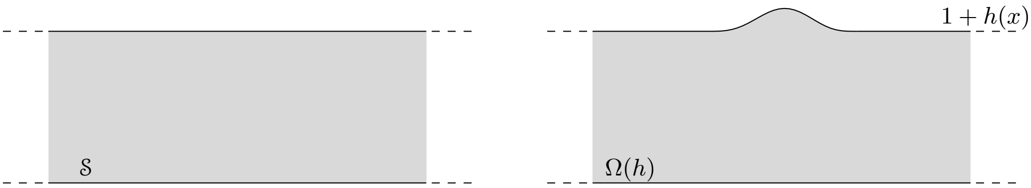


Figure II.1: Reference strip $\mathcal{S}$ (left) and perturbed waveguide $\Omega(h)$ (right).

In this chapter, we work with techniques of perturbations to construct invisible obstacles. The idea, proposed in the article [13], is as follows: in the reference strip $\mathcal{S} = \mathbb{R} \times I$, we have no reflection and perfect transmission, the total field is equal to the incident field, and so $R_\pm = 0$, $T = 1$. How to slightly modify the geometry while keeping these values for the scattering coefficients? To proceed, we will adapt the proof of the implicit function theorem.

1 General scheme

Let us describe the method for the simplest problem, namely obtaining $R_\pm = 0$. At this stage, the approach is the same whether one considers Problem (I.46) with Dirichlet BCs or Problem (I.52) with Neumann BCs (pick $k \in (\pi; 2\pi)$ in the first case, $k \in (0; \pi)$ in the second situation). Let us focus our attention on R_+ and simply write R instead of R_+. Note that from conservation of energy, $R+ = 0$ implies $R_- = 0$. Consider some real valued profile function $h \in \mathscr{C}_0^\infty(\mathbb{R})$ and let $\Omega(h)$ be the waveguide whose upper boundary coincides with the graph of the function $1 + h$ (see Figure II.1 right). Note that we make some assumption of smoothness here for simplicity but $\mathscr{C}_0^2(\mathbb{R})$ would be enough. Let $R(h)$ be the reflection coefficient of the scattering solution u_+ in the geometry $\Omega(h)$. Importantly, we have $R(0) = 0$ because when $h \equiv 0$, $\Omega(h)$ is simply the reference strip $\mathcal{S}$. With this notation, the problem we consider writes

$$\left|\begin{array}{l}\text{Find } h \not\equiv 0 \text{ such that} \\ R(h) = 0.\end{array}\right.$$

Let us look for non reflecting geometries which are small perturbations of $\mathcal{S}$. To proceed, let us look for $h = \varepsilon\mu$ with $\varepsilon > 0$ small and $\mu \in \mathscr{C}_0^\infty(\mathbb{R})$ to be determined. Since ε is small, we can write an asymptotic expansion of R with respect to ε. We obtain

$$\begin{array}{rcl} R(\varepsilon\mu) & = & R(0) + \varepsilon dR(0)(\mu) + \varepsilon^2 \tilde{R}(\varepsilon\mu) \\ & = & \varepsilon dR(0)(\mu) + \varepsilon^2 \tilde{R}(\varepsilon\mu) \end{array} \tag{II.1}$$

where $dR(0)(\mu)$ stands for the differential of R at zero in the direction μ and $\tilde{R}(\varepsilon\mu)$ is an abstract remainder. Since $dR(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ is a linear map from a space of infinite dimension to a space of dimension two (remember that we work with real valued functions h), we have $\dim\ker dR(0) = +\infty$. Therefore, we can pick $\mu_0 \not\equiv 0$ such that

$$dR(0)(\mu_0) = 0. \tag{II.2}$$

By choosing $h = \varepsilon\mu = \varepsilon\mu_0$, we obtain a perturbation of order ε which produces a reflection in $O(\varepsilon^2)$. This is interesting because this is almost zero reflection but not completely satisfactory yet. To compensate for the remainder, we need to work a bit more. Below we will explain how to compute $dR(0)$ and prove that $dR(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ is onto. This allows us to introduce $\mu_1 \in \mathscr{C}_0^\infty(\mathbb{R})$ and $\mu_2 \in \mathscr{C}_0^\infty(\mathbb{R})$ such that

$$dR(0)(\mu_1) = 1 \qquad \text{and} \qquad dR(0)(\mu_2) = i. \tag{II.3}$$

Finally, we look for μ of the form

$$\mu = \mu_0 + \tau_1\mu_1 + \tau_2\mu_2$$

where $\tau_1, \tau_2 \in \mathbb{R}$ are parameters to tune. Inserting this μ in the expansion (II.1), to get $R(\varepsilon\mu) = 0$, we see that we must have

$$0 = \varepsilon dR(0)(\mu_0 + \tau_1\mu_1 + \tau_2\mu_2) + \varepsilon^2\tilde{R}(\varepsilon(\mu_0 + \tau_1\mu_1 + \tau_2\mu_2)).$$

By exploiting (II.3), this yields

$$0 = \tau_1 + i\tau + \varepsilon\tilde{R}(\varepsilon(\mu_0 + \tau_1\mu_1 + \tau_2\mu_2)).$$

In other words, we find that the vector $\vec{\tau} := (\tau_1, \tau_2)^\top \in \mathbb{R}^2$ must satisfy the fixed point equation

$$\vec{\tau} = G^\varepsilon(\vec{\tau}) \quad \text{with} \quad G^\varepsilon(\vec{\tau}) = -\varepsilon(\Re e\,\tilde{R}(\varepsilon(\mu_0 + \tau_1\mu_1 + \tau_2\mu_2)), \Im m\,\tilde{R}(\varepsilon(\mu_0 + \tau_1\mu_1 + \tau_2\mu_2)))^\top. \tag{II.4}$$

Now by proving uniform (with respect to $\vec{\tau}$) error estimates in the asymptotic expansions as ε tends to zero, one can show that for any $r > 0$, there is $\varepsilon_0 > 0$ such that the map $\vec{\tau} \mapsto G^\varepsilon(\vec{\tau})$ is a contraction from $\overline{B(O,r)}$ to $\overline{B(O,r)}$ for all $\varepsilon \in (0;\varepsilon_0]$. Here $B(O,r)$ denotes the open ball of $\mathbb{R}^2$ centered at O of radius r. Therefore the Banach fixed point theorem guarantees that (II.4) admits a unique solution $\vec{\tau}^{\,\text{sol}} \in \overline{B(O,r)}$. Then for $h^{\text{sol}} = \varepsilon(\mu_0 + \tau_1^{\text{sol}}\mu_1 + \tau_2^{\text{sol}}\mu_2)$, we have $R(h^{\text{sol}}) = 0$. This proves the existence of non reflecting geometries.

Let us comment a bit this method.

1) First, it is important to prove that the constructed h^{sol} is not trivial. To proceed, let us work by contradiction and assume that $h^{\text{sol}} \equiv 0$. Then we get

$$0 = dR(0)(h^{\text{sol}}) = dR(0)(\varepsilon(\mu_0 + \tau_1^{\text{sol}}\mu_1 + \tau_2^{\text{sol}}\mu_2)) = \varepsilon(\tau_1^{\text{sol}} + i\tau_2^{\text{sol}}),$$

and so $\tau_1^{\text{sol}} = \tau_2^{\text{sol}} = 0$. This implies $h^{\text{sol}} = \varepsilon\mu_0 \equiv 0$ which is not true due to our choice for μ_0.

2) Observe that this technique guarantees the existence of an infinite number of non reflecting perturbations. Indeed, first, h^{sol} depends on $\varepsilon \in (0;\varepsilon_0]$ (remark that it is not only a scaling because there are non linear terms involved). Additionally, for μ_0 we have some freedom because $\dim\ker dR(0) = +\infty$.

3) We can establish that there is a constant $C > 0$ independent of ε such that we have $|G^\varepsilon(\vec{\tau})| \le C\varepsilon$ in $\overline{B(O,r)}$. As a consequence, as ε tends to zero, we have $|\vec{\tau}^{\,\text{sol}}| = O(\varepsilon)$ and the shape of the perturbation $\varepsilon\mu = \varepsilon(\mu_0 + \tau_1\mu_1 + \tau_2\mu_2)$ is mainly characterized by μ_0.

4) Let us clarify the connection with the implicit function theorem. Introduce the functional

$$\begin{array}{rcl} T : \mathbb{R}\times\mathbb{R}^2 & \to & \mathbb{R}^2 \\ (\tau_0, \vec{\tau}) & \mapsto & (\Re e\,R(\tau_0\mu_0 + \tau_1\mu_1 + \tau_2\mu_2), \Im m\,R(\tau_0\mu_0 + \tau_1\mu_1 + \tau_2\mu_2)) \end{array}$$

which is of class $\mathscr{C}^1$ in a neighbourhood of $0_{\mathbb{R}^3}$. From what will be shown below, we will be able to deduce that $\partial_{\vec{\tau}}T(0,0) : \mathbb{R}^2 \to \mathbb{R}^2$ is well-defined and bijective. On the other hand, we remark that $T(0_{\mathbb{R}^3}) = 0_{\mathbb{R}^2}$. The implicit function theorem applies: there are some neighbourhoods $\mathscr{U} \subset \mathbb{R}$, $\mathscr{V} \subset \mathbb{R}^2$ of 0, $0_{\mathbb{R}^2}$ and a unique function $\varphi : \mathscr{U} \to \mathscr{V}$ of class $\mathscr{C}^1$ such that

$$[\,(\tau_0, \vec{\tau}) \in \mathscr{U} \times \mathscr{V} \quad \text{and} \quad T(\tau_0, \vec{\tau}) = 0\,] \qquad \Leftrightarrow \qquad \vec{\tau} = \varphi(\tau_0).$$

2 Zero reflection for the Dirichlet problem

Let us apply the generic strategy described above to Problem (I.46) with Dirichlet BCs. Pick some $k \in (\pi; 2\pi)$. First, we need to compute $dR(0)$.

Proposition II.1. *For $\mu \in \mathscr{C}_0^\infty(\mathbb{R})$, we have*

$$dR(0)(\mu) \quad = \quad \frac{i\pi^2}{\beta_1} \int_{\mathbb{R}} \mu(x) e^{2i\beta_1 x}\, dx.$$

As a consequence, $dR(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ is onto.

Proof. The quantity $dR(0)$ corresponds to the derivative of R with respect to the geometry. To identify it, we have to understand how R is changed when the boundary of the waveguide is perturbed around the reference situation ($\Omega = \mathcal{S}$). Such results are met classically in shape optimization. To obtain $dR(0)$, we work with techniques of asymptotic analysis.

For $\varepsilon > 0$ small and a given $\mu \in \mathscr{C}_0^\infty(\mathbb{R})$ supported in $(-L; L)$ (change L if necessary), denote by Ω^ε the domain $\Omega(h)$ with $h = \varepsilon\mu$. Let u^ε stand for the solution u_+ introduced in (I.46) in the geometry Ω^ε. It satisfies

$$\left|\begin{array}{l} \text{Find } u^\varepsilon \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega) \text{ such that } u^\varepsilon - w_+ \text{ is outgoing and} \\ \begin{array}{rcll} \Delta u^\varepsilon + k^2 u^\varepsilon & = & 0 & \text{in } \Omega^\varepsilon \\ u^\varepsilon & = & 0 & \text{on } \partial\Omega^\varepsilon. \end{array} \end{array}\right. \tag{II.5}$$

For u^ε, as ε tends to zero, we consider the ansatz

$$u^\varepsilon = u_0 + \varepsilon u_1 + \dots \tag{II.6}$$

where u_0, u_1 are functions to be determined and the dots correspond to higher order terms. On the upper part of $\partial\Omega^\varepsilon$, we have, formally,

$$\begin{array}{rcl} 0 = u^\varepsilon(x, 1 + h(x)) & = & u^\varepsilon(x,1) + \varepsilon h(x) \partial_y u^\varepsilon(x,1) + \dots \\ & = & u_0(x,1) + \varepsilon\,(u_1(x,1) + h(x)\partial_y u_0(x,1)) + \dots. \end{array} \tag{II.7}$$

Now by inserting (II.6) in (II.5) and by exploiting (II.7), collecting the terms of orders ε^0, ε^1, we find that u_0, u_1 satisfy respectively the problems

$$\left|\begin{array}{l} \text{Find } u_0 \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega) \text{ such that } u_0 - w_+ \text{ is outgoing and} \\ \begin{array}{rcll} \Delta u_0 + k^2 u_0 & = & 0 & \text{in } \mathcal{S} \\ u_0 & = & 0 & \text{on } \partial\mathcal{S} \end{array} \end{array}\right. \tag{II.8}$$

$$\left|\begin{array}{l} \text{Find } u_1 \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega) \text{ such that } u_1 \text{ is outgoing and} \\ \begin{array}{rcll} \Delta u_1 + k^2 u_1 & = & 0 & \text{in } \mathcal{S} \\ u_1 & = & 0 & \text{on } \mathbb{R} \times \{0\} \\ u_1 & = & -\mu \partial_y u_0 & \text{on } \mathbb{R} \times \{1\}. \end{array} \end{array}\right. \tag{II.9}$$

Solving (II.8), we obtain

$$u_0(x,y) = w_+(x,y) = e^{i\beta_1 x}\varphi_1(y). \tag{II.10}$$

Denote by R^ε the reflection coefficient of u^ε. According to Proposition I.16, we have

$$R^\varepsilon = \frac{1}{2i\beta_1}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u^\varepsilon}{\partial\nu} w_+ - u^\varepsilon \frac{\partial w_+}{\partial\nu}\,dy.$$

Inserting the expansion (II.6) of u^ε in the above identity and using that $u_0 = w_+$, this gives

$$\begin{aligned} R^\varepsilon &= \frac{1}{2i\beta_1}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u_0}{\partial\nu} w_+ - u_0 \frac{\partial w_+}{\partial\nu}\,dy + \frac{\varepsilon}{2i\beta_1}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u_1}{\partial\nu} w_+ - u_1 \frac{\partial w_+}{\partial\nu}\,dy + \dots \\ &= 0 + \frac{\varepsilon}{2i\beta_1}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u_1}{\partial\nu} w_+ - u_1 \frac{\partial w_+}{\partial\nu}\,dy + \dots, \end{aligned}$$

where again the dots correspond to higher order terms. We deduce that

$$dR(0)(\mu) = \frac{1}{2i\beta_1}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u_1}{\partial\nu} w_+ - u_1 \frac{\partial w_+}{\partial\nu}\,dy.$$

Integrating by parts in $\mathcal{S}_L := (-L;L)\times I$, and using the third line of (II.9), this gives

$$dR(0)(\mu) \;=\; \frac{1}{2i\beta_1}\int_{\partial\mathcal{S}} u_1 \frac{\partial w_+}{\partial y}\,dx = \frac{i}{2\beta_1}\int_{\mathbb{R}} \mu(x)\Big(\frac{\partial w_+}{\partial y}\Big)^2 (x,1)\,dx.$$

Finally, from (II.10) (remember that $\varphi_1(y) = \sqrt{2}\sin(\pi y)$), we obtain the formula

$$dR(0)(\mu) \;=\; \frac{i\pi^2}{\beta_1}\int_{\mathbb{R}} \mu(x) e^{2i\beta_1 x}\,dx.$$

Since the real part of $x\mapsto e^{2i\beta_1 x}$ is even while its imaginary part is odd, we see that it is easy to find functions μ_1, μ_2 satisfying the relations (II.3), which ensures that $dR(0) : \mathscr{C}^\infty_0(\mathbb{R}) \to \mathbb{C}$ is onto.

The formal calculus above can be rigorously justified by proving error estimates. To proceed, one method consists in rectifying the boundary of Ω^ε using "almost identical" diffeomorphisms to transform the perturbed domain into the reference strip $\mathcal{S}$ (see e.g. [23, Chap. 7, §6.5]). Then one can prove the estimate, for ε small enough,

$$\|u^\varepsilon - (u_0 + \varepsilon u_1)\|_{\mathrm{H}^1(\mathcal{S}_L\setminus\overline{\omega})} \le C\varepsilon^2,$$

where C is a constant independent of ε and ω is a neighbourhood of $\mathrm{supp}(\mu)\times\{1\}$. □

3 Numerical implementation

The theoretical approach presented above leads very naturally to an algorithm to construct numerically non reflecting perturbations. Let us describe the strategy.

The main idea consists in solving the fixed point equation

$$\vec{\tau} = G^\varepsilon(\vec{\tau})$$

(see (II.4)) using an iterative procedure. First, we choose μ_0, μ_1, μ_2 once for all. Then we start with $\vec{\tau}^0 = (0,0)$ and for $p\in\mathbb{N}$, we set $\vec{\tau}^{p+1} = G^\varepsilon(\vec{\tau}^p)$. Denote $\mu^p := \mu_0 + \tau_1^p\mu_1 + \tau_2^p\mu_2$. From (II.4), we have

$$\begin{aligned} G^\varepsilon(\vec{\tau}^p) &= -\varepsilon(\Re e\,\tilde{R}(\varepsilon\mu^p), \Im m\,\tilde{R}(\varepsilon\mu^p))^\top \\ &= \vec{\tau}^p - \varepsilon^{-1}(\Re e\,R(\varepsilon\mu^p), \Im m\,R(\varepsilon\mu^p)). \end{aligned}$$

Therefore, for $\vec{\tau}^p$ we obtain the recursive equation

$$\vec{\tau}^{p+1} = \vec{\tau}^p - \varepsilon^{-1}(\Re e\, R(\varepsilon\mu^p), \Im m\, R(\varepsilon\mu^p)). \tag{II.11}$$

We stop the loop when we have $|R(\varepsilon\mu^p)| \leq \eta$ where $\eta > 0$ is a small given criterion. We then define $\vec{\tau}^{\text{sol}}$ as the last value of $\vec{\tau}^p$. If the iterative process does not converge, we try again with a smaller value of $\varepsilon > 0$. Note that at each step $p \geq 0$, we need to solve a scattering problem of the form

$$\left|\begin{array}{rl} \multicolumn{2}{l}{\text{Find } u \text{ such that } u - w_+ \text{ is outgoing and}} \\ \Delta u + k^2 u = 0 & \text{in } \Omega^p := \Omega(\varepsilon\mu^p) \\ u = 0 & \text{on } \partial\Omega^p. \end{array}\right. \tag{II.12}$$

To proceed, we approximate the solution of (II.12) by working with Formulation (I.51). We use a P2 finite element method in $\Omega_5^p := \{(x,y) \in \Omega^p \,|\, |x| < 5\}$. At $x = \pm 5$, a truncated Dirichlet-to-Neumann map with 10 terms serves as a transparent boundary condition. In other words, we take $q = 2$, $L = 5$, $M = 10$ in (I.51). Note that at each step, it is necessary to mesh a new domain. For the computations, we use the `FreeFem++`[1] software while we display the results with `Paraview`[2].

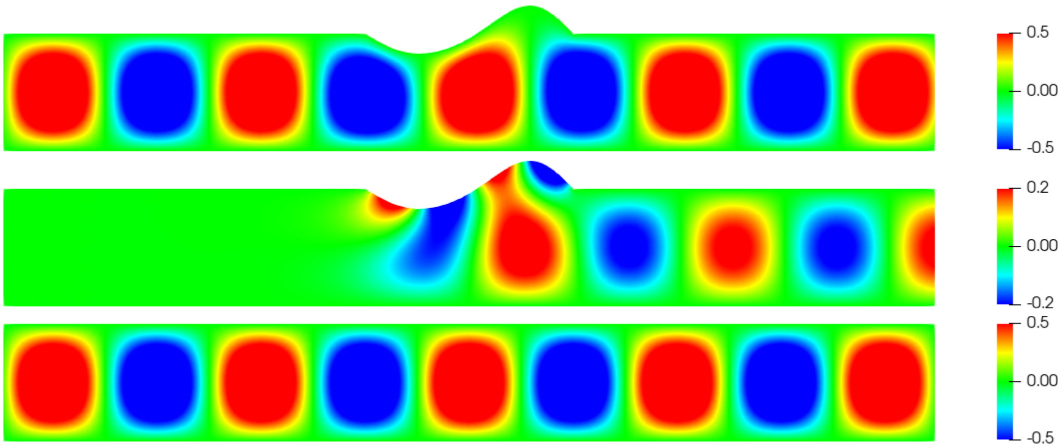


Figure II.2: Example of non reflecting obstacle for the problem (I.46) with Dirichlet BCs. Top: $\Re e\, u$. Middle: $\Re e\, u_s$. Bottom: $\Re e\, w_+$ in the reference strip. Here $k = 1.5\pi$.

Let us give a concrete application. Set $\delta := \pi/\beta_1$, $I_{\text{defect}} := (-\delta; \delta)$ and for μ_0, μ_1, μ_2, let us work with the functions such that

$$\mu_0(x) = \sin(\beta_1 x), \quad \mu_1(x) = -\frac{\beta_1^2}{\pi^3}\sin(2\beta_1 x), \quad \mu_2(x) = \frac{7\beta_1^2}{12\pi^2}\cos(3\beta_1 x/2), \qquad \text{for } x \in I_{\text{defect}},$$

$\mu_0 = \mu_1 = \mu_2 = 0$ in $\mathbb{R} \setminus \overline{I_{\text{defect}}}$. These functions are continuous and compactly supported but do not belong to $\mathscr{C}_0^\infty(\mathbb{R})$. However this is not actually needed for the above theory. On the other hand, one can check that they indeed satisfy relations (II.2), (II.3) (remark in particular that μ_0, μ_1 are odd while μ_2 is even). We set $\eta = 10^{-4}$, $k = 1.5\pi$. In Figure II.2, we display the geometry at the end of the iterative procedure for $\varepsilon = 0.2$. We have obtained $|R| \approx 8.10^{-5}$ in 24 iterations. As expected, we observe that the scattered field is exponentially decaying as $x \to -\infty$ (the incident wave comes from the left). We note that for the transmitted field, there is a small shift of phase. This is not surprising because $R = 0$ only implies $|T| = 1$ and not $T = 1$. Interestingly, since there is only one complex coefficient to cancel, the algorithm converges though ε is not that small. This allows us to get not so small non reflecting perturbations of the reference strip.

[1] `FreeFem++`, https://freefem.org/.

[2] `Paraview`, http://www.paraview.org/.

4 Perfect transmission for the Dirichlet problem

Can we hope for more and obtain $T = 1$ with the above approach? To proceed, a natural idea is to work with the quantity $T - 1$. More precisely, in the reference strip we have $T - 1 = 0$. Is it possible to perturb the geometry while keeping $T - 1 = 0$? Let us compute the differential of T with respect to the geometry.

Proposition II.2. *For $\mu \in \mathscr{C}_0^\infty(\mathbb{R})$, we have*

$$dT(0)(\mu) \quad = \quad \frac{i\pi^2}{\beta_1}\int_{\mathbb{R}} \mu(x)\,dx.$$

As a consequence, $dT(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ is not onto.

Proof. To show this result, one works as for R in Proposition II.1. Let us keep the same notation. According to Proposition I.16, we have

$$T^\varepsilon - 1 = \frac{1}{2i\beta_1}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u^\varepsilon}{\partial\nu}w_- - u^\varepsilon\frac{\partial w_-}{\partial\nu}\,dy.$$

Inserting the expansion (II.6) of u^ε in the above identity and using that $u_0 = w_+$, this gives

$$\begin{aligned} T^\varepsilon - 1 \quad &= \quad \frac{1}{2i\beta_1}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u_0}{\partial\nu}w_- - u_0\frac{\partial w_-}{\partial\nu}\,dy + \frac{\varepsilon}{2i\beta_1}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u_1}{\partial\nu}w_- - u_1\frac{\partial w_-}{\partial\nu}\,dy + \dots \\ &= \quad 0 + \frac{\varepsilon}{2i\beta_1}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u_1}{\partial\nu}w_- - u_1\frac{\partial w_-}{\partial\nu}\,dy + \dots, \end{aligned}$$

where the dots correspond to higher order terms. We deduce that

$$dT(0)(\mu) = \frac{1}{2i\beta_1}\int_{\Sigma_L\cup\Sigma_{-L}} \frac{\partial u_1}{\partial\nu}w_- - u_1\frac{\partial w_-}{\partial\nu}\,dy.$$

Integrating by parts in $\mathcal{S}_L = (-L; L) \times I$, and using the third line of (II.9), this gives

$$dT(0)(\mu) = \frac{1}{2i\beta_1}\int_{\partial\mathcal{S}} u_1\frac{\partial w_-}{\partial y}\,dx = \frac{i}{2\beta_1}\int_{\mathbb{R}} \mu(x)\frac{\partial w_+}{\partial y}(x,1)\frac{\partial w_-}{\partial y}(x,1)\,dx = \frac{i\pi^2}{\beta_1}\int_{\mathbb{R}} \mu(x)\,dx.$$

Since the real part of $dT(0)(\mu)$ is null for all $\mu \in \mathscr{C}_0^\infty(\mathbb{R})$, this shows that $dT(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ is not onto. □

Remark II.3. *The fact that the real part of $dT(0)$ is necessarily null could have been guessed from conservation of conservation. Indeed, otherwise by linearity of $dT(0)$, we could find some $\mu \in \mathscr{C}_0^\infty(\mathbb{R})$ such that $dT(0)(\mu) > 0$. Then for $\varepsilon > 0$ small enough, $T(\varepsilon\mu)$ would have a real part larger than one. This is impossible due to conservation of energy.*

Though $dT(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ is not onto, Proposition II.2 proves that we can control the imaginary part of T. Let us exploit this property.

Define the map

$$\begin{array}{rccl} \mathscr{F} : & \mathscr{C}_0^\infty(\mathbb{R}) & \to & \mathbb{R}^3 \\ & h & \mapsto & (\Re e\, R(h), \Im m\, R(h), \Im m\, T(h))^\top. \end{array} \tag{II.13}$$

We have $\mathscr{F}(0) = 0$ and we wish to find some $h \not\equiv 0$ such that $\mathscr{F}(h) = 0$. From Propositions II.1 and II.2, we know that $d\mathscr{F}(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{R}^3$ is onto. Therefore there are $\mu_0, \dots, \mu_3 \in \mathscr{C}_0^\infty(\mathbb{R})$ such that

$$d\mathscr{F}(0)(\mu_0) = 0, \qquad [d\mathscr{F}(0)(\mu_1), d\mathscr{F}(0)(\mu_2), d\mathscr{F}(0)(\mu_3)] = \mathrm{Id}_{3\times 3}.$$

Set $\vec{\tau} := (\tau_1, \tau_2, \tau_3)^\top \in \mathbb{R}^3$ and consider the new fixed point equation

$$\vec{\tau} = F^\varepsilon(\vec{\tau}) \tag{II.14}$$

with

$$\begin{array}{rcl} F^\varepsilon(\vec{\tau}) & := & -\varepsilon(\Re e\,\tilde{R}(\varepsilon\mu(\vec{\tau})), \Im m\,\tilde{R}(\varepsilon\mu(\vec{\tau})), \Im m\,\tilde{T}(\varepsilon\mu(\vec{\tau}))^\top \\ \mu(\vec{\tau}) & := & \mu_0 + \tau_1\mu_1 + \tau_2\mu_2 + \tau_3\mu_3. \end{array}$$

Here $\tilde{R}$, $\tilde{T}$ are the remainders in the expansions

$$\begin{array}{rcl} R(\varepsilon\mu) & = & \varepsilon dR(0)(\mu) + \varepsilon^2\tilde{R}(\varepsilon\mu) \\ T(\varepsilon\mu) & = & 1 + \varepsilon dT(0)(\mu) + \varepsilon^2\tilde{T}(\varepsilon\mu). \end{array}$$

Then for any given $r > 0$, one can show that (II.14) admits a unique solution $\vec{\tau}^{\,\mathrm{sol}} \in \overline{B(O,r)}$ for ε small enough (here $B(O,r)$ denotes the open ball of $\mathbb{R}^3$ centered at O of radius r). Defining

$$h^{\,\mathrm{sol}} = \varepsilon\Big(\mu_0 + \sum_{p=1}^{3} \tau_p^{\mathrm{sol}}\mu_p\Big),$$

we obtain $\mathscr{F}(h^{\,\mathrm{sol}}) = 0$.

Why does this imply $T^\varepsilon = 1$? Conservation of energy imposes $|R^\varepsilon|^2 + |T^\varepsilon|^2 = 1$. Therefore when $R^\varepsilon = 0$ and $\Im m\,T^\varepsilon = 0$, the only possibility is to have either $T^\varepsilon = -1$ or $T^\varepsilon = 1$. Since we made a small perturbation of the reference strip, error estimates for the asymptotic expansion of u^ε imply that for ε small, T^ε is close too one. Therefore, for ε small enough, necessarily we must have $T^\varepsilon = 1$ exactly.

In Figure II.3, we give an example of perfectly invisible perturbation of the reference strip for the problem (I.46). This time, compared to Figure II.2 where we imposed only zero reflection, we observe that the scattered field is indeed exponentially decaying both as $x \to -\infty$ and as $x \to +\infty$.

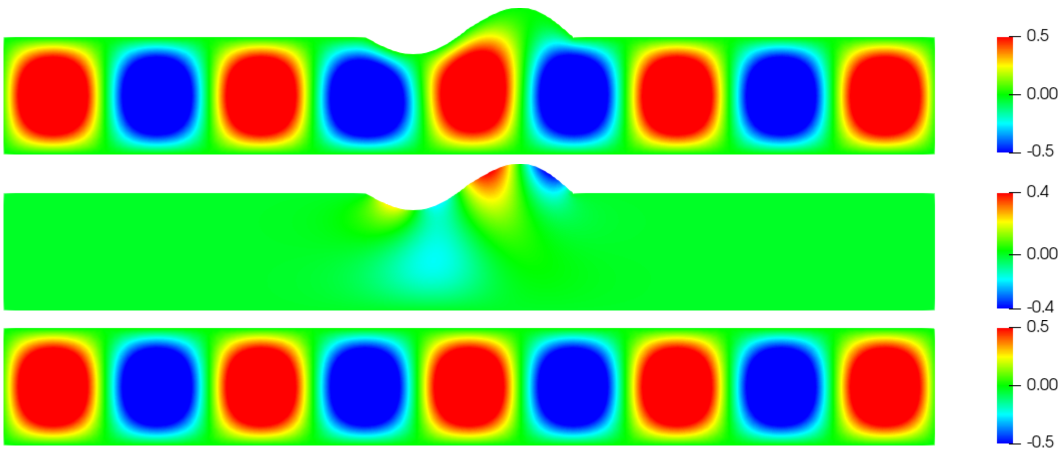


Figure II.3: Example of perfectly invisible obstacle for the problem (I.46) with Dirichlet BCs. Top: $\Re e\, u$. Middle: $\Re e\, u_s$. Bottom: $\Re e\, w_+$ in the reference strip. Here $k = 1.5\pi$.

5 Study of the Neumann problem

Assume now that we wish to apply the strategy described in §1 to Problem (I.52) with Neumann BCs. Pick some $k \in (0;\pi)$ so that only $w_\pm(x,y) = e^{\pm ikx}$ are propagating modes.

Let us compute $dR(0)$ and $dT(0)$. As above, for $\varepsilon > 0$ and a given $\mu \in \mathscr{C}_0^\infty(\mathbb{R})$, denote by Ω^ε the domain $\Omega(\varepsilon\mu)$. Let u^ε stand for the solution introduced in (I.55) corresponding to the scattering of the incident rightgoing wave w_+ in the geometry Ω^ε. It satisfies

$$\left|\begin{array}{rcl} \Delta u^\varepsilon + k^2 u^\varepsilon &=& 0 \quad \text{in } \Omega^\varepsilon \\ \partial_{\nu^\varepsilon} u^\varepsilon &=& 0 \quad \text{on } \partial\Omega^\varepsilon. \end{array}\right. \tag{II.15}$$

Let us emphasize that ν^ε, the outward unit normal vector to $\partial\Omega^\varepsilon$, depends on ε. For u^ε, as ε tends to zero, we consider the ansatz

$$u^\varepsilon = u_0 + \varepsilon u_1 + \dots \tag{II.16}$$

where u_0, u_1 are functions to be determined and the dots correspond to higher order terms. On $\partial\Omega^\varepsilon$, we have the expansions

$$\nu^\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2(\mu'(x))^2}} \left(\begin{array}{c} -\varepsilon\mu'(x) \\ 1 \end{array}\right) = \left(\begin{array}{c} 0 \\ 1 \end{array}\right) + \varepsilon \left(\begin{array}{c} -\mu'(x) \\ 0 \end{array}\right) + \dots \tag{II.17}$$

$$\nabla u^\varepsilon(x, 1+\varepsilon\mu(x)) \quad = \quad \nabla u^\varepsilon(x,1) + \varepsilon\mu(x) \left(\begin{array}{c} \partial^2_{xy} u_\varepsilon(x,1) \\ \partial^2_{yy} u_\varepsilon(x,1) \end{array}\right) + \dots . \tag{II.18}$$

Now we insert (II.16) in (II.15) and exploit (II.17), (II.18). Collecting the terms of orders ε^0, ε^1, we find that u_0, u_1 satisfy respectively the problems

$$\left|\begin{array}{rcl} \Delta u_0 + k^2 u_0 &=& 0 \quad \text{in } \mathcal{S} \\ \partial_\nu u_0 &=& 0 \quad \text{on } \partial\mathcal{S} \end{array}\right. \qquad \left|\begin{array}{rcll} \Delta u_1 + k^2 u_1 &=& 0 & \text{in } \mathcal{S} \\ \partial_\nu u_1 &=& 0 & \text{on } \mathbb{R}\times\{0\} \\ \partial_\nu u_1 &=& \mu'(x)\partial_x u_0 - \mu(x)\partial^2_{yy} u_0 & \text{on } \mathbb{R}\times\{1\}. \end{array}\right.$$

Using additionally that $u^\varepsilon - w_+$ is outgoing, we get first

$$u_0(x,y) = w_+(x,y) = e^{ikx}.$$

Since w_+ is independent of y, we deduce that u_1 satisfies the condition

$$\partial_\nu u_1 = \mu'(x)\partial_x w_+ \quad \text{on } \mathbb{R}\times\{1\}. \tag{II.19}$$

Denote by R^ε, T^ε the scattering coefficients of u^ε. Assuming that μ is supported in $(-L;L)$, from Proposition I.17, we know that

$$2ikR^\varepsilon = \int_{\Sigma_{-L}\cup\Sigma_L} \frac{\partial u^\varepsilon}{\partial\nu} u^\varepsilon w_+ - u^\varepsilon \frac{\partial w_+}{\partial\nu}\, dy, \qquad 2ikT^\varepsilon = \int_{\Sigma_{-L}\cup\Sigma_L} \frac{\partial u^\varepsilon}{\partial\nu} w_- - u^\varepsilon \frac{\partial w_-}{\partial\nu}\, dy. \tag{II.20}$$

Inserting the expansion (II.16) of u^ε in (II.20), integrating by parts and exploiting (II.19), we obtain

$$R^\varepsilon = 0 - \frac{\varepsilon}{2ik}\int_{\mathbb{R}} \mu'(x)\partial_x w_+(x) w_+(x)\, dx, \qquad T^\varepsilon = 1 + \frac{\varepsilon}{2ik}\int_{\mathbb{R}} \mu'(x)\partial_x w_+(x) w_-(x)\, dx.$$

This gives the formulas

$$\begin{array}{rcl} dR(0)(\mu) &=& \dfrac{-1}{2ik}\displaystyle\int_{\mathbb{R}} \mu'(x)\partial_x w_+ w_+\, dx = \dfrac{-1}{2}\int_{\mathbb{R}} \mu'(x) e^{2ikx}\, dx = ik\int_{\mathbb{R}} \mu(x) e^{2ikx}\, dx \\ dT(0)(\mu) &=& \dfrac{1}{2ik}\displaystyle\int_{\mathbb{R}} \mu'(x)\partial_x w_+ w_-\, dx = \dfrac{1}{2}\int_{\mathbb{R}} \mu'(x)\, dx = 0. \end{array} \tag{II.21}$$

As in the Dirichlet case, exploiting that the real part of $x \mapsto e^{2ikx}$ is even while its imaginary part is odd, we see that it is easy to find functions μ_1, μ_2 satisfying the relations (II.3), which ensures that $dR(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ is onto. Then by applying what has been done in §1, one can construct perturbations of the reference strip which are non reflecting at a given $k \in (0;\pi)$.

Let us give two examples. Set $\delta := \pi/k$, $I_{\text{defect}} := (-\delta;\delta)$ and for μ_0, μ_1, μ_2, let us work first with the functions such that

$$\mu_0(x) = \sin(kx), \quad \mu_1(x) = -\frac{1}{\pi}\sin(2kx), \quad \mu_2(x) = \frac{7}{12}\cos(3kx/2), \qquad \text{for } x \in I_{\text{defect}}, \tag{II.22}$$

$\mu_0 = \mu_1 = \mu_2 = 0$ in $\mathbb{R} \setminus \overline{I_{\text{defect}}}$. One can check that they indeed satisfy relations (II.2), (II.3). We set $\eta = 10^{-4}$, $k = 0.8\pi$. In Figure II.4, we display the geometry at the end of the iterative procedure for $\varepsilon = 0.4$. For this choice of functions μ_0, μ_1, μ_2, we are able to obtain a rather large non reflecting perturbation of the reference strip. Here $|R| \approx 4.10^{-5}$ in 15 iterations.

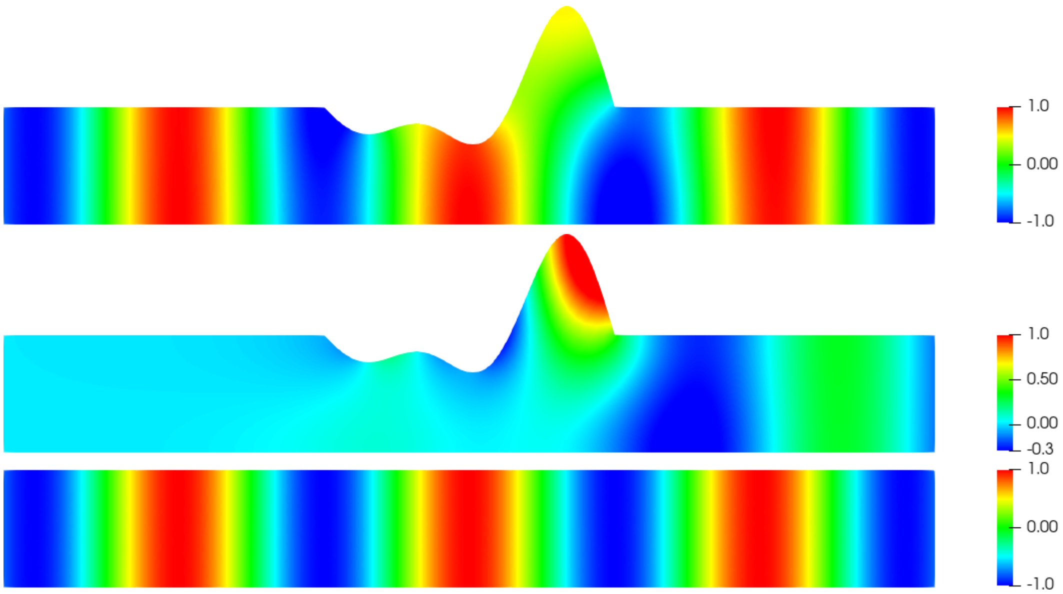


Figure II.4: Example of non reflecting geometry for Problem (I.52) with Neumann BCs. Top: $\Re e\, u$. Middle: $\Re e\, u_s$. Bottom: $\Re e\, w_+$ in the reference strip. Here $k = 0.8\pi$.

In Figure II.5, we display another example of non reflecting defect. It has been obtained by changing the μ_0 in (II.22) to

$$\mu_0(x) = |x| - \delta, \qquad \text{for } x \in I_{\text{defect}},$$

$\mu_0 = 0$ in $\mathbb{R} \setminus \overline{I_{\text{defect}}}$. Here the defect lies entirely in the region $y < 1$. Because of this property, we can use symmetry with respect to the line $y = 1$ to create a non reflecting obstacle completely embedded in the waveguide (see Figure II.6).

In §4 for the Dirichlet problem, by exploiting conservation of energy, we explained how to construct waveguide where $T = 1$. Can we adapt this to the Neumann problem? Well, from the computation of $dT(0)$ in (II.21) we see that it is impossible because $dT(0)$ is null. It means that a perturbation of order ε of the reference strip gives a transmission coefficient T^ε such that $T^\varepsilon - 1$ is in $O(\varepsilon^2)$. This looks appealing for perfect transmission. The problem is that since $dT(0)$ is null, one cannot use this term which has a linear dependence with respect to the perturbation of the reference strip to cancel the whole (non linear) expansion of T^ε via the resolution of the fixed point problem. As a consequence, our technique fails to design perfectly invisible defects in the Neumann case.

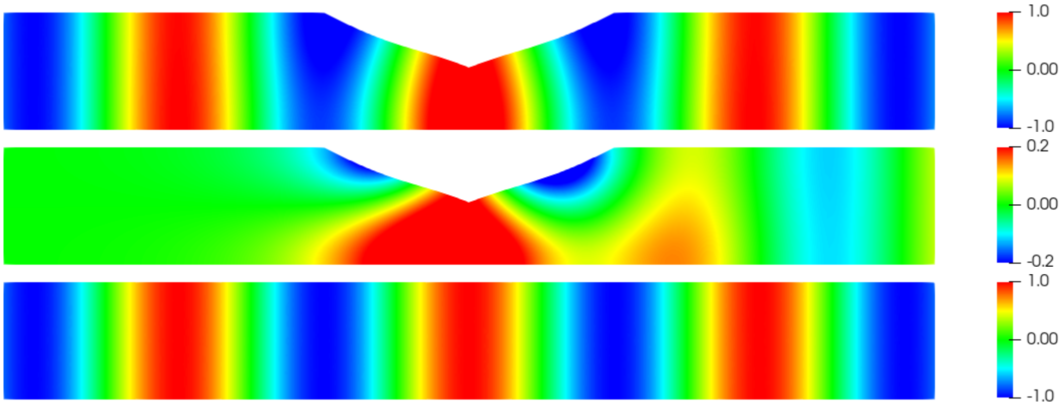


Figure II.5: Example of non reflecting obstacle for Problem (I.52) with Neumann BCs. Top: $\Re e\, u$. Middle: $\Re e\, u_s$. Bottom: $\Re e\, w_+$ in the reference strip. Here $k = 0.8\pi$.

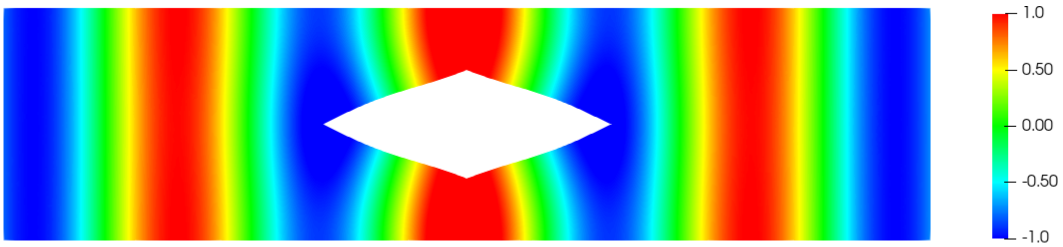


Figure II.6: Example of non reflecting obstacle for Problem (I.52) with Neumann BCs. Here $k = 0.8\pi$.

Above we showed how to construct invisible smooth perturbations of the reference strip $\mathcal{S}$. In the remaining part of this chapter, we wish to explain how to design invisible non smooth perturbations of $\mathcal{S}$. The terminology smooth/not smooth here does not refer to the regularity of the domain but to the form of the asymptotic expansion involved in the asymptotic procedure. In the non smooth case, the field u^ε exhibits rapid variations in a neighborhood of the perturbation that must be caught with adapted variables. In that situation, the asymptotics of the scattering coefficients is in general more complicated to obtain. The interesting point is that it can bring new useful terms, for example non zero differentials for T for the Neumann problem. Below we consider two types of non smooth perturbations of the reference strip.

6 Non reflecting clouds of small obstacles

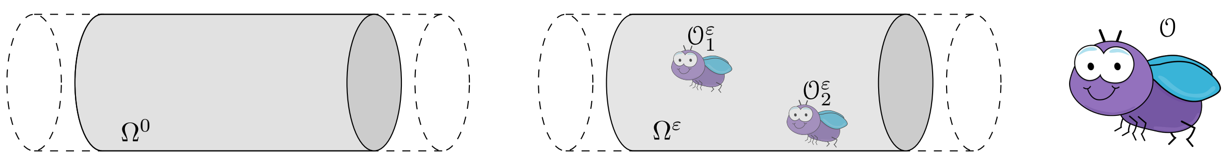


Figure II.7: Unperturbed waveguide (left), perturbed waveguide with two small obstacles (middle), set $\mathcal{O}$ (right).

Let us work first with clouds of small obstacles. To simplify the asymptotic analysis, we work in 3D with Dirichlet boundary conditions. Note that in 2D, the Green function of the Laplace operator, whose behavior dictates the form of the asymptotic expansions as ε tends to zero, has a logarithmic singularity which does not appear in 3D. Let

$$\Omega^0 := \{z = (x, y) \,|\, x \in \mathbb{R} \text{ and } y \in \omega\}$$

be a cylinder of $\mathbb{R}^3$ whose transverse section $\omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary (see Figure II.7 left). Consider $\mathcal{O} \subset \mathbb{R}^3$ a bounded domain with Lipschitz boundary and for M_1 a point located in Ω^0, set, for ε small,

$$\mathcal{O}_1^\varepsilon := \{z \in \mathbb{R}^3 \,|\, \varepsilon^{-1}(z - M_1) \in \mathcal{O}\}.$$

Finally define the perturbed waveguide

$$\Omega^\varepsilon := \Omega^0 \setminus \overline{\mathcal{O}_1^\varepsilon}.$$

To begin with, for the moment we assume that there is only one obstacle and not a cloud. The problem we consider writes

$$\left|\begin{array}{rcl} \Delta u^\varepsilon + k^2 u^\varepsilon &=& 0 \quad \text{in } \Omega^\varepsilon \\ u^\varepsilon &=& 0 \quad \text{on } \partial\Omega^\varepsilon. \end{array}\right. \tag{II.23}$$

We fix the wavenumber $k > 0$ such that only two modes $w_\pm$ can propagate in Ω^ε. These $w_\pm$ are not exactly the same as the ones in (I.54), they involve the eigenfunctions associated with the first eigenvalue of the Dirichlet Laplacian in ω, but the situation is similar. As in §3.6, Problem (II.23) admits a solution with the expansion

$$u^\varepsilon = \left|\begin{array}{rl} w_+ + R^\varepsilon w_- + \tilde{u}^\varepsilon & \text{for } x \le -d \\ T^\varepsilon w_+ + \tilde{u}^\varepsilon & \text{for } x \ge d, \end{array}\right.$$

where $R^\varepsilon, T^\varepsilon \in \mathbb{C}$ and $\tilde{u}^\varepsilon$ decays exponentially at infinity (we denote R^ε instead of R_+^ε to simplify). Again $d > 0$ is such that $\Omega^\varepsilon = \Omega^0$ for $|x| > d$. The first step in the approach is to compute an asymptotic expansion of u^ε as ε tends to zero. This is a rather long work that we will not present here (one may consult the reference [31, §2.2] for more details). Let us simply stress that it appears that u^ε has a rapid variation in a neighborhood of the obstacle to satisfy the constraint of being null on $\partial\mathcal{O}_1^\varepsilon$. This rapid variation must be caught with adapted variables. Then it is necessary to work both with some inner field and outer field expansions of u^ε that we match in some intermediate region. This is the method of matched asymptotic expansions which is well documented in the literature. At the end of the procedure, when ε tends to zero, we obtain

$$R^\varepsilon = 0 + \varepsilon\, 4i\pi \mathrm{cap}(\mathcal{O}) w_+(M_1)^2 + O(\varepsilon^2), \qquad T^\varepsilon = 1 + \varepsilon\, 4i\pi \mathrm{cap}(\mathcal{O}) |w_+(M_1)|^2 + O(\varepsilon^2).$$

Here $\mathrm{cap}(\mathcal{O})$ stands for the capacity of the domain $\mathcal{O}$, a constant which appears classically in asymptotic analysis. An important point for our study is that we always have $\mathrm{cap}(\mathcal{O}) > 0$. Additionally, one finds that w_+ does not vanish in Ω^0. From these two properties, we infer that one single small obstacle cannot even be non reflecting: whatever the choice for M_1 or for the shape $\mathcal{O}$, $\mathcal{O}_1^\varepsilon$ will always generate a reflection whose amplitude is of order ε.

Let us add a second small obstacle

$$\mathcal{O}_2^\varepsilon := \{z \in \mathbb{R}^3 \,|\, \varepsilon^{-1}(z - M_2) \in \mathcal{O}\}$$

in the waveguide, centred at the point $M_2 \in \Omega^0$ with $M_2 \neq M_1$. Still denoting by R^ε, T^ε the scattering coefficients in this new geometry, we obtain the expansions, as $\varepsilon \to 0^+$,

$$R^\varepsilon = 0 + \varepsilon\, 4i\pi \mathrm{cap}(\mathcal{O}) \sum_{j=1}^{2} w_+(M_j)^2 + O(\varepsilon^2), \quad T^\varepsilon = 1 + \varepsilon\, 4i\pi \mathrm{cap}(\mathcal{O}) \sum_{j=1}^{2} |w_+(M_j)|^2 + O(\varepsilon^2). \tag{II.24}$$

Observe that the interactions between the small objects do not appear at order ε, only at higher orders. The interesting point is that now, using the known expression of w_+, we can find positions M_1, M_2 of the obstacles such that

$$\sum_{j=1}^{2} w_+(M_j)^2 = 0. \tag{II.25}$$

In that case, we have a perturbation of Ω^0 of order ε which produces a reflection in $O(\varepsilon^2)$. As mentioned above, this is almost no reflection. But by working a bit harder, we can achieve more. Pick M_1, M_2 such that relation (II.25) is satisfied. Then by slightly perturbing the position of one obstacle, we can get $R^\varepsilon = 0$ exactly. More precisely, for $\tau \in \mathbb{R}^3$, define $M_1^{\varepsilon\tau} = M_1 + \varepsilon\tau$ and

$$\mathcal{O}_1^{\varepsilon\tau} := \{z \in \mathbb{R}^3 \,|\, \varepsilon^{-1}(z - M_1^{\varepsilon\tau}) \in \mathcal{O}\}.$$

One can show that there is τ, which is defined as the solution of a fixed point problem similar to (II.4), such that the reflection coefficient in the corresponding waveguide is zero (for more details, see [17]). Note that since $\tau \in \mathbb{R}^3$, we have enough degrees of freedom to cancel one single complex number.

Can we get perfect invisibility, *i.e.* $T^\varepsilon = 1$, with this approach? From (II.24) we see that the answer is no. Indeed, whatever the position of the small obstacles or their number, we always have a phase shift for the transmitted wave.

One can also study what happens at higher wavenumbers k. In that situation, as can be seen in the particular case (I.22), more modes can propagate. The reflection and transmission coefficients then become respectively reflection and transmission matrices $\mathcal{R}^\varepsilon$, $\mathcal{T}^\varepsilon$. One can prove that by working with a sufficiently large number of small obstacles, we can cancel exactly the whole reflection matrix. The idea is the same as above. First we compute an asymptotic expansion of $\mathcal{R}^\varepsilon$ as $\varepsilon \to 0$. Then we find positions M_1, M_2,... , M_k of the obstacles to cancel the term of order ε in the expansion of $\mathcal{R}^\varepsilon$. Then by slightly perturbing the position of one group of obstacles by solving a fixed point problem in $\mathbb{R}^N$, for a certain N depending on the number of propagating modes, we get $\mathcal{R}^\varepsilon = 0$. The higher the number of propagating modes, the more obstacles we need.

To conclude this section, let us mention that in this work, the main difficulty mathematically consists in proving error estimates in the asymptotic expansions which are uniform with respect to the parameter τ in a closed ball to justify that the map appearing in the fixed point problem is indeed a contraction for ε small enough. We will not elaborate more on that topic here and refer the interesting reader to [17].

7 Perfect transmission for the Neumann problem

The different strategies presented above fail to provide examples of waveguides where $T = 1$ (perfect transmission without phase shift) for the problem with Neumann BCs. Is there some fundamental obstruction to get $T = 1$ in that case? In this section, we answer negatively to that question. To proceed, we work with another singular perturbation of the reference strip.

Consider the waveguide Ω^ε pictured in Figure II.8. It is made of the reference strip $\mathcal{S}$ to which we have glued at the points $M_1 := (x_1, 1)$, $M_2 := (x_2, 1)$, $M_3 := (x_3, 1)$, thin chimneys of width $\varepsilon > 0$ small and heights respectively equal to h_1, h_2, h_3. We fix $k \in (0; \pi)$ so that only the modes $w_\pm(x, y) = e^{ikx}$ can propagate. The first step is to compute an asymptotic expansion of the scattering coefficients as ε, the width of the thin rectangles, tends to zero. Again, this is a rather long work that we will not present here, the reason being that the fields vary rapidly in some zone around the M_n, $n = 1, 2, 3$. We refer the reader to [11] for more details. In this article, by using the method of matched asymptotic expansions, it is shown that when the h_n are such that $kh_n \in \{\pi/2 + \pi\mathbb{N}\}$, when ε tends to zero, we have

$$\begin{aligned} R^\varepsilon &= 0 + \varepsilon\Big(ik\sum_{n=1}^{3}(w_+(M_n))^2\tan(kh_n)\Big) + O(\varepsilon^2) \\ T^\varepsilon &= 1 + \varepsilon\Big(ik\sum_{n=1}^{3}\tan(kh_n)\Big) + O(\varepsilon^2). \end{aligned} \tag{II.26}$$

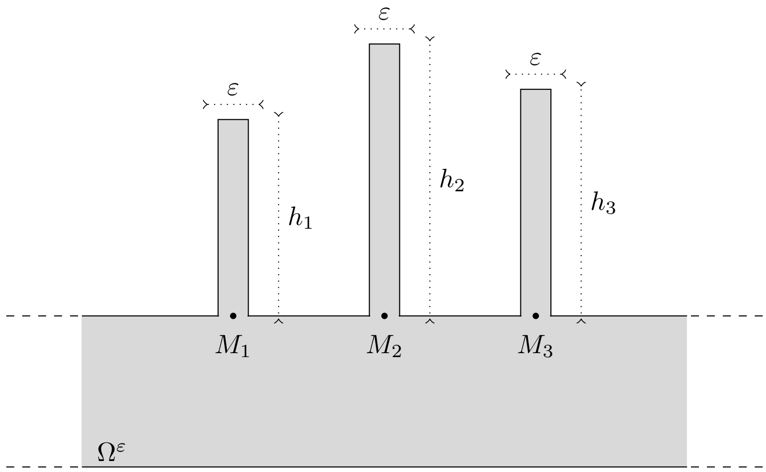


Figure II.8: Geometry of Ω^ε.

Remark II.4. *When $kh_n \in \{\pi/2 + \pi\mathbb{N}\}$, resonant phenomena appears in the chimneys and the asymptotics (II.26) is no more valid. This regime will be exploited in Chapter III (see Section 2).*

The crucial point to observe in (II.26) compared to the other perturbations of the reference strip above is the appearance of the term of order ε in the expansion of T^ε. Its imaginary part is non zero and can change sign according to the value of the h_n. Thus by considering another type of perturbation of $\mathcal{S}$, we have been able to obtain a non zero $dT(0)$.

By using that $(w_+(M_n))^2 = e^{2ikx_n}$, we can find positions and heights of the chimneys such that $R^\varepsilon = O(\varepsilon^2)$ and $T^\varepsilon - 1 = O(\varepsilon^2)$. Then perturbing slightly the h_n around these particular values, by solving a fixed point problem in $\mathbb{R}^3$ similar to (II.4), we can achieve

$$R^\varepsilon = 0 \qquad \text{and} \qquad \Im m\, T^\varepsilon = 0$$

in the new geometry. Finally, by exploiting the relation of conservation of energy $|R^\varepsilon|^2 + |T^\varepsilon|^2 = 1$ as in (4), one shows that this implies $T^\varepsilon = 1$ for $\varepsilon > 0$ sufficiently small.

Initially we have to control two complex numbers (R^ε and T^ε), so a priori we need four real degrees of freedom. But due to the constraint of conservation of energy, three parameters are sufficient. This explains why three chimneys are involved here. We could also have probably worked with only two chimneys, by perturbing their heights and the gap between them.

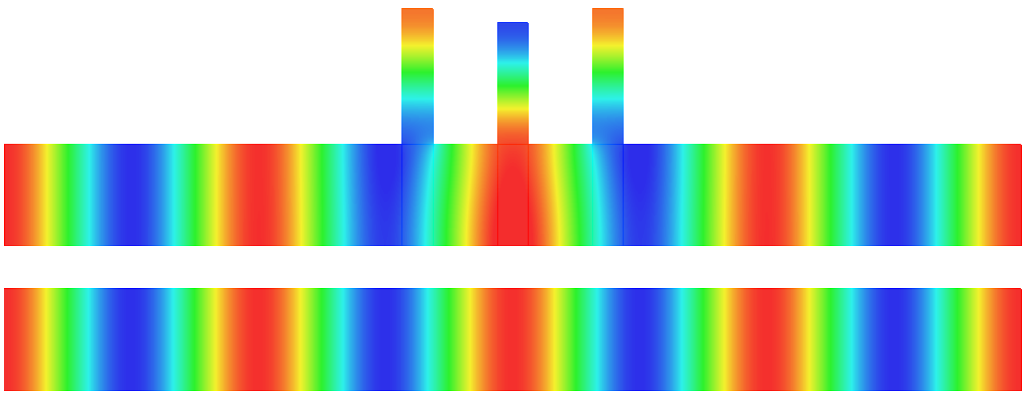


Figure II.9: Top: real part of u^ε in the geometry obtained at the end of the iterative procedure. Bottom: real part of w_+ in the reference strip.

This approach can be implemented numerically very naturally. First we set the M_n and the h_n to kill the terms of order ε in (II.26). Then we tune slightly the length of the ligaments by solving the corresponding fixed point problem iteratively as in (II.11). At each step, we solve a scattering problem in a new geometry. One can see this procedure as acting on the pistons on a trumpet to achieve $T^\varepsilon = 1$. In Figure II.9 we display a geometry obtained with this method. A bit far from the chimneys so that evanescent terms can be decently neglected, we remark that the field is the same as in the reference strip. Again, numerically one observes that the algorithm converges though ε, the width of the ligaments, is not that small.

8 Concluding remarks

In this chapter, we constructed smooth and non-smooth perturbations of the reference geometry which are invisible, in a broad sense (non reflecting or perfectly invisible). We worked at a given wavenumber k. We could proceed similarly to impose invisibility at given $k_1, \dots, k_N$. However we emphasize that the set of wavenumbers must be discrete and finite. Imposing zero reflection for a continuum of k is probably impossible in general due to the analyticity of the map $k \mapsto R(k)$. In the numerical results we presented, we were interested in controlling only R and sometimes $\Im m\, T$ at one k. As a consequence, we had very few constraints and for this reason, the fixed point algorithm converges with not so small values of ε, which allowed us to obtain rather large invisible defects. When the number of constraints increases, for example when working at higher k so that there are more scattering coefficients to control, in practice we find that ε must be chosen smaller to have convergence of the method. A natural idea then is to try to reiterate the procedure to obtain larger invisible defects: once an invisible obstacle has been constructed, we can consider it as a new starting configuration and perturb it while keeping the same scattering coefficients. This is usually called a continuation method which allows one to explore the variety (of infinite dimension) of invisible obstacles. We will not implement it here (see [8] for more details). Instead, we simply illustrate the method in Figure II.10 in finite dimension. More precisely, instead of working with $\mathscr{F} : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{R}^3$ as in the case of perfect invisibility (see (II.13)), we consider some smooth

$$\begin{array}{rccc} \mathscr{F} : & \mathbb{R}^2 & \to & \mathbb{R} \\ & (h_1, h_2) & \mapsto & \mathscr{F}(h_1, h_2) \end{array}$$

such that $\mathscr{F}(0) = 0$, $\nabla\mathscr{F}(0) \neq 0$.

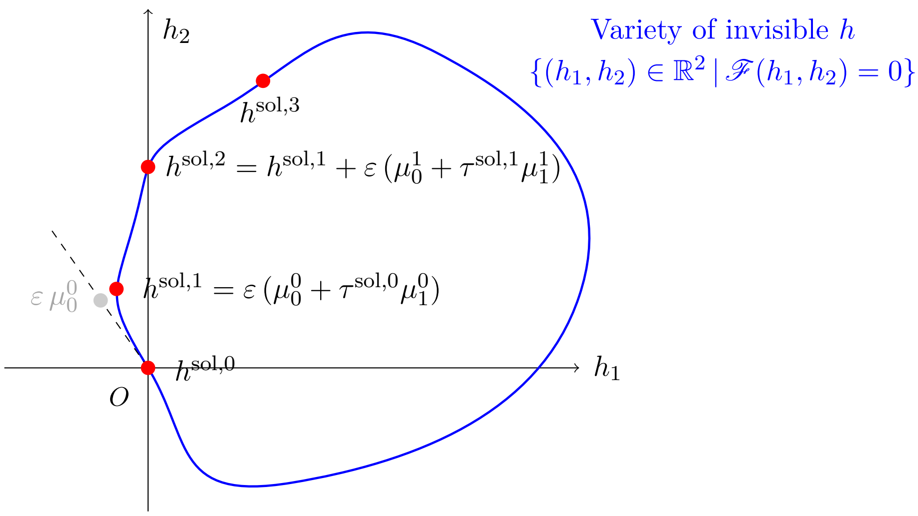


Figure II.10: Illustration of the continuation method. Here for $n \in \mathbb{N}$, $\mu_0^n, \mu_1^n \in \mathrm{span}(h_0, h_1)$ are such that $d\mathscr{F}(h^{\mathrm{sol},n})(\mu_0^n) = 0$ and $d\mathscr{F}(h^{\mathrm{sol},n})(\mu_1^n) = 1$.

Chapter III

Playing with resonances to reach invisibility

1 Playing with the Fano resonance

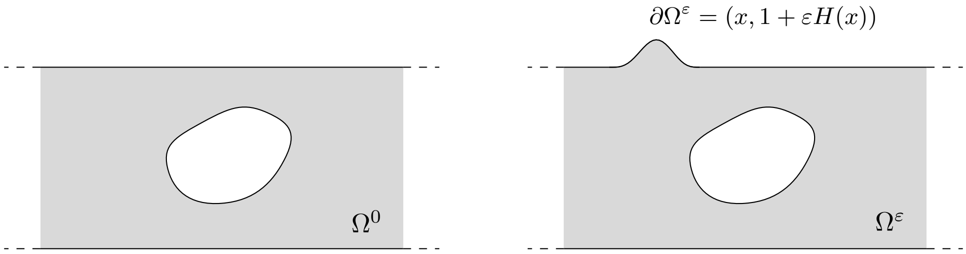


Figure III.1: Original waveguide Ω^0 (left) and perturbed geometry Ω^ε (right).

The Fano resonance, named from the physicist Ugo Fano (1912-2001), is a classical phenomenon that arises in many situations in physics. For our particular concern, it appears as follows. Assume that the geometry of our waveguide is characterized by a real parameter ε. Below, ε will be the amplitude of a local perturbation of the walls (see Figure III.1). To simplify notation, set $\lambda := k^2$ so that Problem (I.3) writes

$$\left|\begin{array}{rcll} \Delta u + \lambda u & = & 0 & \text{in } \Omega^\varepsilon \\ \partial_\nu u & = & 0 & \text{on } \partial\Omega^\varepsilon. \end{array}\right. \tag{III.1}$$

Assume that trapped modes exist for Problem (III.1) with $\varepsilon = 0$ and $\lambda = \lambda^0 \in \mathbb{R}$. Then, for $\varepsilon \neq 0$ small, the scattering matrix $\mathcal{S}$, which is of size 2×2 in monomode regime, exhibits a rapid change for real λ varying in a neighbourhood of λ^0. The justification of this result requires adapted tools (see [36, 35, 37, 1, 18]) and we will not detail this aspect. Our goal is to exploit this rapid change together with symmetry considerations of the geometry to provide examples of waveguides where $R = 0$ or $T = 0$. Note that the case $T = 0$, that we will call zero transmission, is not related to invisibility. It corresponds to a situation where the energy of an incident wave is completely backscattered, like for a mirror.

To understand more this Fano resonance phenomenon, let us work on a 1D toy problem.

1.1 A 1D toy problem

Consider the geometry

$$\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3 \qquad \text{with } \Omega_1 := (-\infty; 0) \times \{0\},\ \Omega_2 := \{0\} \times (0; 1),\ \Omega_3 := (0; 1) \times \{0\}$$

Figure III.2: A 1D geometry.

(see Figure III.2). For a function φ defined in Ω, set $\varphi_i := \varphi|_{\Omega_i}$. Working in suitable coordinates, we can see the Ω_i as 1D domains. Similarly to (III.1), we study the Helmholtz problem with Neumann boundary conditions

$$-\varphi'' = k^2\varphi \text{ in } \Omega, \qquad \left|\begin{array}{l}\varphi_1(0) = \varphi_2(0) = \varphi_3(0),\\ \varphi_1'(0) = \varphi_2'(0) + \varphi_3'(0),\\ \varphi_2'(1) = \varphi_3'(1) = 0.\end{array}\right. \tag{III.2}$$

In particular, at the junction point O, we impose continuity of the field and conservation of the flux (Kirchhoff law). We are interested in the scattering of the rightgoing incident wave $\varphi_i(x) = e^{ikx}$. We denote by φ and $\varphi_s = \varphi - \varphi_i$ the corresponding total and scattered fields. We impose that φ_s is outgoing at infinity. For the simple problem considered here, the radiation condition boils down to assume that φ_s writes as $\varphi_s(x) = R\,e^{-ikx}$ where $R \in \mathbb{C}$ is the reflection coefficient. Using the two boundary conditions of (III.2), we are led to look for a solution φ such that

$$\varphi_1(x) = e^{ikx} + R\,e^{-ikx}, \quad \varphi_2(y) = A\cos(k(y-1)), \quad \varphi_3(x) = B\cos(k(x-1)),$$

where A, $B \in \mathbb{C}$. Writing the transmission conditions at the junction point O, we obtain that R, A, B must solve the system

$$\mathbb{M}\Phi = F \qquad \text{with } \mathbb{M} := \begin{pmatrix} 1 & -\cos(k) & 0 \\ 0 & \cos(k) & -\cos(k) \\ i & \sin(k) & \sin(k)\end{pmatrix}, \; \Phi := \begin{pmatrix} R \\ A \\ B\end{pmatrix} \; F := \begin{pmatrix} -1 \\ 0 \\ i\end{pmatrix}. \tag{III.3}$$

One finds that this system (and so Problem (III.2) with the above mentioned radiation condition) is uniquely solvable if and only $k \notin (2\mathbb{N}+1)\pi/2$. When $k \in (2\mathbb{N}+1)\pi/2$, the kernel of Problem (III.2) coincides with $\mathrm{span}(\varphi_{\mathrm{tr}})$ where φ_{tr} is the trapped mode such that $\varphi_{\mathrm{tr}}(x) = 0$ in Ω_1, $\varphi_{\mathrm{tr}}(y) = \sin(ky)$ in Ω_2 and $\varphi_{\mathrm{tr}}(x) = -\sin(kx)$ in Ω_3. On the other hand, for any $k > 0$, one can check that System (III.3) (and so Problem (III.2)) admits a solution. Moreover, as in (I.48) the coefficient R is always uniquely defined (even when $k \in (2\mathbb{N}+1)\pi/2$) and such that

$$R = \frac{\cos(k) + 2i\sin(k)}{\cos(k) - 2i\sin(k)}. \tag{III.4}$$

The map $k \mapsto R(k)$ is π-periodic and $|R(k)| = 1$. The latter relation, which is due to conservation of energy, guarantees that $R(k) = e^{i\theta(k)}$ for some phase $\theta(k) \in \mathbb{R}/(2\pi\mathbb{Z})$.

Now we consider the same problem in the perturbed geometry $\Omega^\varepsilon := \Omega_1 \cup \Omega_2 \cup \Omega_3^\varepsilon$ with $\Omega_3^\varepsilon := (0; 1+\varepsilon) \times \{0\}$ and $\varepsilon \in \mathbb{R}$ small. We denote with a superscript ε all the above quantities. In Ω^ε, the resolution of the previous scattering problem leads to solve the system

$$\mathbb{M}^\varepsilon\Phi^\varepsilon = F \qquad \text{with } \mathbb{M}^\varepsilon := \begin{pmatrix} 1 & -\cos(k) & 0 \\ 0 & \cos(k) & -\cos(k(1+\varepsilon)) \\ i & \sin(k) & \sin(k(1+\varepsilon))\end{pmatrix}, \; \Phi^\varepsilon := \begin{pmatrix} R^\varepsilon \\ A^\varepsilon \\ B^\varepsilon\end{pmatrix}.$$

The vector F is the same as in (III.3). For $\varepsilon \neq 0$ small, we find that the determinant of $\mathbb{M}^\varepsilon$ does not vanish. As a consequence, Problem (III.2) set in Ω^ε has a unique solution. One finds

$$R^\varepsilon = \frac{\cos(k)\cos(k(1+\varepsilon)) + i\sin(k(2+\varepsilon))}{\cos(k)\cos(k(1+\varepsilon)) - i\sin(k(2+\varepsilon))}.$$

Again, we have $|R^\varepsilon(k)| = 1$ (conservation of energy) so we can write $R^\varepsilon(k) = e^{i\theta^\varepsilon(k)}$ for some $\theta^\varepsilon(k) \in \mathbb{R}/(2\pi\mathbb{Z})$. Note that $\theta^0 = \theta$ where θ appears after (III.4). The map $k \mapsto \theta^\varepsilon(k)$ is displayed in Figure III.3 for several values of ε (see also the alternative representation (III.4)). We observe that for $\varepsilon \neq 0$, the curve $k \mapsto \theta^\varepsilon(k)$ has a fast variation for k close to $\pi/2$. The variation is even faster as $\varepsilon \neq 0$ gets small. On the other hand, for $\varepsilon = 0$ the curve $k \mapsto \theta^0(k)$ has a very smooth behaviour. We emphasize that for $(\varepsilon, k) = (0, \pi/2)$, as mentioned above, trapped modes exist for Problem (III.2).

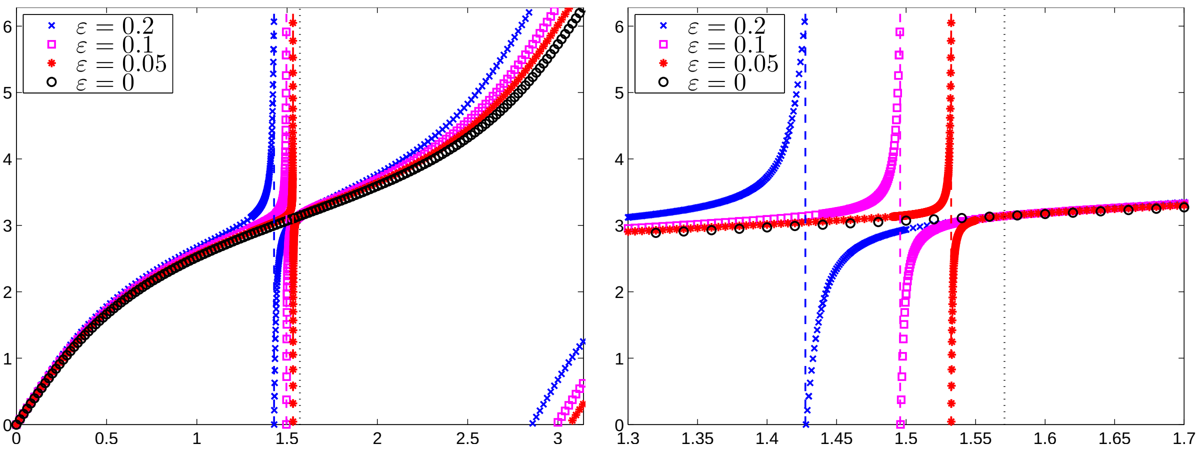


Figure III.3: Maps $k \mapsto \theta^\varepsilon(k)$ for several values of ε. The right picture is a zoom on the left picture around $k = \pi/2$ (marked by the vertical black dotted line). The vertical coloured dashed lines indicate the values of k such that $\theta^\varepsilon(k) = 0$.

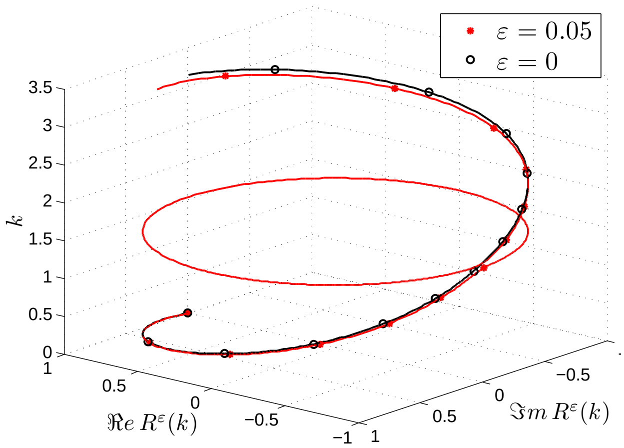


Figure III.4: Parametric curves $k \mapsto (\Re e\, R^\varepsilon(k), \Im m\, R^\varepsilon(k))$ for $k \in (0;\pi)$.

In order to study the variations of the reflection coefficient with respect to the frequency and the geometry, we define the map $\mathcal{R} : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\mathcal{R}(\varepsilon, k) = \frac{\cos(k)\cos(k(1+\varepsilon)) + i\sin(k(2+\varepsilon))}{\cos(k)\cos(k(1+\varepsilon)) - i\sin(k(2+\varepsilon))}. \tag{III.5}$$

With such a notation, we have $R^\varepsilon(k) = \mathcal{R}(\varepsilon, k)$ and $R(k) = \mathcal{R}(0, k)$. For all $k \in (0;\pi)$, there holds $\lim_{\varepsilon\to 0} \mathcal{R}(\varepsilon, k) = \mathcal{R}(0, k)$. Now assume that the frequency and the geometry are related by some prescribed law in a neighbourhood of the point $(\varepsilon, k) = (0, \pi/2)$ corresponding to a setting supporting trapped modes. For example, assume that

$$k = \pi/2 + \varepsilon k'$$

for a given $k' \in \mathbb{R}$. Then for $k' \neq -\pi/4$, starting from expression (III.5), we find as $\varepsilon \to 0$ the expansion

$$\mathcal{R}(\varepsilon, \pi/2 + \varepsilon k') = -1 + \varepsilon\left(\frac{-2ik'(\pi + 2k')}{\pi + 4k'}\right) + O(\varepsilon^2). \tag{III.6}$$

Note that we have $\mathcal{R}(0, \pi/2) = -1$. For $k' = -\pi/4$ and more generally, for

$$k = \pi/2 - \varepsilon\pi/4 + \varepsilon^2\mu$$

with $\mu \in \mathbb{R}$, we obtain

$$\mathcal{R}(\varepsilon, \pi/2 - \varepsilon\pi/4 + \varepsilon^2\mu) = g(\mu) + O(\varepsilon) \qquad \text{with} \qquad g(\mu) = \frac{\pi^2 + i(32\mu - 4\pi)}{\pi^2 - i(32\mu - 4\pi)}. \tag{III.7}$$

Classical results concerning the Möbius transform (see e.g. [20, Chap. 5]) guarantee that g is a bijection between $\mathbb{R}$ and $\mathscr{C}(0,1) \setminus \{-1 + 0i\}$ ($\mathscr{C}(0,1)$ is the unit circle of the complex plane). Thus for any $z_0 \in \mathscr{C}(0,1)$, we can find a path $\{\gamma(s),\ s \in (0;1)\} \subset \mathbb{R}^2$ such that

$$\lim_{s\to 1} \gamma(s) = (0, \pi/2) \qquad \text{and} \qquad \lim_{s\to 1} \mathcal{R}(\gamma(s)) = z_0$$

(see Figure III.5 right). This proves that the map $\mathcal{R}(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{C}$ is not continuous at $(0, \pi/2)$. This shows also that for $\varepsilon_0 \neq 0$ small fixed (see the vertical red dashed line in Figure III.5 left), the curve $k \mapsto \mathcal{R}(\varepsilon_0, k)$ must exhibit a rapid change. Indeed, μ varying in $[-C\varepsilon_0^{-1}; C\varepsilon_0^{-1}]$ for some arbitrary $C > 0$ (which is only a small change for k) leads to a large change for $\mathcal{R}(\varepsilon_0, \pi/2 - \varepsilon_0\pi/4 + \varepsilon_0^2\mu)$. This is exactly what we observed in Figure III.3.

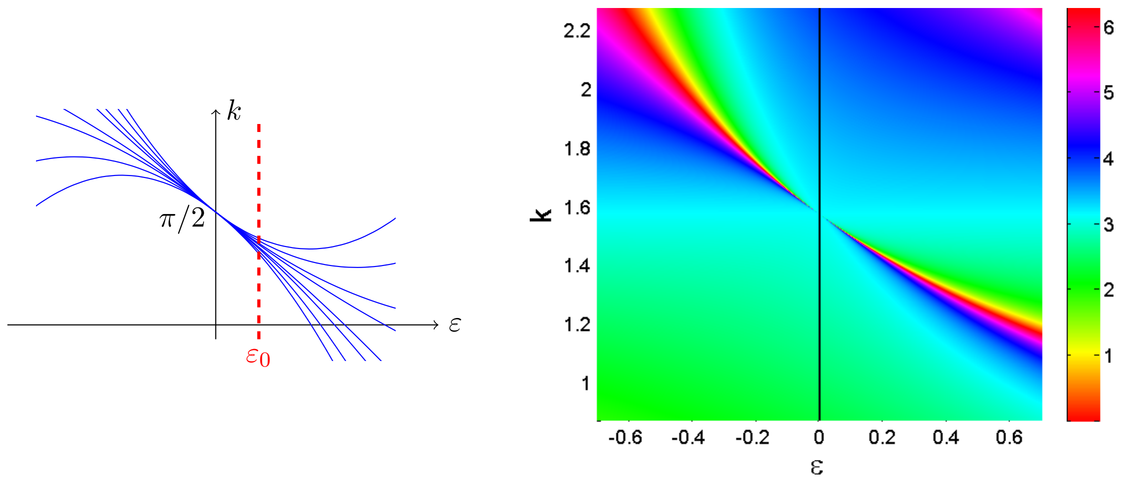


Figure III.5: Left: several parabolic paths $\{\gamma(s),\ s \in (0;1)\} \subset \mathbb{R}^2$ such that $\lim_{s\to 1} \gamma(s) = (0, \pi/2)$. According to the path, the limit of the coefficient $\mathcal{R}(\gamma(s))$ defined in (III.5) as $s \to 1$ is different. Right: the colours indicate the phase of $\mathcal{R}(\varepsilon, k)$. This phase is valued in $[0; 2\pi)$.

1.2 Fano resonance in the 2D waveguide

Let us come back to the problem (III.1) in 2D. We assume that Ω^0 is such that the Neumann Laplacian has a simple eigenvalue $\lambda_0 \in (0;\pi)$ (geometric multiplicity equal to one). We perturb the geometry from some smooth compactly supported profile function H with amplitude $\varepsilon \geq 0$ as in Figure III.1 right. We denote by Ω^ε the new waveguide and $\mathbb{S}(\varepsilon,\lambda)$, $T(\varepsilon,\lambda)$, $R_\pm(\varepsilon,\lambda)$ the scattering matrix/coefficients in the geometry Ω^ε at frequency λ. For short, we set $\mathbb{S}^0 = \mathbb{S}(0,\lambda^0)$, $T^0 = T(0,\lambda^0)$, $R^0_\pm = R_\pm(0,\lambda^0)$. Introduced u_0 an eigenfunction associated with λ^0 such that $\|u_0\|_{\mathrm{L}^2(\Omega)} = 1$. Decomposition in Fourier series as in Chapter I guarantees that as $|x| \to +\infty$, we have the expansion

$$u_0(x,y) = K_\pm e^{-\sqrt{\pi^2-\lambda^0}|x|}\cos(\pi y) + \dots$$

where $K_\pm \in \mathbb{C}$. In [18], the following theorem is proved.

Theorem III.1. *Assume that $(K_+,K_-) \neq (0,0)$. There is a quantity $\ell(H) \in \mathbb{R}$, which depends linearly on H, such that when $\varepsilon \to 0$,*

$$\mathbb{S}(\varepsilon,\lambda^0+\varepsilon\lambda') = \mathbb{S}^0 + O(\varepsilon) \qquad \text{for } \lambda' \neq \ell(H), \tag{III.8}$$

and, for any $\mu \in \mathbb{R}$,

$$\mathbb{S}(\varepsilon,\lambda^0+\varepsilon\ell(H)+\varepsilon^2\mu) = \mathbb{S}^0 + \frac{\tau^\top\tau}{i\tilde{\mu}-|\tau|^2/2} + O(\varepsilon). \tag{III.9}$$

In this expression $\tau = (a,b) \in \mathbb{C}\times\mathbb{C}$ depends only on Ω and $\tilde{\mu} = A\mu + B$ for some unessential real constants A, B with $A \neq 0$.

Observe that (III.8) and (III.9) are respectively the analogous of (III.6) and (III.7). As explained above, Theorem III.1 shows that the mapping $\mathbb{S}(\cdot,\cdot)$ is not continuous at $(0,\lambda^0)$ (setting where trapped modes exist). Moreover for ε_0 small fixed, it proves that the scattering matrix $\lambda \mapsto \mathbb{S}(\varepsilon_0,\lambda)$ exhibits a quick change in a neighbourhood of $\lambda^0+\varepsilon_0\ell(H)$: this is the Fano resonance phenomenon. When $(K_+,K_-) = (0,0)$ a faster Fano resonance phenomenon occurs.

1.3 Zero reflection and zero transmission

In the sequel, to simplify we denote by $\mathbb{S}^\varepsilon(\mu)$, $T^\varepsilon(\mu)$, $R^\varepsilon_\pm(\mu)$ the values of $\mathbb{S}$, T, $R_\pm$ in Ω^ε at the frequency $\lambda = \lambda^0 + \varepsilon\ell(H) + \varepsilon^2\mu$.

Assume now that Ω^ε is symmetric with respect to the vertical axis, *i.e.* such that

$$\Omega^\varepsilon = \{(-x,y)\,|\,(x,y) \in \Omega^\varepsilon\}.$$

Then we can decompose the problem into two half-waveguide problems with Neumann/Dirichlet boundary conditions at $x = 0$.

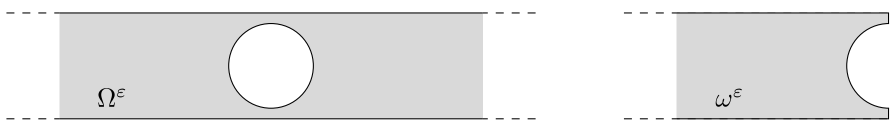


Figure III.6: Domain Ω^ε (left) and ω^ε (right).

More precisely, define the half-waveguide

$$\omega^\varepsilon := \{(x,y) \in \Omega^\varepsilon\,|\,x < 0\}$$

(see Figure III.6 right). Introduce the problem with Neumann BCs

$$\left|\begin{array}{rcll} \Delta v + k^2 v &=& 0 & \text{in } \omega^\varepsilon \\ \partial_\nu v &=& 0 & \text{on } \partial\omega^\varepsilon \end{array}\right. \tag{III.10}$$

as well as the problem with mixed BCs

$$\left|\begin{array}{rcll}\Delta V + k^2 V & = & 0 & \text{in } \omega^\varepsilon \\ \partial_\nu V & = & 0 & \text{on } \partial\omega^\varepsilon \cap \partial\Omega_L \\ V & = & 0 & \text{on } \Sigma_L := \partial\omega^\varepsilon \setminus \partial\Omega_L.\end{array}\right. \tag{III.11}$$

Problems (III.10) and (III.11) admit respectively the solutions

$$v^\varepsilon = w_+ + R_N^\varepsilon\, w_- + \tilde{v}^\varepsilon,$$

$$V^\varepsilon = w_+ + R_D^\varepsilon\, w_- + \tilde{V}^\varepsilon,$$

where R_N^ε, $R_D^\varepsilon \in \mathbb{C}$ and $\tilde{v}^\varepsilon$, $\tilde{V}^\varepsilon \in \mathrm{H}^1(\omega^\varepsilon)$. Due to conservation of energy, one has

$$|R_N^\varepsilon| = |R_D^\varepsilon| = 1$$

(since there is only one output in ω^ε, all the energy propagated by the incident wave is backscattered). Now, direct inspection shows that if u^ε is a solution of Problem (III.1) associated to an incident wave coming from left or right, then we have

$$u^\varepsilon(x,y) = \frac{v^\varepsilon(x,y) + V^\varepsilon(x,y)}{2} \quad \text{in } \omega^\varepsilon, \qquad u^\varepsilon(x,y) = \frac{v^\varepsilon(-x,y) - V^\varepsilon(-x,y)}{2} \quad \text{in } \Omega^\varepsilon \setminus \overline{\omega^\varepsilon}$$

(up possibly to a term which is exponentially decaying at $\pm\infty$ if there are trapped modes at the given wavenumber k). We deduce that the scattering coefficients $R_\pm^\varepsilon$, T^ε for Problem (III.1) are such that

$$R_+^\varepsilon = R_-^\varepsilon = \frac{R_N^\varepsilon + R_D^\varepsilon}{2} \qquad \text{and} \qquad T^\varepsilon = \frac{R_N^\varepsilon - R_D^\varepsilon}{2}.$$

If we indicate the dependence with respect to μ as in §1.2, this writes

$$R_+^\varepsilon(\mu) = R_-^\varepsilon(\mu) = \frac{R_N^\varepsilon(\mu) + R_D^\varepsilon(\mu)}{2} \qquad \text{and} \qquad T^\varepsilon(\mu) = \frac{R_N^\varepsilon(\mu) - R_D^\varepsilon(\mu)}{2}\,. \tag{III.12}$$

To set ideas assume that the trapped modes associated with λ^0 are even. In that case, $(\varepsilon,\lambda) \mapsto R_D(\varepsilon,\lambda)$ is smooth at $(0,\lambda_0)$. As a consequence, for ε small, $\mu \mapsto R_D^\varepsilon(\mu)$ does not vary much on the unit circle $\mathscr{C}(0,1)$ for $\mu \in (-\varepsilon^{-1/2};\varepsilon^{-1/2})$. On the other hand, by adapting the result of Theorem III.1, one establishes that $\mu \mapsto R_D^\varepsilon(\mu)$ runs once on $\mathscr{C}(0,1)$ for $\mu \in (-\varepsilon^{-1/2};\varepsilon^{-1/2})$. From formulas (III.12), this ensures that the curves $\mu \mapsto T^\varepsilon(\mu)$, $\mu \mapsto R_+^\varepsilon(\mu)$ for $\mu \in (-\varepsilon^{-1/2};\varepsilon^{-1/2})$, pass exactly through zero for ε small enough. This provides examples of geometries where we have either zero reflection or zero transmission.

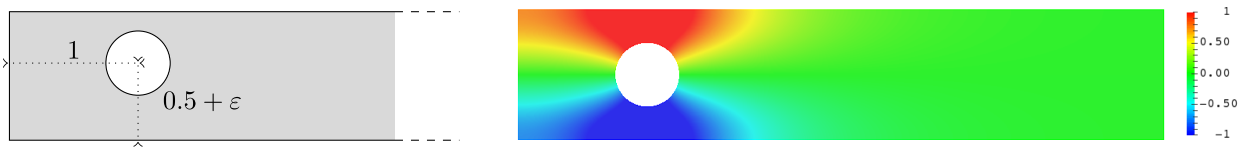


Figure III.7: Left: geometry of the half waveguide. Right: real part of a trapped mode for $\varepsilon = 0$ and $k^0 := \sqrt{\lambda^0} \approx 2.7403$. The computation has been realized by using Perfectly Matched Layers (see Chapter Chapter IV Section 2).

Let us illustrate this numerically. In Figure III.7 right, we give an example of geometry supporting trapped modes for the Neumann problem (III.10) at a particular $\lambda = \lambda_0 \in (0;\pi)$. Note that the waveguide is symmetric with respect to the line of equation $y = 1/2$ which can be used to give some proofs of existence of such trapped modes in certain circumstances. Then we perturb the domain by slightly shifting vertically the disk[1]. Then the symmetry is broken and from the analysis above, we know that when we sweep in λ in a small neighbourhood of λ_0, there is one λ for which one

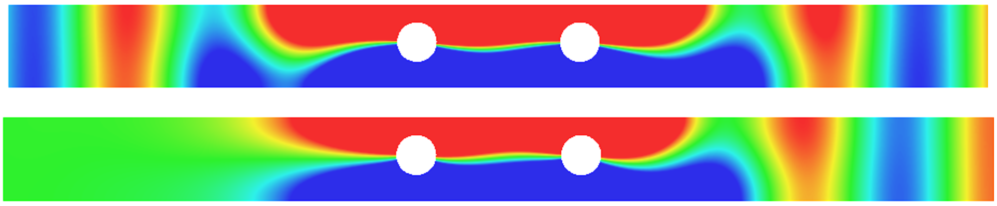


Figure III.8: $\Re e\, u^\varepsilon$ (top) and $\Re e\,(u^\varepsilon - w_+)$ in a setting where $R_\pm^\varepsilon = 0$ ($\varepsilon = 0.05$ and $k = \sqrt{\lambda} = 2.751$).

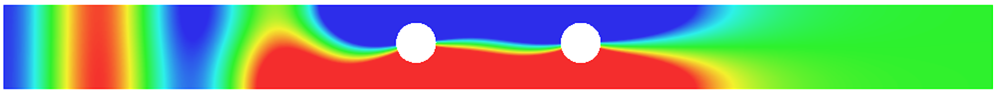


Figure III.9: $\Re e\, u^\varepsilon$ in a setting where $T^\varepsilon = 0$ ($\varepsilon = 0.05$ and $k = \sqrt{\lambda} = 2.75495$).

has zero reflection (see Figure III.8) and one λ for which one gets zero transmission (see Figure III.9). Let us stress that the smaller ε, the more delicate the adjustment of λ.
When the domain Ω^ε is not symmetric with respect to the vertical axis, we cannot use the decomposition with the two half-waveguide problems. In that case, for a fixed small $\varepsilon > 0$, in general the curves $\mu \mapsto R_\pm^\varepsilon(\mu)$ do not pass through zero in the complex plane and we do not observe zero reflection. However we can show, quite surprisingly, that $\mu \mapsto T^\varepsilon(\mu)$ always vanishes for some particular μ. Let us give the main ingredients of the proof.

Theorem III.2. *Assume that $T^0 = T(0, \lambda^0) \neq 0$. Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0; \varepsilon_0]$, there is $\mu \in \mathbb{R}$ such that $T^\varepsilon(\mu) = 0$.*

Proof. Theorem III.1 provides the estimate

$$|T^\varepsilon(\mu) - T^{\mathrm{asy}}(\mu)| \leq C\,\varepsilon \tag{III.13}$$

$$\text{with} \qquad T^{\mathrm{asy}}(\mu) = T^0 + \frac{ab}{i\tilde{\mu} - (|a|^2 + |b|^2)/2}.$$

For any compact set $I \subset \mathbb{R}$, the constant $C > 0$ in (III.13) can be chosen independent of $\mu \in I$.
$\star$ First, we study the set $\{T^{\mathrm{asy}}(\mu),\ \mu \in \mathbb{R}\}$. Classical results concerning the Möbius transform guarantee that $\{T^{\mathrm{asy}}(\mu),\ \mu \in \mathbb{R}\}$ coincides with $\mathscr{C}^{\mathrm{asy}} \setminus \{T^0\}$ where $\mathscr{C}^{\mathrm{asy}}$ is a circle passing through T^0. Let us show that $\mathscr{C}^{\mathrm{asy}}$ also passes through zero. One finds that $T^{\mathrm{asy}}(\mu) = 0$ for some $\mu \in \mathbb{R}$ if and only if there holds

$$\frac{|a|^2 + |b|^2}{2} = \Re e\left(\frac{ab}{T^0}\right). \tag{III.14}$$

An intermediate calculus of [18] implies $R_+^0\,\overline{a} + T^0\,\overline{b} = a$ and $T^0\,\overline{a} + R_-^0\,\overline{b} = b$. From this and the unitarity of $\mathbb{S}^0$ which imposes $R_-^0 = -\overline{R_+^0}\,T^0/\overline{T^0}$, we can obtain (III.14). Denote $\mu_\star$ the value of μ such that $T^{\mathrm{asy}}(\mu_\star) = 0$ and for $\varepsilon > 0$, define the interval $I^\varepsilon = (\mu_\star - \sqrt{\varepsilon}; \mu_\star + \sqrt{\varepsilon})$. From (III.13), for $\varepsilon > 0$ small, we know that the curve $\{T^\varepsilon(\mu),\ \mu \in I^\varepsilon\}$ passes close to zero. Now, using the unitary structure of $\mathbb{S}^\varepsilon(\mu)$, we show that this curves passes exactly through zero for ε small.
$\star$ Assume by contradiction that for all $\varepsilon > 0$, $\mu \mapsto T^\varepsilon(\mu)$ does not pass through zero in I_ε. Since $\mathbb{S}^\varepsilon(\mu)$ is unitary, there holds $R_+^\varepsilon(\mu)\,\overline{T^\varepsilon(\mu)} + T^\varepsilon(\mu)\,\overline{R_-^\varepsilon(\mu)} = 0$ and so

$$-R_+^\varepsilon(\mu)/\overline{R_-^\varepsilon(\mu)} = T^\varepsilon(\mu)/\overline{T^\varepsilon(\mu)} \qquad \forall \mu \in I^\varepsilon.$$

[1]Observe that a horizontal shift of the position of the disk would maintain the decoupling between symmetric and skew-symmetric modes. As a consequence, the eigenvalue embedded in the continuous spectrum would remain an eigenvalue embedded in the continuous spectrum and no Fano resonance phenomenon would be observed.

But if $\mu \mapsto T^\varepsilon(\mu)$ does not pass through zero on I^ε, one can verify that the point $T^\varepsilon(\mu)/\overline{T^\varepsilon(\mu)} = e^{2i\arg(T^\varepsilon(\mu))}$ must run rapidly on the unit circle for $\mu \in I_\varepsilon$ as $\varepsilon \to 0$. On the other hand, $R_+^\varepsilon(\mu)/\overline{R_-^\varepsilon(\mu)}$ tends to a constant on I_ε as $\varepsilon \to 0$. This way we obtain a contradiction. $\square$

Remark III.3. *The fact that $\mathscr{C}^{\mathrm{asy}}$ passes through zero is quite mysterious. It is related to the rigid structure of the scattering matrix. Without assumption of symmetry, we do not have physical reason to explain this miracle.*

We illustrate this result in Figure III.10. First we find that trapped modes exist for $\varepsilon = 0$ and $\sqrt{\lambda^0} \approx 1.2395 \in (0;\pi)$. Then we compute $T(\varepsilon,\lambda)$ ($\times$) and $R_+(\varepsilon,\lambda)$ ($\bullet$) for $\sqrt{\lambda} \in (1.2;1.3)$ and $\varepsilon = 0.05$. As predicted, we observe that $\lambda \mapsto T(\varepsilon,\lambda)$ passes through zero around λ^0. Finally, we display the real part of u_+ in Ω^ε for $\varepsilon = 0.05$ and $\sqrt{\lambda} = 1.2449$, a configuration where $T(\varepsilon,\lambda) \approx 0$.

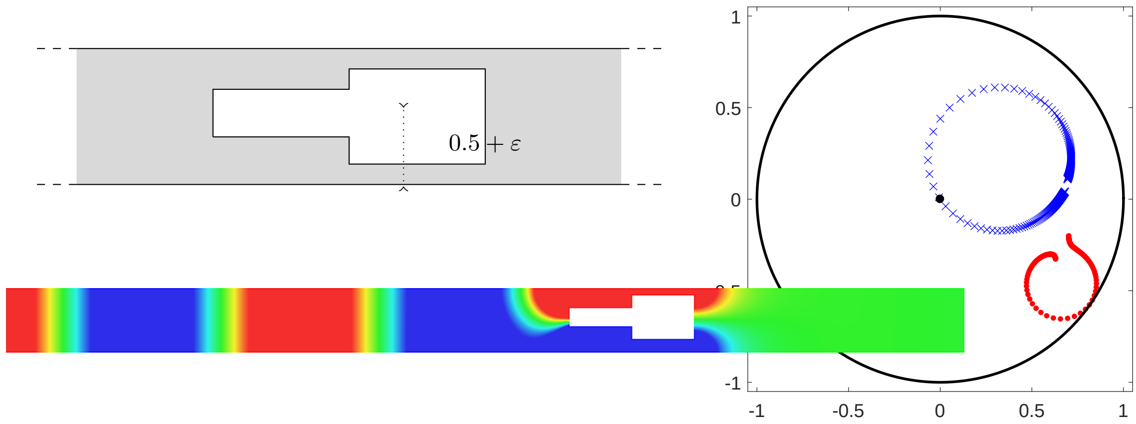


Figure III.10: Zero transmission in a waveguide via the Fano resonance mechanism.

In this study concerning the exploitation of the Fano resonance mechanism to obtain zero reflection/zero transmission, we considered the case of Neumann BCs. Let us mention that Dirichlet BCs can be studied completely similarly.

2 Cloaking of a given obstacle by using thin resonant ligaments

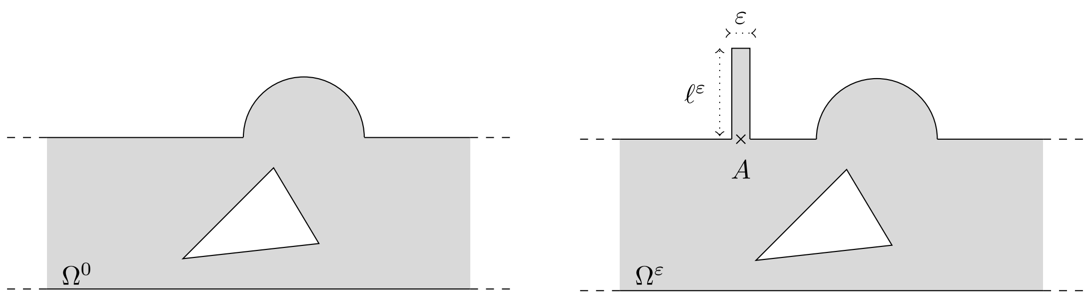


Figure III.11: Left: initial setting. Right: geometry with one thin outer resonator.

In this section, we change the point of view. Instead of constructing invisible objects, we assume that some obstacle is given and explain how to hide it (or to cloak if we use the terminology of physicists). More precisely, starting from a setting where $T \neq 1$, we show how to perturb the initial geometry by working with ligaments as pictured in Figure III.11 to obtain a new waveguide where $T \approx 1$. Let us mention that we do not work with ad hoc penetrable materials as in transformation optics. Additionally, we do not add active sources in the system as people do in active cloaking [32, 15]. What we realize is passive cloaking at infinity by perturbing the shape of the waveguide.

As in Chapter I, the main difficulty of the problem lies in the fact that the dependence of the scattering coefficients with respect to the geometry is not explicit and not linear. In order to address it, techniques of optimization have been considered. We refer the reader in particular to [3, 26, 27]. However, due to the features of the Helmholtz equation, the functionals involved in the analysis are non convex and unsatisfying local minima exist. Moreover, these methods do not allow the user to control the main features of the shape compare to the approach we present and which has been developed in [16].

To cloak obstacles, we work with thin outer resonators, that we also call ligaments, of width $\varepsilon > 0$ small compared to the wavelength (see again Figure III.11 right). These ligaments are interesting because they are almost 1D objects, which allows us to explicit their influence on the fields and so on the scattering coefficients. However in general, *i.e.* for most lengths, they produce only perturbations of order ε which is not sufficient to compensate for the scattering due to the initial object. But by working around the resonance lengths (see (III.16)) of the resonators, we can get effects of order one. This is a key aspect in our approach which makes it in particular different from the technique presented in Chapter II, Section 7. Note that thin ligaments around resonance lengths have been studied for example in [25, 14, 30, 29] in a context close to ours, namely in the analysis of the scattering of an incident wave by a periodic array of subwavelength slits. The core of our approach is based on an asymptotic expansion of the scattering solutions with respect to ε as ε tends to zero. This allow us to derive formula for the scattering coefficients with a relatively explicit dependence on the geometrical features. To obtain the expansions, we apply again techniques of matched asymptotic expansions. For related methods, we refer the reader to [6, 24].

Let us describe the general strategy. We consider the acoustic problem (I.52) with Neumann BCs and assume that $k \in (0;\pi)$ (the height of the guide is stille one outside of some compact region). Denote by u_+^ε the solution of (I.52) corresponding to the scattering of the rightgoing plane wave w_+ in Ω^ε. The first step is to compute an asymptotic expansion of u_+^ε as ε tends to zero. As usual in asymptotic analysis, we work with different ansatz for u_+^ε, depending on the region. More precisely, we consider the outer expansions

$$\left|\begin{array}{ll} u_+^\varepsilon(x,y) = u^0(x,y) + \dots & \text{in } \Omega^0 \\ u_+^\varepsilon(x,y) = \varepsilon^{-1}v^{-1}(y) + v^0(y) + \dots & \text{in the resonator} \end{array}\right. \tag{III.15}$$

where Ω^0 denotes the initial waveguide without the ligament and u^0, v^{-1}, v^0 denote unknown functions which are independent of ε. To begin with, suppose that the ligament has a length $\ell > 0$ independent of ε. Considering the restriction of Problem (I.3) in Ω^ε to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous 1D problem

$$(\mathscr{P}_{1\mathrm{D}}) \left|\begin{array}{ll} \partial_y^2 v + k^2 v = 0 & \text{in } (1;1+\ell) \\ v(1) = \partial_y v(1+\ell) = 0. & \end{array}\right.$$

An important message is that the features of $(\mathscr{P}_{1\mathrm{D}})$ play a key role in the physical phenomena and so in the asymptotic analysis. We denote by ℓ_{res} (resonance lengths), the values of ℓ, given by

$$\ell_{\mathrm{res}} := \pi(m+1/2)/k, \qquad m \in \mathbb{N}, \tag{III.16}$$

such that $(\mathscr{P}_{1\mathrm{D}})$ admits a non zero solution. Note that the non zero functions solving $(\mathscr{P}_{1\mathrm{D}})$ coincide, up to a multiplicative constant, with $\sin(k(y-1))$.

Assume first that $\ell \neq \ell_{\mathrm{res}}$. Then we find $v^{-1} \equiv 0$ in (III.15) and when $\varepsilon \to 0$, we can show that

$$\left|\begin{array}{ll} u_\pm^\varepsilon(x,y) = u_\pm + o(1) & \text{in } \Omega^0 \\ u_\pm^\varepsilon(x,y) = u_\pm(A)\, v_0(y) + o(1) & \text{in the resonator} \end{array}\right. \tag{III.17}$$

where $u_\pm$ stands for the scattering solution corresponding to an incident plane wave coming from $\mp\infty$ and $A = (x_A, 1)$ is the attachment point of the ligament. Moreover in (III.17),

$$v_0(y) = \cos(k(y-1)) + \tan(k(y-\ell))\sin(k(y-1))$$

(observe that we have $\partial_y^2 v + k^2 v = 0$ in $(1; 1+\ell)$ as well as $v_0(1) = 1$ and $\partial_x v_0(1+\ell) = 0$). From this, we deduce that

$$R_\pm^\varepsilon = R_\pm + o(1), \qquad\qquad T^\varepsilon = T + o(1),$$

where $R_\pm$, T are the scattering coefficients in the geometry without the ligament. In that situation, we see that the resonator has no influence at order ε^0, which is not interesting for our purpose.

Assume now that $\ell = \ell_{\text{res}}$. In that case, the asymptotic analysis is more involved. At the end of the (long) procedure, see [16] for the details, we obtain, when $\varepsilon \to 0$,

$$\left|\begin{array}{ll} u_+^\varepsilon(x,y) = u_+(x,y) + ak\gamma(x,y) + o(1) & \text{in } \Omega^0 \\ u_+^\varepsilon(x,y) = \varepsilon^{-1} a \sin(k(y-1)) + O(1) & \text{in the resonator} \end{array}\right. \tag{III.18}$$

where γ denotes the outgoing Green function such that

$$\left|\begin{array}{rcll} \Delta\gamma + k^2\gamma &=& 0 & \text{in } \Omega \\ \partial_\nu \gamma &=& \delta_A & \text{on } \partial\Omega. \end{array}\right.$$

Moreover in (III.18), we find that a is given by

$$ak = -\frac{u_+(A)}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi}$$

where Γ and C_Ξ are some constants which depend only on the initial geometry Ω^0. Observe that in (III.18), the field blows up as $O(\varepsilon^{-1})$ in the resonator, which is directly related to the fact that there is a complex resonance close to the considered real k. From (III.18), we obtain

$$R_+^\varepsilon = R_+ + iau_+(A)/2 + o(1), \qquad T^\varepsilon = T + iau_-(A)/2 + o(1).$$

This time the thin resonator has an influence at order ε^0. Let us make a small variant by assuming that the length of the ligament is equal to $\ell^\varepsilon = \ell_{\text{res}} + \varepsilon\eta$, where $\eta \in \mathbb{R}$ is a parameter that we set as we wish. In that situation the analysis is very similar to the previous case and when $\varepsilon \to 0$, we find

$$\left|\begin{array}{ll} u_+^\varepsilon(x,y) = u_+(x,y) + a(\eta)k\gamma(x,y) + o(1) & \text{in } \Omega^0 \\ u_+^\varepsilon(x,y) = \varepsilon^{-1} a(\eta) \sin(k(y-1)) + O(1) & \text{in the resonator} \end{array}\right. \tag{III.19}$$

with

$$a(\eta)k = -\frac{u_+(A)}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi + \eta}.$$

This gives

$$R_+^\varepsilon = R_+^{\text{asy}}(\eta) + o(1), \qquad T^\varepsilon = T^{\text{asy}}(\eta) + o(1) \tag{III.20}$$

with

$$R_+^{\text{asy}}(\eta) := R_+ + ia(\eta)u_+(A)/2, \qquad T^{\text{asy}}(\eta) := T + ia(\eta)u_-(A)/2 + o(1).$$

Thus, not only the resonator has an influence at order ε^0, but additionally the latter depends on the choice made for η. We get something similar to what has been shown in Section 1 (see in particular Figure III.5): for all $\eta \in \mathbb{R}$, the resonator tends to the 1D segment $(1; 1+\ell_{\text{res}})$, but depending on the choice of η, the limit of the corresponding scattering coefficients is not the same. As a consequence, the scattering coefficients, considered as functions of the two variables (ε, ℓ), are not continuous at the point $(0, \ell_{\text{res}})$. Moreover, for ε_0 fixed small, varying slightly ℓ around ℓ_{res}, which corresponds for example to sweep $\eta \in [-\varepsilon_0^{-1/2}; \varepsilon_0^{-1/2}]$, we obtain a large variation for $R_+^{\varepsilon_0}$, T^{ε_0} (again see the illustration of Figure III.5). More precisely, working with the Möbius transform, one shows that the sets

$$\{R_+^{\mathrm{asy}}(\eta),\, \eta \in \overline{\mathbb{R}}\}, \qquad \{T^{\mathrm{asy}}(\eta),\, \eta \in \overline{\mathbb{R}}\}$$

where $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, coincide with circles. We deduce that asymptotically, when $\varepsilon \to 0$, when perturbing the length of the ligament around ℓ_{res}, R_+^{ε}, T^{ε} run on circles. Interestingly for our purpose, the features of these circles depend on A, the attachment point of the ligament.

In view of achieving zero reflection, by using the expansions of $u_\pm(A)$ far from the obstacle, the following statement is established in [16]:

Proposition III.4. *Assume that $R_+ \neq 0$ and $T \neq 0$. There are some $A = (x_A, 1)$ such that there exists η such that $R_+^{\mathrm{asy}}(\eta) = 0$. In that case, when $\varepsilon \to 0$, we have $R_+^{\varepsilon} = 0 + o(1)$.*

Remark III.5. *Note that we exclude the case $R_+ = 0$ because in this situation we already have zero reflection in the initial geometry and there is no need for adding a resonator. In the case $T = 0$, i.e. $|R_+| = 1$ due to conservation of energy, the most challenging situation, our approach does not work. However, one possibility to get zero reflection is to add first one or several resonators to obtain a transmission coefficient quite different from zero. And then to add another well-tuned resonator to kill the reflection. Let us mention that this strategy is also interesting when T is small but non zero because in this case achieving almost zero reflection with only one resonator is quite unstable.*

Remark III.6. *It is important to emphasize that compared to what we presented in the previous sections, we do not reach exactly $R_+^{\varepsilon} = 0$ but simply get $R_+^{\mathrm{asy}}(\eta) = 0$. In other words, there is a residue due to the error in the expansion. This analysis encourages us to take ε as small as possible. However when ε becomes small, the amplitude of the field in the resonator gets very high and the tuning procedure of the features of the ligament is very sensitive. Thus one must find a compromise between small reflection and robustness with respect to perturbations of the geometry.*

In Figures III.12–III.13, we consider the scattering of the rightgoing plane wave by a sound hard obstacle (the fish). We have added a well-tuned ligament to obtain almost zero reflection. In Figure III.13, we observe that by working with a smaller ε, as expected we can reduce the reflection.

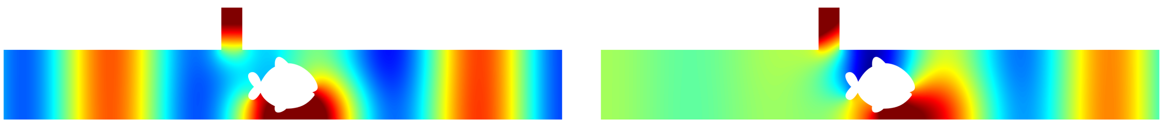


Figure III.12: Real parts of u_+^{ε} (left) and of $u_+^{\varepsilon} - w_+$ (right). The length of the resonator is tuned to get almost zero reflection. Here $\varepsilon = 0.3$.

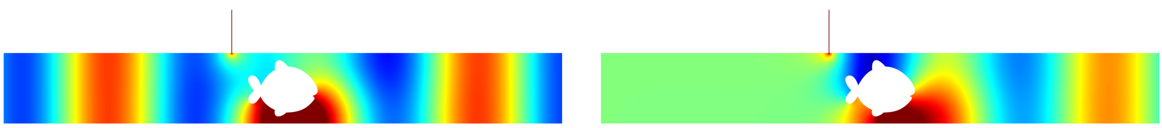


Figure III.13: Same quantities as in Figure III.12 but with a thinner resonator (here $\varepsilon = 0.01$).

Once almost zero reflection has been obtained, it remains to compensate for the phase shift. To proceed, one method consists in coupling the previous waveguide with what we call a phase-shifter. This is a device where one has zero reflection and any prescribed phase. We have shown that such phase shifters can be designed by working with two well-tuned resonant ligaments added to the reference strip $\mathbb{R} \times (0; 1)$. At the end, we obtain $T^{\varepsilon} \approx 1$ with three well-tuned resonant ligaments.

On the other hand, by exploiting again the results of the asymptotic analysis (III.19), (III.20), we have established that we can get $T^\varepsilon \approx 1$ with only two well-tuned ligaments. Let us assess the degrees of freedom which are involved. We wish to control two complex coefficients, R^ε, T^ε, and so four real parameters. The relation of conservation of energy imposes one constraint. As a consequence, there are three real degrees of freedom. In our strategy here, we play with the two lengths of the resonators and with the distance between them.

In Figures III.14, III.15, we cloak two different obstacles/defects by working with two resonant ligaments. In each case, on the first line we display the field corresponding to the scattering of a rightgoing plane wave without the resonators. The setting of Figure III.15 is particularly challenging because the initial transmission coefficient is very small. To restore a good transmission, we observe that we have to excite strongly the resonances. Practically, this is probably a limitation because we can imagine that dissipation will then become important.

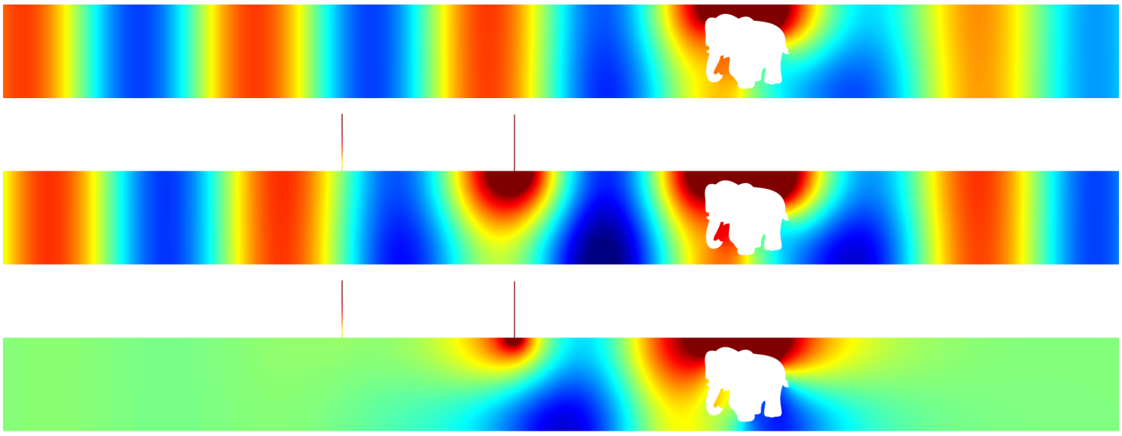


Figure III.14: Real parts of u_+ (top), u_+^ε (middle) and $u_+^\varepsilon - w_+$ (bottom). The resonators are tuned to get $T^\varepsilon \approx 1$. Here $\varepsilon = 0.01$.

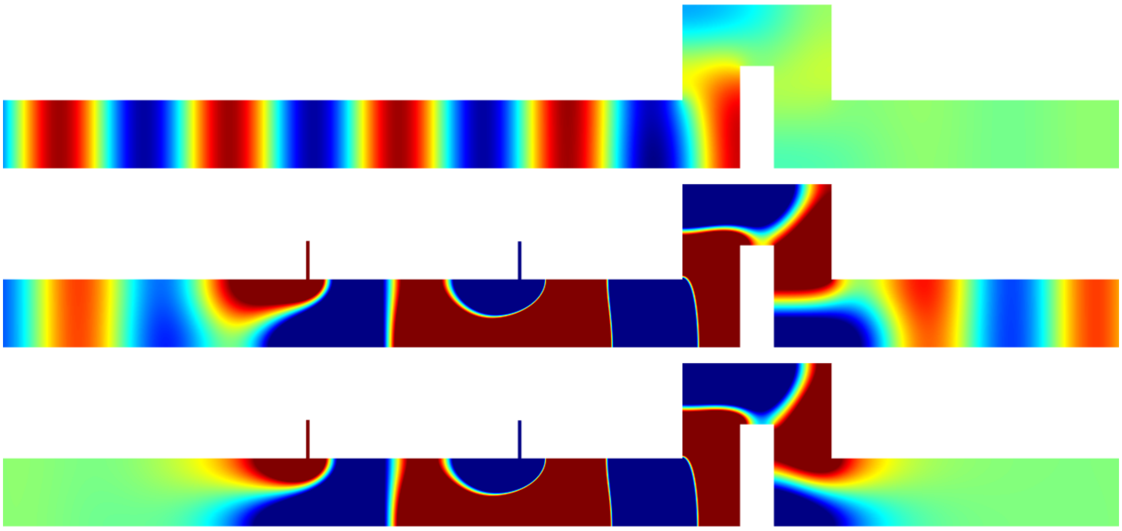


Figure III.15: Real parts of u_+ (top), u_+^ε (middle) and $u_+^\varepsilon - w_+$ (bottom). The resonators are tuned to get $T^\varepsilon \approx 1$. Here $\varepsilon = 0.05$.

Chapter IV

A spectral problem characterizing zero reflection

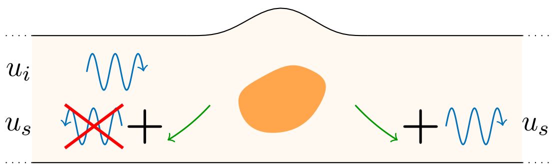


Figure IV.1: Schematic picture of a reflectionless mode. The propagating wave (blue) is not reflected. The backscattered field is purely evanescent (green).

Let us adopt another point of view concerning questions of invisibility. Instead of considering the wavenumber k fixed in the problem and try to find geometries where we have zero reflection or perfect invisibility, we assume that the geometry is fixed and we look for k such that there is an incident field whose energy is completely transmitted through the waveguide. To proceed, we present the results of [12] (see also the related works ([21, 39]) where it is shown that such reflectionless modes can be characterized as eigenfunctions of an original non-selfadjoint spectral problem. The approach is based on the following basic observation: if for an incident wave, the backscattered field is evanescent, then the total field is ingoing in the input lead and outgoing in the output lead. To select ingoing waves on one side of the obstacle and outgoing waves on the other side, we use complex scalings [2, 5] (or Perfectly Matched Layers [9]) with imaginary parts of different signs. We prove that the real eigenvalues of the obtained spectrum correspond either to trapped modes (also called Bound States in the Continuum, BSCs or BICs, in quantum mechanics) or to reflectionless modes. Interestingly, complex eigenvalues also contain useful information on weak reflection cases. When the geometry has certain symmetries, the new spectral problem enters the class of $\mathcal{PT}$-symmetric problems. Let us describe this in more details.

1 Setting

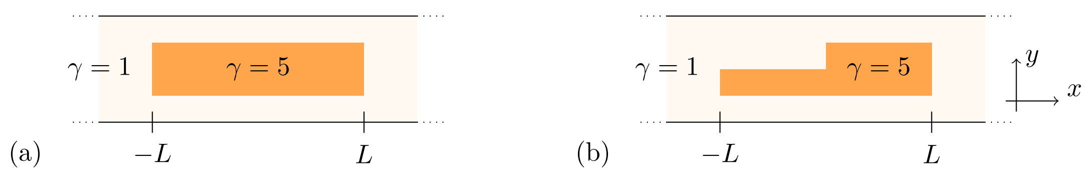


Figure IV.2: Symmetric (a) and non-symmetric (b) obstacles considered in the numerics below.

To make the presentation as simple as possible, we study the scattering of waves in 2D by a penetrable obstacle. The waveguide coincides with the region $\Omega := \{(x, y) \in \mathbb{R}^2 \,|\, 0 < y < 1\}$ and we consider the problem

$$\left|\begin{array}{rcl}\Delta u+k^2\gamma u &=& 0 \quad \text{in } \Omega\\ \partial_y u &=& 0 \quad \text{on } \partial\Omega\end{array}\right. \tag{IV.1}$$

with Neumann BCs. The coefficient γ corresponds to the material index of the medium filling Ω. We assume that γ is a positive and bounded function such that $\gamma = 1$ for $|x| \geq L$ where $L > 0$ is given. In other words, the obstacle is located in the region $\Omega_L := \{(x, y) \in \Omega \,|\, |x| < L\}$ (see Figure IV.2). Pick $k \in (N\pi; (N + 1)\pi)$, with $N \in \mathbb{N} := \{0, 1, \dots\}$. Let us change a bit the definition of the modes (I.53) by modifying the normalization and set

$$w_n^{\pm}(x,y)=\frac{e^{\pm i\beta_n x}\varphi_n(y)}{(2|\beta_n|)^{1/2}},\quad \beta_n:=\sqrt{k^2-n^2\pi^2},\quad \varphi_n(y)=\left|\begin{array}{ll}1 & \text{if } n=0\\ \sqrt{2}\cos(n\pi y) & \text{for } n>0.\end{array}\right. \tag{IV.2}$$

For $n = 0, \dots, N$, the wave $w_n^{\pm}$ propagates along the (Ox) axis from $\mp\infty$ to $\pm\infty$. On the other hand, for $n > N$, $w_n^{\pm}$ is exponentially growing at $\mp\infty$ and exponentially decaying at $\pm\infty$. For $n = 0, \dots, N$, we consider the scattering of the wave w_n^+ by the obstacle located in Ω. By adapting Proposition I.14, one shows that Problem (IV.1) admits a solution $u_n = w_n^+ + u_n^s$ with the outgoing scattered field u_n^s written as

$$u_n^s=\sum_{p=0}^{+\infty}s_{np}^{\pm}w_p^{\pm}\qquad \text{for } \pm x\geq L \tag{IV.3}$$

with $(s_{np}^{\pm}) \in \mathbb{C}^{\mathbb{N}}$. The solution u_n is uniquely defined if and only if Trapped Modes (TMs) do not exist at the wavenumber k. We remind the reader that trapped modes are non zero functions $u \in \mathrm{L}^2(\Omega)$ satisfying (IV.1). We denote by $\mathscr{K}_{\mathrm{t}}$ the set of k^2 such that TMs exist at the wavenumber k. On the other hand, as already mentioned after (I.48), the scattering coefficients $s_{np}^{\pm}$ in (IV.3) are always uniquely defined, including for $k^2 \in \mathscr{K}_{\mathrm{t}}$. In the following, we will be particularly interested in the features of the reflection matrix (whose size, determined by the number of propagative modes, depends on k)

$$R(k) := (s_{np}^-)_{0\leq n,p\leq N} \in \mathbb{C}^{N+1\times N+1}. \tag{IV.4}$$

Definition IV.1. *We say that the wavenumber $k \in (0; +\infty) \setminus \pi\mathbb{N}$ is reflectionless if* $\ker R(k) \neq \{0\}$.

Let us explain this definition. In general, by linearity, for an incident field (coming from the left)

$$u_i=\sum_{n=0}^{N}a_n w_n^+,\qquad (a_n)_{n=0}^N\in\mathbb{C}^{N+1}, \tag{IV.5}$$

Problem (IV.1) admits a solution u such that $u = u_i + u_s$ with

$$u_s=\sum_{p=0}^{+\infty}b_p^{\pm}w_p^{\pm} \text{ for } \pm x\geq L \qquad \text{and} \qquad b_p^{\pm}=\sum_{n=0}^{N}a_n s_{np}^{\pm}\in\mathbb{C}. \tag{IV.6}$$

The above definition says that, if k is reflectionless, then there are $(a_n)_{n=0}^N \in \mathbb{C}^{N+1} \setminus \{0\}$ such that the b_p^- in (IV.6) satisfy $b_p^- = 0$, $p = 0, \dots, N$. In other words, the scattered field is exponentially decaying for $x \leq -L$. Finally notice that the corresponding total field $u = u_i + u_s$ decomposes as

$$\begin{array}{ll} u=\displaystyle\sum_{n=0}^{N}a_n w_n^+ +\tilde{u} & \text{for } x\leq -L\\ u=\displaystyle\sum_{n=0}^{N}t_n w_n^+ +\tilde{u} & \text{for } x\geq L\end{array} \tag{IV.7}$$

where $t_n = a_n + b_n^+$ and where $\tilde{u}$ decays exponentially for $\pm x \geq L$. In other words, the total field is ingoing for $x \leq -L$ and outgoing for $x \geq L$.

In the following, we call Reflectionless Modes (RMs) the functions u satisfying (IV.1) and admitting expansion (IV.7). We denote by $\mathscr{K}_{\mathrm{r}}$ the set of k^2 such that the wavenumber k is reflectionless. Our objective is to explain how to determine directly the set $\mathscr{K}_{\mathrm{r}}$ and the corresponding RMs by solving an eigenvalue problem, instead of computing the reflection matrix for all values of k.

2 Classical complex scaling

As a first step, we remind briefly how to use a complex scaling to compute trapped modes. Define the unbounded operator A of $\mathrm{L}^2(\Omega)$ such that

$$Au = -\frac{1}{\gamma}\Delta u$$

with Neumann boundary conditions $\partial_y u = 0$ on $y = 0$ and $y = 1$. It is known that A is a selfadjoint operator ($\mathrm{L}^2(\Omega)$ is endowed with the inner product $(\gamma\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$) whose spectrum $\sigma(A)$ coincides with $[0;+\infty)$. More precisely, we have $\sigma_{\mathrm{ess}}(A) = [0;+\infty)$ where $\sigma_{\mathrm{ess}}(A)$ denotes the essential spectrum of A. By definition, $\sigma_{\mathrm{ess}}(A)$ corresponds to the set of $\lambda \in \mathbb{C}$ for which there exists a so-called singular sequence $(u^{(m)})$, that is an orthonormal sequence $(u^{(m)}) \in \mathrm{L}^2(\Omega)^{\mathbb{N}}$ such that $((A-\lambda)u^{(m)})$ converges to 0 in $\mathrm{L}^2(\Omega)$. Besides, $\sigma(A)$ may contain eigenvalues (at most a sequence accumulating at $+\infty$) corresponding to TMs. In order to reveal these eigenvalues which are embedded in $\sigma_{\mathrm{ess}}(A)$, one can use a complex change of variables. For $0 < \theta < \pi/2$, set $\eta = e^{i\theta}$ and define the function $\mathcal{I}_\theta : \mathbb{R} \to \mathbb{C}$ such that

$$\mathcal{I}_\theta(x) = \left|\begin{array}{ll} -L + (x+L)\,\eta & \text{for } x \le -L \\ x & \text{for } |x| < L \\ +L + (x-L)\,\eta & \text{for } x \ge L. \end{array}\right. \tag{IV.8}$$

For the sake of simplicity, we will use abusively the same notation $\mathcal{I}_\theta$ for the following map: $\{\Omega \to \mathbb{C}\times(0;1),\ (x,y) \mapsto (\mathcal{I}_\theta(x), y)\}$. Note that with this definition, the left inverse $\mathcal{I}_\theta^{-1}$ of $\mathcal{I}_\theta$, acting from $\mathcal{I}_\theta(\Omega)$ to Ω, is equal to $\mathcal{I}_{-\theta}$. One can easily check that for all $n \ge 0$, $w_n^+ \circ \mathcal{I}_\theta$ is exponentially decaying for $x \ge L$, while $w_n^- \circ \mathcal{I}_\theta$ is exponentially decaying for $x \le -L$. As a consequence, defining from expansion (IV.6) the function $v_\theta = u_s \circ \mathcal{I}_\theta$, one has $v_\theta = u_s$ for $|x| < L$ and $v_\theta \in \mathrm{L}^2(\Omega)$ (which is in general not true for u_s). Moreover v_θ satisfies the following equation in Ω:

$$\alpha_\theta \frac{\partial}{\partial x}\Big(\alpha_\theta \frac{\partial v_\theta}{\partial x}\Big) + \frac{\partial^2 v_\theta}{\partial y^2} + k^2\gamma v_\theta = k^2(1-\gamma)u_i \tag{IV.9}$$

with $\alpha_\theta(x) = 1$ for $|x| < L$ and $\alpha_\theta(x) = \eta^{-1} = \overline{\eta}$ for $\pm x \ge L$. In particular, for a TM, v_θ solves (IV.9) with $u_i = 0$. This leads us to consider the unbounded operator A_θ of $\mathrm{L}^2(\Omega)$ such that

$$A_\theta v_\theta = -\frac{1}{\gamma}\left(\alpha_\theta \frac{\partial}{\partial x}\Big(\alpha_\theta \frac{\partial v_\theta}{\partial x}\Big) + \frac{\partial^2 v_\theta}{\partial y^2}\right) \tag{IV.10}$$

again with homogeneous Neumann boundary conditions. Since α_θ is complex valued, the operator A_θ is not selfadjoint. However, we use the same definition as above for $\sigma_{\mathrm{ess}}(A_\theta)$, which is licit for this operator. We recall below the main spectral properties of A_θ [38]:

Theorem IV.2. *i) There holds*

$$\sigma_{\mathrm{ess}}(A_\theta) = \bigcup_{n\in\mathbb{N},\, t\ge 0} \{n^2\pi^2 + te^{-2i\theta}\}. \tag{IV.11}$$

ii) The spectrum of A_θ satisfies $\sigma(A_\theta) \subset \mathscr{R}_\theta^-$ with

$$\mathscr{R}_\theta^- := \{z \in \mathbb{C}\,|\, -2\theta \le \arg(z) \le 0\}.$$

iii) $\sigma(A_\theta) \setminus \sigma_{\mathrm{ess}}(A_\theta)$ *is discrete and contains only eigenvalues of finite multiplicity.*

iv) Assume that $k^2 \in \sigma(A_\theta) \setminus \sigma_{\mathrm{ess}}(A_\theta)$*. Then* k^2 *is real if and only if* $k^2 \in \mathscr{K}_{\mathrm{t}}$*. Moreover if* v_θ *is an eigenfunction associated to* k^2 *such that* $\Im m\, k^2 < 0$*, then* $v_\theta \circ \mathrm{J}_{-\theta}$ *is a solution of the original problem (IV.1) whose amplitude is exponentially growing at* $+\infty$ *or at* $-\infty$*.*

The interesting point is that now TMs correspond to isolated eigenvalues of A_θ, and as such, they can be computed numerically as illustrated below. Note that the elements k^2 of $\sigma(A_\theta) \setminus \sigma_{\mathrm{ess}}(A_\theta)$ such that $\Im m\, k^2 < 0$, if they exist, correspond to complex resonances (quasi normal modes). Let us point out that the complex scaling is just a technique to reveal them. Indeed complex resonances are intrinsic objects defined as the poles of the meromorphic extension from $\{z \in \mathbb{C} \,|\, \Im m\, z > 0\}$ to $\{z \in \mathbb{C} \,|\, \Im m\, z \leq 0\}$ of the operator valued map $z \mapsto (\Delta + z\gamma)^{-1}$. For more details, we refer the reader to [4].

3 Conjugated complex scaling

Now, we show that replacing the classical complex scaling by an unusual conjugated complex scaling, and proceeding as in the previous section, we can define a new complex spectrum which contains the reflectionless values $k^2 \in \mathscr{K}_{\mathrm{r}}$ we are interested in. We define the map $\mathcal{J}_\theta : \Omega \to \mathbb{C}\times(0;1)$ using the following complex change of variables

$$\mathcal{J}_\theta(x) = \left|\begin{array}{ll} -L + (x+L)\,\overline{\eta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x-L)\,\eta & \text{for } x \geq L, \end{array}\right. \tag{IV.12}$$

with again $\eta = e^{i\theta}$ $(0 < \theta < \pi/2)$. Note the important difference in the definitions of $\mathcal{J}_\theta$ and J_θ for $x \leq -L$: η has been replaced by the conjugated parameter $\overline{\eta}$ to select the ingoing modes instead of the outgoing ones in accordance with (IV.7). Now, if u is a RM associated to $k^2 \in \mathscr{K}_{\mathrm{r}}$, setting $w_\theta = u \circ \mathcal{J}_\theta$, one has $w_\theta = u$ for $|x| < L$ and $w_\theta \in \mathrm{L}^2(\Omega)$ (which is not the case for u). The function w_θ satisfies the following equation in Ω:

$$\beta_\theta \frac{\partial}{\partial x}\Big(\beta_\theta \frac{\partial w_\theta}{\partial x}\Big) + \frac{\partial^2 w_\theta}{\partial y^2} + k^2 \gamma w_\theta = 0 \tag{IV.13}$$

with $\beta_\theta(x) = 1$ for $|x| < L$, $\beta_\theta(x) = \eta$ for $x \leq -L$ and $\beta_\theta(x) = \overline{\eta}$ for $x \geq L$. This leads us to define the unbounded operator B_θ of $\mathrm{L}^2(\Omega)$ such that

$$B_\theta w_\theta = -\frac{1}{\gamma}\left(\beta_\theta \frac{\partial}{\partial x}\Big(\beta_\theta \frac{\partial w_\theta}{\partial x}\Big) + \frac{\partial^2 w_\theta}{\partial y^2}\right) \tag{IV.14}$$

with homogeneous Neumann boundary conditions. As A_θ, the operator B_θ is not selfadjoint. Its spectral properties are summarized in the following theorem.

Theorem IV.3. *i) There holds*

$$\sigma_{\mathrm{ess}}(B_\theta) = \bigcup_{n\in\mathbb{N},\, t\geq 0} \{n^2\pi^2 + te^{-2i\theta},\, n^2\pi^2 + te^{+2i\theta}\}. \tag{IV.15}$$

ii) The spectrum of B_θ *satisfies* $\sigma(B_\theta) \subset \mathscr{R}_\theta$ *with*

$$\mathscr{R}_\theta := \{z \in \mathbb{C} \,|\, -2\theta \leq \arg(z) \leq 2\theta\}. \tag{IV.16}$$

iii) Assume that $k^2 \in \sigma(B_\theta)\setminus\sigma_{\mathrm{ess}}(B_\theta)$*. Then* k^2 *is real if and only if* $k^2 \in \mathscr{K}_{\mathrm{t}} \cup \mathscr{K}_{\mathrm{r}}$*. Moreover if* w_θ *is an eigenfunction associated to* k^2 *such that* $\pm\Im m\, k^2 < 0$*, then* $w_\theta \circ \mathcal{J}_{-\theta}$ *is a solution of (IV.1) whose amplitude is exponentially growing at* $\pm\infty$ *and exponentially decaying at* $\mp\infty$*.*

The important result is that isolated real eigenvalues of B_θ correspond precisely to TMs and RMs. The following proposition provides a criterion to determine whether an eigenfunction associated to a real eigenvalue of B_θ is a TM or a RM.

Proposition IV.4. *Assume that $(k^2, w_\theta) \in \mathbb{R} \times \mathrm{L}^2(\Omega)$ is an eigenpair of B_θ such that $k \in (N\pi; (N+1)\pi)$, $N \in \mathbb{N}$. Set*

$$\rho(w_\theta) = \sum_{n=0}^{N} \Big| \int_0^1 w_\theta(-L, y)\varphi_n(y)\, dy \Big|^2 \tag{IV.17}$$

where φ_n is defined in (IV.2). If $\rho(w_\theta) = 0$ then $w_\theta \circ \mathcal{J}_{-\theta}$ is a TM ($k^2 \in \mathscr{K}_{\mathrm{t}}$). If $\rho(w_\theta) > 0$ then $w_\theta \circ \mathcal{J}_{-\theta}$ is a RM ($k^2 \in \mathscr{K}_{\mathrm{r}}$). In this case, the incident field u_i defined in (IV.5) with

$$a_n = \int_0^1 w_\theta(-L, y)\varphi_n(y)\, dy, \quad n = 0, \dots, N,$$

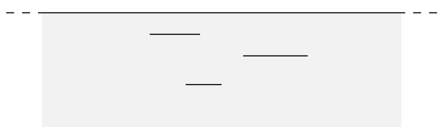
yields a scattered field which decays exponentially for $x \le -L$.

The next proposition tells that B_θ satisfies the celebrated $\mathcal{PT}$ symmetry property when the obstacle is symmetric with respect to the (Oy) axis. This ensures in particular the stability of simple real eigenvalues, with respect to perturbations of the obstacle satisfying the same symmetry constraint.

Proposition IV.5. *Assume that γ satisfies $\gamma(x, y) = \gamma(-x, y)$ for all $(x, y) \in \Omega$. Then the operator B_θ is $\mathcal{PT}$-symmetric ($\mathcal{PT}B_\theta\mathcal{PT} = B_\theta$) with $\mathcal{P}\varphi(x, y) = \varphi(-x, y)$, $\mathcal{T}\varphi(x, y) = \overline{\varphi(x, y)}$ for $\varphi \in \mathrm{L}^2(\Omega)$. Therefore, we have $\sigma(B_\theta) = \overline{\sigma(B_\theta)}$.*

The proof is straightforward observing that the β_θ defined after (IV.13) satisfies $\beta_\theta(-x, y) = \overline{\beta_\theta(x, y)}$.

Finally let us mention a specific difficulty which appears in the spectral analysis of B_θ. While Theorem IV.2 guarantees that $\sigma(A_\theta) \setminus \sigma_{\mathrm{ess}}(A_\theta)$ is discrete, we do not write such a statement for the operator B_θ in Theorem IV.3. A major difference between both operators is that $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(A_\theta)$ is connected whereas $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(B_\theta)$ has a countably infinite number of connected components. As a consequence, to prove that $\sigma(B_\theta) \setminus \sigma_{\mathrm{ess}}(B_\theta)$ is discrete using the Fredholm analytic theorem, it is necessary to find one λ such that $B_\theta - \lambda$ is invertible in each of the components of $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(B_\theta)$. In general, in presence of an obstacle, such a λ probably exists (proofs for certain classes of γ can be obtained working as in [10]). But for this problem, we can have surprising perturbation results. Thus, if there is no obstacle ($\gamma \equiv 1$ in Ω), then there holds $\sigma(B_\theta) = \mathscr{R}_\theta$ (see (IV.16)): all connected components of $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(B_\theta)$, except the one containing the complex half-plane $\Re e\, \lambda < 0$, are filled with eigenvalues. To show this result, observe that for $k^2 \in \sigma(B_\theta) \setminus \sigma_{\mathrm{ess}}(B_\theta)$, the function $u \circ \mathcal{J}_\theta$, with $u(x, y) = e^{ikx}$, is a non-zero element of $\ker B_\theta$. Notice that this pathological property is also true when Ω contains a family of sound hard cracks (homogeneous Neumann boundary condition) parallel to the (Ox) axis (see the illustration of Figure IV.3).



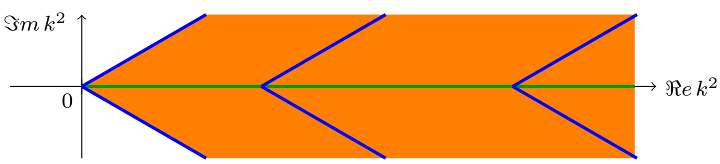


Figure IV.3: Left: waveguide with horizontal cracks. Right: the spectrum of the corresponding B_θ fills the whole sector delimited by $\sigma_{\mathrm{ess}}(B_\theta)$. Here the blue lines represent $\sigma_{\mathrm{ess}}(B_\theta)$.

4 Numerical experiments

4.1 Classical complex scaling: classical complex resonance modes

We first compute the spectrum of the operator A_θ defined in (IV.10) with a classical complex scaling (complex resonance spectrum). For the numerical experiments, we truncate the computational domain at some distance of the obstacle and use finite elements. This corresponds to the so-called Perfectly Matched Layers (PMLs) method. We refer the reader to [22] for the numerical analysis of the error due to truncation of the waveguide and discretization. The setting is as follows. We take γ such that $\gamma = 5$ in $\mathscr{O} = (-1; 1) \times (0.25; 0.75)$ and $\gamma = 1$ in $\Omega \setminus \overline{\mathscr{O}}$ (see Figure IV.2 (a)). In the definition of the maps $\mathcal{I}_\theta$, α_θ (see (IV.8), (IV.9)), we take $\theta = \pi/4$ (so that $\eta = e^{i\pi/4}$) and $L = 1$. In practice, we use a P2 finite element method in the bounded domain $\Omega_{12} = \{(x, y) \in \Omega \,|\, -12 < x < 12\}$ with Dirichlet boundary condition at $x = \pm 12$.

In Figure IV.4 and below, we display the square root of the spectrum (k instead of k^2). The vertical marks on the real axis correspond to the thresholds (0, π, 2π, ...). In accordance with Theorem IV.2, we observe that $\sqrt{\sigma(A_\theta)}$ is located in the region $\sqrt{\mathscr{R}_\theta^-} = \{z \in \mathbb{C} \,|\, -\theta \leq \arg(z) \leq 0\}$. Moreover, the discretisation of the essential spectrum $\sigma_{\mathrm{ess}}(A_\theta)$ defined in (IV.11) and forming branches starting at the threshold points appears clearly. Note that a simple calculation shows that $\sqrt{\{n^2\pi^2 + te^{-2i\theta},\, t \geq 0\}}$ is a half-line for $n = 0$ and a piece of hyperbola for $n \geq 1$. This is precisely what we get. Eigenvalues located on the real axis correspond to trapped modes ($k^2 \in \mathscr{K}_{\mathrm{t}}$). In the chosen setting, which is symmetric with respect to the axis $\mathbb{R} \times \{0.5\}$, one can prove that trapped modes exist [19]. On the other hand, the eigenvalues in the complex plane which are not the discretisation of the essential spectrum correspond to complex resonances.

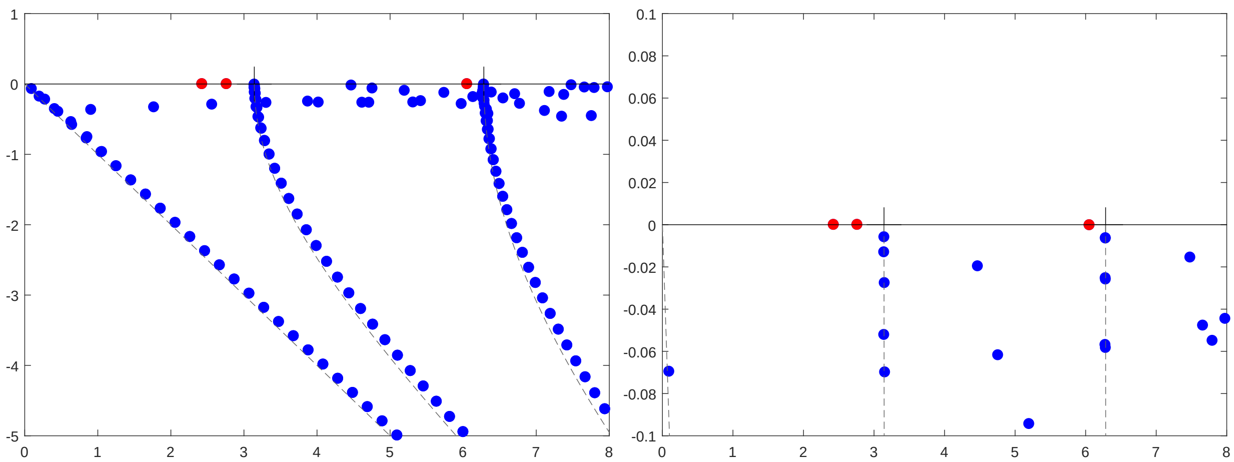


Figure IV.4: Classical complex resonances in the complex k plane corresponding to the spectrum of A_θ for a symmetric obstacle (Figure IV.2 (a)). The trapped modes are in red, the dashed lines represent the essential spectrum of A_θ (see (IV.11)). The picture on the right is a zoom-in of that on the left.

4.2 Conjugated complex scaling: reflectionless modes

Now we compute the spectrum of the operator B_θ defined in (IV.14) with a conjugated complex scaling. First, we use exactly the same symmetric setting (see Figure IV.2 (a)) as in the previous paragraph. In Figure IV.5, we display the square root of the spectrum $\sqrt{\sigma(B_\theta)}$. Since γ satisfies $\gamma(x, y) = \gamma(-x, y)$, according to Proposition IV.5 we know that B_θ is $\mathcal{PT}$-symmetric and that therefore its spectrum is stable by conjugation ($\sigma(B_\theta) = \overline{\sigma(B_\theta)}$). This is indeed what we obtain. Note that the mesh has been constructed so that $\mathcal{PT}$-symmetry is preserved at the discrete level.

$\mathcal{PT}$-symmetry is an interesting property in our case because it guarantees that eigenvalues located close to the real axis which are isolated (no other eigenvalue in a vicinity) are real. Therefore, according to Theorem IV.3, they correspond to trapped modes or to reflectionless modes. Remark that, for the same geometry, the spectrum of B_θ (Figure IV.5) contains more elements on the real axis than the spectrum of A_θ (Figure IV.4): the additional elements (green points in Figure IV.5) correspond to reflectionless modes.

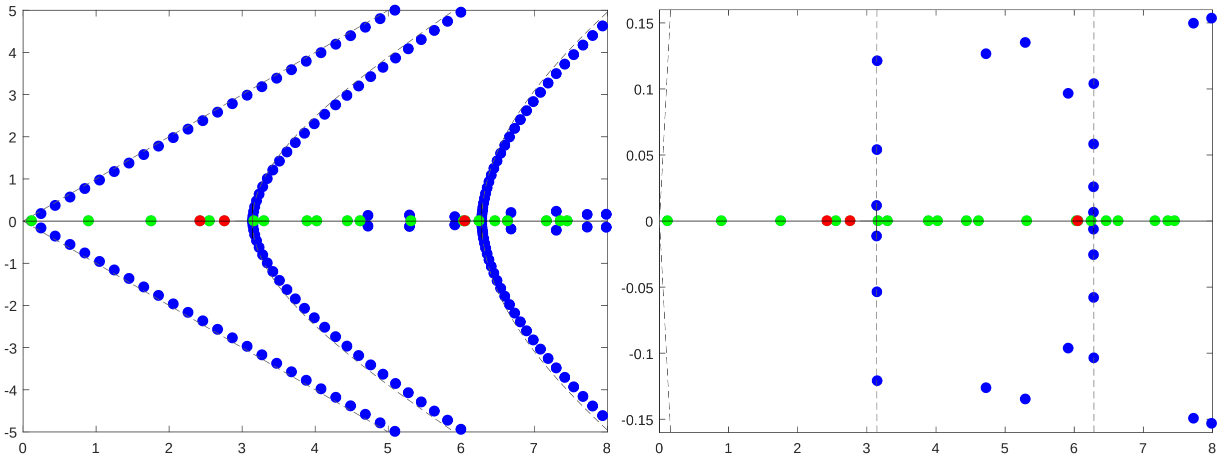
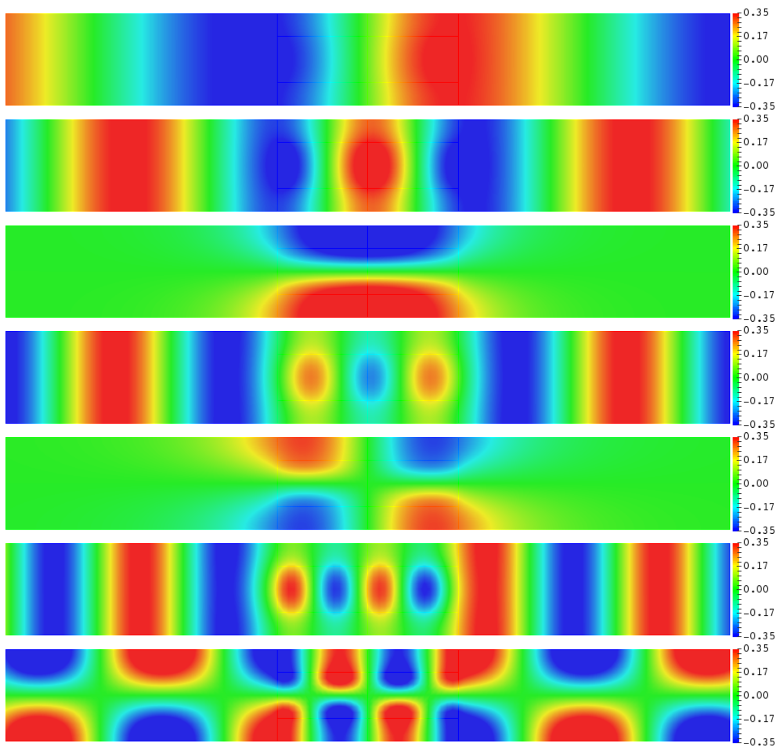


Figure IV.5: Reflectionless eigenvalues in the complex k plane corresponding to the spectrum of B_θ for a symmetric obstacle (Figure IV.2 (a)). The trapped modes in red are the same as in Figure IV.4. The reflectionless modes are in green and the dashed lines represent the essential spectrum of B_θ (see (IV.15)). The picture on the right is a zoom-in of that on the left.



k	0.9	1.8	2.4	2.6	2.8	3.3	3.9
$\rho(w_\theta)$	0.14	0.14	$8.0\,10^{-10}$	0.14	$4.3\,10^{-9}$	0.14	0.14

Figure IV.6: Top: real part of eigenmodes associated with real eigenvalues of B_θ from Figure IV.5. Bottom: value of k and of the indicator function ρ for each of these 7 eigenmodes. The 3rd and the 5th eigenmodes are trapped modes, the five others are reflectionless modes.

In Figure IV.6 top, we represent the real part of eigenfunctions associated with seven real eigenvalues of B_θ. To obtain these pictures, we take $L = 4$ in the definition of $\mathcal{J}_\theta$ in (IV.12) and we display only the restrictions of the eigenfunctions to $\Omega_L = \{(x, y) \in \Omega \,|\, -L < x < L\}$. We recognize two trapped modes (images 3 and 5). The other modes are reflectionless modes. In Figure IV.6 bottom, we provide the value of the indicator function ρ defined in (IV.17) for the seven eigenmodes. We have to mention that eigenmodes are normalized so that their L^2 norm is equal to one. The indicator function ρ offers a clear criterion to distinguish between trapped modes and reflectionless modes. Moreover, in order to inspect the scattering coefficient, we remark that for reflectionless modes associated with wavenumbers k smaller than π, the incident field u_i in (IV.5) decomposes only on the piston mode $w_0^+(x, y) = e^{ikx}/\sqrt{2k}$ (monomode regime). In this case the reflection matrix $R(k)$ in (IV.4) is nothing but the usual reflection coefficient. In Figure IV.7, we thus display the modulus of this coefficient $R(k)$ with respect to $k \in (0.1; 3.1)$. As expected, we observe that R vanishes for the values of k obtained in Figure IV.6 solving the spectral problem for B_θ. Of course obtaining the curve $k \mapsto |R(k)|$ is relatively costly and it is precisely what we want to avoid by computing the reflectionless k as eigenvalues. Here it is simply a way to check our results.

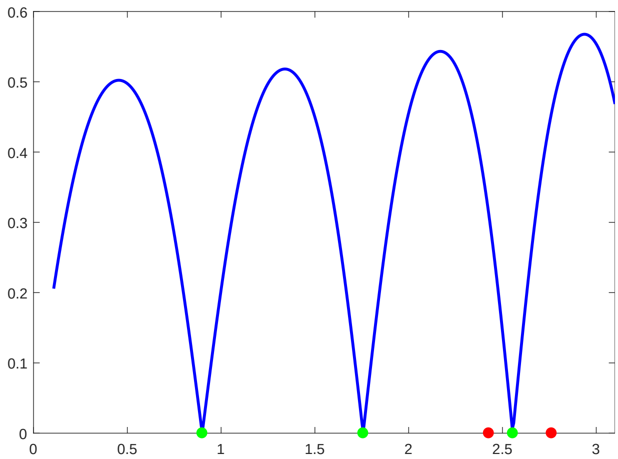


Figure IV.7: Curve $k \mapsto |R(k)|$ (modulus of the reflection coefficient) for $k \in (0.1; 3.1)$. The green and red dots represent respectively the reflectionless modes and the trapped modes computed in Figure IV.5. We indeed observe that $R(k)$ is null for reflectionless k.

In Figure IV.8, we represent the modulus of reflectionless mode eigenfunctions of B_θ associated with one real eigenvalue and two complex conjugated eigenvalues. We observe, and this is true in general, a symmetry with respect to the (Oy) axis for modes corresponding to real eigenvalues which disappears for complex ones. This is the so-called broken symmetry phenomenon which is well-known for $\mathcal{PT}$-symmetric operators (see e.g. the review [7]).

Now, we use the non-symmetric setting (see Figure IV.2 (b)), and we display the square root of the spectrum of B_θ (in Figure IV.9) for a coefficient γ which is not symmetric in x nor in y. More precisely, we take γ such that $\gamma = 5$ in $\mathscr{O} = (-1; 0] \times (0.25; 0.5) \cup [0; 1) \times (0.25; 0.75)$ and $\gamma = 1$ in $\Omega \setminus \overline{\mathscr{O}}$. We observe that the spectrum is no longer stable by conjugation ($\sigma(B_\theta) \neq \overline{\sigma(B_\theta)}$) since the operator B_θ is not $\mathcal{PT}$-symmetric, and there is no "help" for the eigenvalues to be real. However, a closer look shows the presence of eigenvalues close to the real axis, in particular for $k \approx 1.0 + 0.13i$, $k \approx 1.9 + 0.005i$, $k \approx 2.5 + 0.02i$, $k \approx 2.8 + 0.08i$ and $k \approx 3.0 - 0.008i$. In Figure IV.10, we represent $k \mapsto |R(k)|$ for $k \in (0.1; 3.1)$ where there is only one propagating mode in the leads. It is interesting to note that the above computed complex reflectionless modes (located close to the real axis) have an influence on this curve. More precisely, $k \mapsto |R(k)|$ attains minima for $k \in (0.1; 3.1)$ close to the real part of these complex reflectionless modes. Therefore complex reflectionless modes also have significance for scattering at real frequncies.

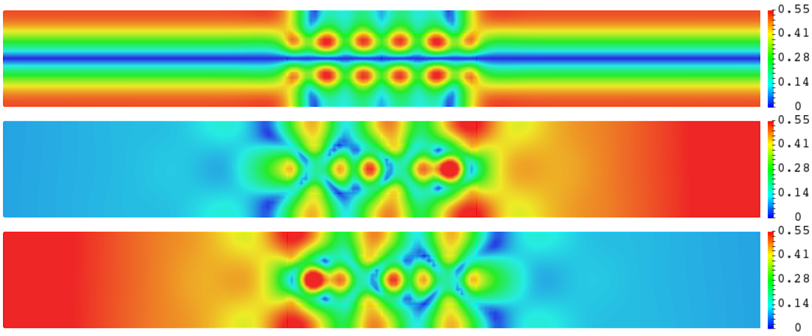


Figure IV.8: Modulus of eigenfunctions of B_θ associated to the eigenvalues $k \approx 5.31$ (top), $k \approx 5.29 - 0.13i$ (middle) and $k \approx 5.29 + 0.13i$ (bottom) obtained in Figure IV.5. The symmetry $x \to -x$ for real k (due to $\mathcal{PT}$-symmetry) disappears for complex k.

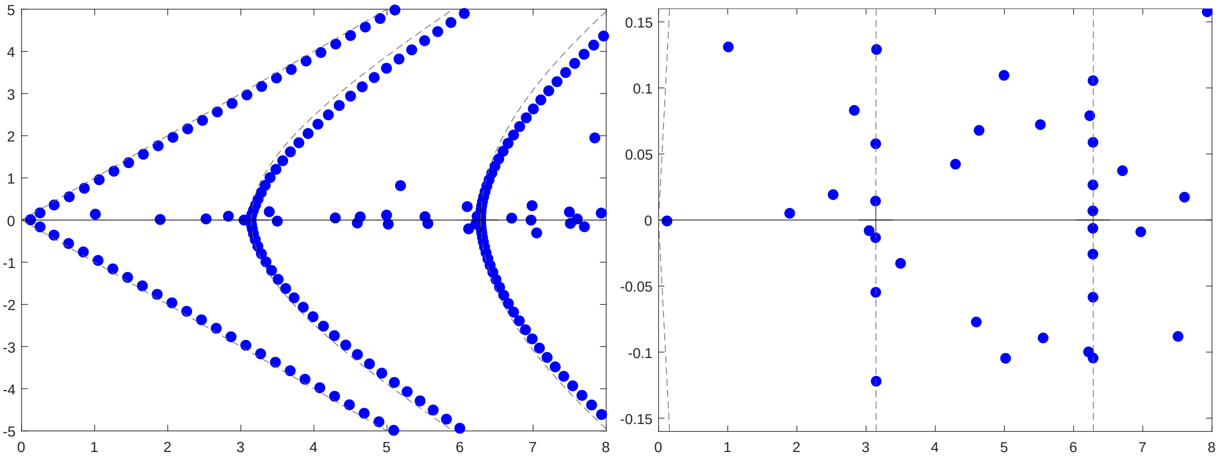


Figure IV.9: Spectrum of B_θ in the complex k plane for a non symmetric obstacle (Figure IV.2 (b)). The dashed lines represent the essential spectrum of B_θ (see (IV.15)). The spectrum is not stable by conjugation. The picture on the right is a zoom-in of that on the left.

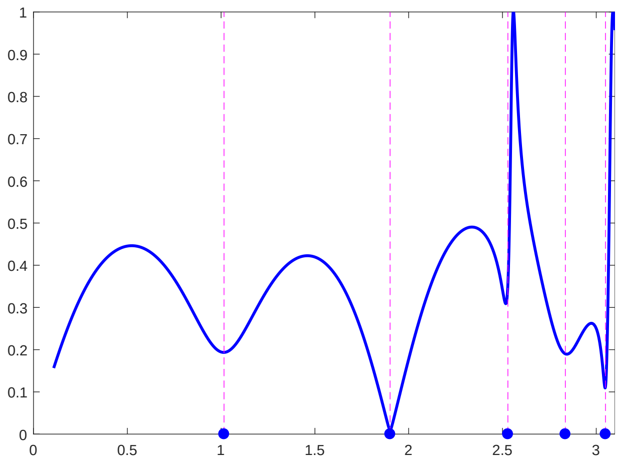


Figure IV.10: Curve $k \mapsto |R(k)|$ (modulus of the reflection coefficient) for $k \in (0.1; 3.1)$ and a non symmetric obstacle. The blue dots and the vertical dashed lines correspond to the real parts of the eigenvalues of B_θ located close to the real axis computed in Figure IV.9. We observe that $|R(k)|$ is minimal for these particular k.

5 Concluding remarks

Determining the scattering coefficients for a range of frequencies to identify the k for which there are incident fields which produce zero reflection is a tedious work. This chapter shows that reflectionless frequencies can be directly computed as the eigenvalues of a non-selfadjoint operator B_θ (see (IV.14)) with conjugated complex scalings enforcing ingoing behaviour in the incident lead and outgoing behaviour in the other lead. The reflectionless spectrum of this operator B_θ provides a complementary information to the one contained in the classical complex resonance spectrum associated with leaky modes which decompose only on outgoing waves (see the operator A_θ in (IV.10)). Note that eigenvalues corresponding to trapped modes belong to both the reflectionless spectrum and to the classical complex resonance spectrum because trapped modes do not excite propagating waves. Let us make a few additional comments and highlight future directions as well as open questions.

i) We have seen that the non-selfadjoint operator B_θ is $\mathcal{PT}$ symmetric when the structure has mirror symmetry. Interestingly, a direct calculus shows that in the very simple case of a 1D transmission problem through a slab of constant index, reflectionless frequencies are all real. This gives an example of a non-selfadjoint $\mathcal{PT}$ symmetric operator with only real eigenvalues.

ii) In this work, we investigated scattering problems in waveguides with $N = 2$ leads for which two reflectionless spectra exist: one associated with incident waves propagating from the left and another corresponding to incident waves propagating from the right. The more general case with N ($N \geq 2$) leads can be considered as well. Among the total of 2^N different spectra with an ingoing or an outgoing complex scaling in each lead, two spectra correspond to eigenmodes which decompose on waves which are all outgoing or all ingoing. As a consequence, there are $2^N - 2$ reflectionless spectra.

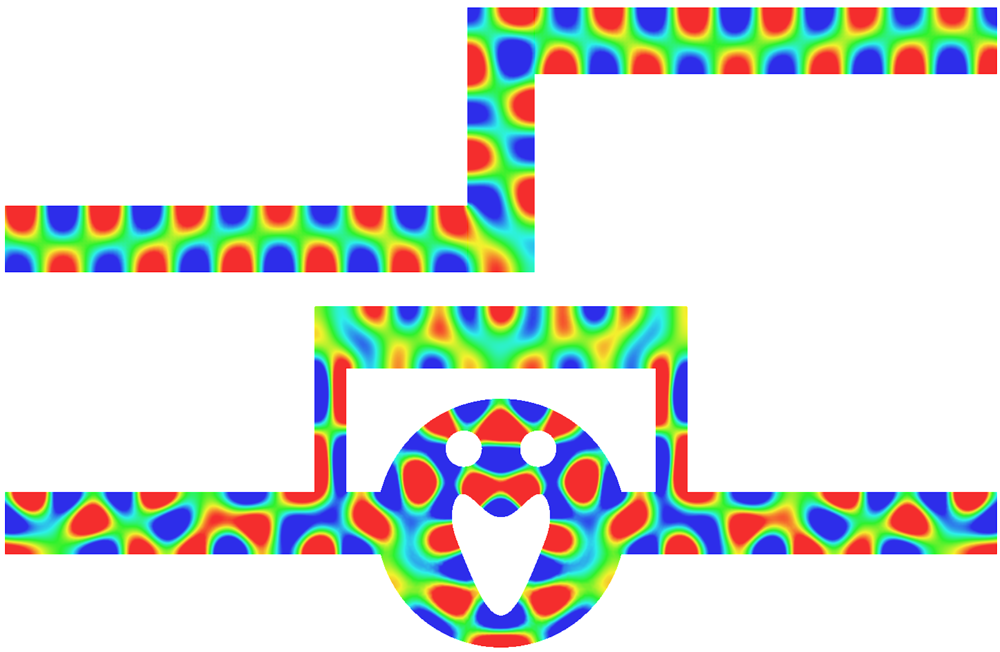


Figure IV.11: Examples of reflectionless modes in waveguides with sound hard walls (Neumann).

iii) Above we computed reflectionless modes in waveguides containing penetrable obstacles. We can work completely similarly with perturbations of the geometry. In Figure IV.11, we give two examples of reflectionless modes in such structures. Note that in each case, B_θ is $\mathcal{PT}$ symmetric due to the symmetry of the geometry. Other kinds of BCs (Dirichlet, ...) and higher dimension (3D) can be dealt with similarly.

There are many open questions with this work. Here we list just of few of them.

iv) First, it would be nice to obtain criteria on the index material/geometry ensuring that the spectrum of B_θ is discrete outside of $\sigma_{\text{ess}}(B_\theta)$. In 1D, we have been able to prove rather general results. In higher dimension this is quite open.

v) On the other hand, could we show that B_θ has always real eigenvalues corresponding to reflectionless modes, at least for $\mathcal{PT}$ symmetric problems?

vi) The question of the approximation of the spectrum of B_θ is a field of research in itself, the reason being that B_θ is a non selfadjoint operator. Indeed, the theory of perturbations of non selfadjoint operators is not well developed and many phenomena can occur. Here it is easy to see in the situation where the waveguide contains only (Neumann) horizontal cracks. In that case we explained before Figure IV.3 that $\sigma(B_\theta)$ fills a whole sector. However when we truncate the domain at some distance L, we can show that the corresponding spectrum is discrete. This seems particularly pathological. But even when $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete, proving that the spectrum of the problem in the truncated waveguide converges to the one of B_θ when L tends to $+\infty$ is a challenging task. Additionally, it would be interesting to prove that spurious eigenvalues, which would converge to some non physical values, do not exist. In practice, we not only truncate the domain but also approximate the problem by working in finite dimension with finite elements. This is also an approximation which has to be studied. In this context, it seems that the notion of pseudo-spectrum, see e.g. [40], which has been developed to understand the properties of non normal operators, may provide useful information.

Bibliography

[1] G.S. Abeynanda and S.P. Shipman. Dynamic resonance in the high-Q and near-monochromatic regime. *MMET, IEEE*, 10.1109/MMET.2016.7544100, 2016.

[2] J. Aguilar and J.-M. Combes. A class of analytic perturbations for one-body schrödinger hamiltonians. *Comm. Math. Phys.*, 22(4):269–279, 1971.

[3] G.V. Alekseev, A.V. Lobanov, and Y.E. Spivak. Optimization method in problems of acoustic cloaking of material bodies. *Comput. Math. Math. Phys.*, 57(9):1459–1474, 2017.

[4] A. Aslanyan, L. Parnovski, and D. Vassiliev. Complex resonances in acoustic waveguides. *Quart. J. Mech. Appl. Math.*, 53(3):429–447, 2000.

[5] E. Balslev and J.-M. Combes. Spectral properties of many-body Schrödinger operators with dilatation-analytic interactions. *Comm. Math. Phys.*, 22(4):280–294, 1971.

[6] J.T. Beale. Scattering frequencies of resonators. *Comm. Pure Appl. Math.*, 26(4):549–563, 1973.

[7] C.M. Bender. Making sense of non-Hermitian Hamiltonians. *Rep. Prog. Phys.*, 70(6):947, 2007.

[8] A. Bera, A.-S. Bonnet-Ben Dhia, and L. Chesnel. A continuation method for building invisible obstacles in waveguides. *Q. J. Mech. Appl. Math.*, 74(1):83–116, 2021.

[9] J.-P. Berenger. A perfectly matched layer for the absorption of electromagnetic waves. *J. Comput. Phys.*, 114(2):185–200, 1994.

[10] A.-S. Bonnet-Ben Dhia, L. Chesnel, and S.A. Nazarov. Non-scattering wavenumbers and far field invisibility for a finite set of incident/scattering directions. *Inverse Problems*, 31(4):045006, 2015.

[11] A.-S. Bonnet-Ben Dhia, L. Chesnel, and S.A. Nazarov. Perfect transmission invisibility for waveguides with sound hard walls. *J. Math. Pures Appl.*, 111:79–105, 2018.

[12] A.-S. Bonnet-Ben Dhia, L. Chesnel, and V. Pagneux. Trapped modes and reflectionless modes as eigenfunctions of the same spectral problem. *Proc. R. Soc. A*, 474(2213):20180050, 2018.

[13] A.-S. Bonnet-Ben Dhia and S.A. Nazarov. Obstacles in acoustic waveguides becoming "invisible" at given frequencies. *Acoust. Phys.*, 59(6):633–639, 2013.

[14] É. Bonnetier and F. Triki. Asymptotic of the Green function for the diffraction by a perfectly conducting plane perturbed by a sub-wavelength rectangular cavity. *Math. Method. Appl. Sci.*, 33(6):772–798, 2010.

[15] M. Cassier, T. Degiovanni, S. Guenneau, and F.-G. Vasquez. Active thermal cloaking and mimicking. *Proc. R. Soc. A*, 477:20200941, 2021.

[16] L. Chesnel, J. Heleine, and S.A. Nazarov. Acoustic passive cloaking using thin outer resonators. *Z. Angew. Math. Phys.*, 73(3):98, 2022.

[17] L. Chesnel and S.A. Nazarov. Team organization may help swarms of flies to become invisible in closed waveguides. *Inverse Problems and Imaging*, 10(4):977–1006, 2016.

[18] L. Chesnel and S.A. Nazarov. Non reflection and perfect reflection via Fano resonance in waveguides. *Comm. Math. Sci.*, 16(7):1779–1800, 2018.

[19] D.V. Evans, M. Levitin, and D. Vassiliev. Existence theorems for trapped modes. *J. Fluid. Mech.*, 261:21–31, 1994.

[20] P. Henrici. *Applied and computational complex analysis.* Wiley-Interscience, 1974. Volume 1: Power series—integration—conformal mapping—location of zeros, Pure and Applied Mathe-

matics.

[21] H. Hernandez-Coronado, D. Krejčiřík, and Siegl P. Perfect transmission scattering as a $\mathcal{PT}$-symmetric spectral problem. *Phys. Lett. A*, 375(22):2149–2152, 2011.

[22] V. Kalvin. Analysis of perfectly matched layer operators for acoustic scattering on manifolds with quasicylindrical ends. *J. Math. Pures Appl.*, 100(2):204–219, 2013.

[23] T. Kato. *Perturbation theory for linear operators.* Springer-Verlag, Berlin, reprint of the corr. print. of the 2nd ed. 1980 edition, 1995.

[24] V.A. Kozlov, V.G. Maz'ya, and A.B. Movchan. Asymptotic analysis of a mixed boundary value problem in a multi-structure. *Asymptot. Anal.*, 8(2):105–143, 1994.

[25] G.A. Kriegsmann. Complete transmission through a two-dimensional difffraction grating. *SIAM J. Appl. Math.*, 65(1):24–42, 2004.

[26] N. Lebbe, C. Dapogny, E. Oudet, K. Hassan, and A. Gliere. Robust shape and topology optimization of nanophotonic devices using the level set method. *J. Comput. Phys.*, 395(0):710–746, 2019.

[27] N. Lebbe, A. Glière, K. Hassan, C. Dapogny, and E. Oudet. Shape optimization for the design of passive mid-infrared photonic components. *Opt. Quant. Electron.*, 51(5):166, 2019.

[28] U. Leonhardt. Optical conformal mapping. *Science*, 312(5781):1777–1780, 2006.

[29] J. Lin, S. Shipman, and H. Zhang. A mathematical theory for Fano resonance in a periodic array of narrow slits. *SIAM J. Appl. Math.*, 80(5):2045–2070, 2020.

[30] J. Lin and H. Zhang. Scattering and field enhancement of a perfect conducting narrow slit. *SIAM J. Appl. Math.*, 77(3):951–976, 2017.

[31] V.G. Maz'ya, S.A. Nazarov, and B.A. Plamenevskiĭ. *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains, Vol. 1, 2.* Birkhäuser, Basel, 2000.

[32] D.A.B. Miller. On perfect cloaking. *Opt. Express*, 14(25):12457–12466, 2006.

[33] J. Peetre. Another approach to elliptic boundary problems. *Commun. Pure Appl. Math.*, 14:711–731, 1961.

[34] J.B. Pendry, D. Schurig, and D.R. Smith. Controlling electromagnetic fields. *Science*, 312(5781):1780–1782, 2006.

[35] S.P. Shipman and H. Tu. Total resonant transmission and reflection by periodic structures. *SIAM J. Appl. Math.*, 72(1):216–239, 2012.

[36] S.P. Shipman and S. Venakides. Resonant transmission near nonrobust periodic slab modes. *Phys. Rev. E*, 71(2):026611, 2005.

[37] S.P. Shipman and A.T. Welters. Resonant electromagnetic scattering in anisotropic layered media. *J. Math. Phys.*, 54(10):103511, 2013.

[38] B. Simon. Resonances and complex scaling: a rigorous overview. *Int. J. of Quantum Chem.*, XIV(22):529–542, 1978.

[39] W.R. Sweeney, C.W. Hsu, and A.D. Stone. Theory of reflectionless scattering modes. *Phys. Rev. A*, 102(6):063511, 2020.

[40] L.N Trefethen. Spectra and pseudospectra: the behavior of nonnormal matrices and operators. 2020.

[41] J. Wloka. *Partial Differential Equations.* Cambridge Univ. Press, 1987.