Invisibility and complete reflectivity in waveguides with finite length branches

Lucas Chesnel[1]

Coll. with S.A. Nazarov[2] and V. Pagneux[3].

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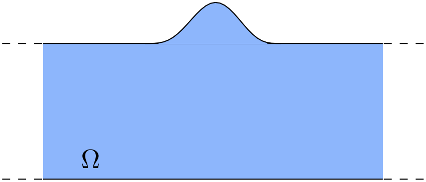
[3]LAUM, Université du Maine, France

Waveguide problem

► Scattering in time-harmonic regime of a plane wave in the acoustic waveguide Ω coinciding with $\{(x, y) \in \mathbb{R} \times (0; 1)\}$ outside a compact region.



Find $v = v_\mathrm{i} + v_\mathrm{s}$ s. t.

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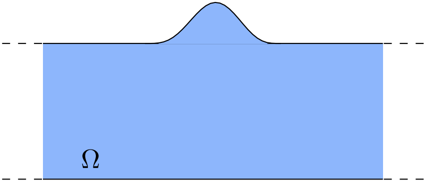

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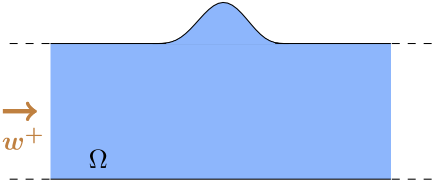

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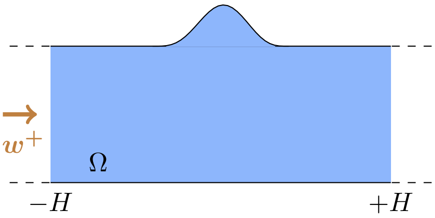

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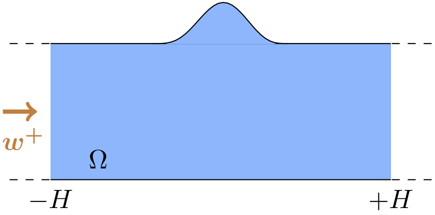

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DEFINITION: v_i = incident field
v = total field
v_s = scattered field.

Invisibility and complete reflectivity

- At infinity, one measures the reflection coefficient $R = s^-$ and/or the transmission coefficient $T = 1 + s^+$ (other terms are too small).

- From conservation of energy, one has

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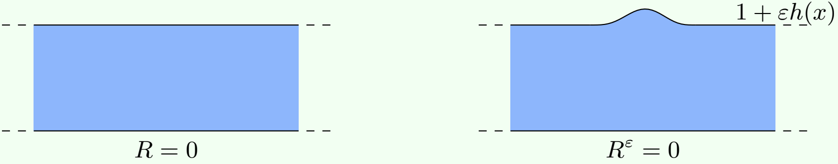
- We shall assume that the wavenumber k is given.

Existing methods

- Implicit functions theorem: we can construct small non reflective defects (see A. Bera's talk on Thursday).



$\Rightarrow$ We obtain small defects such that $R = 0$ (harder to get $T = 1$).

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- Fano resonance: if for a setting trapped modes exist, then perturbing slightly the geometry and k, we can get $R = 0$ or $T = 0$.

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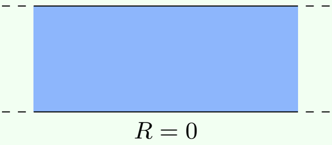

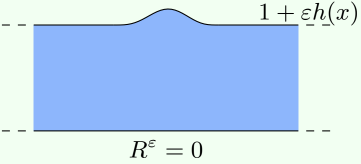

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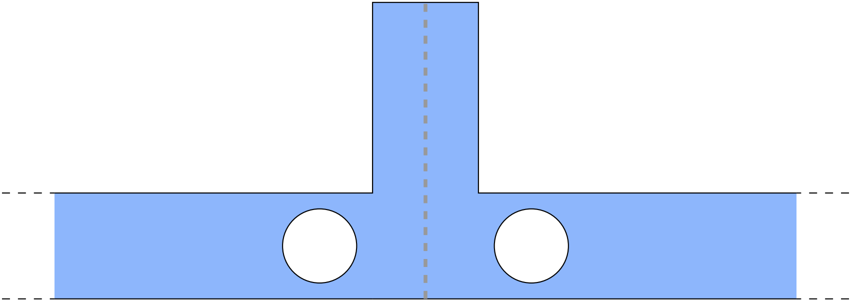
TALK

We propose another mechanism to get **large defects** s. t.

$$R = 0 \ (|T| = 1), \quad T = 1 \ (R = 0) \quad \text{or} \quad T = 0 \ (|R| = 1).$$

Geometrical setting

- We work in waveguides which are symmetric with respect to (Oy) and which contain a branch of finite height.



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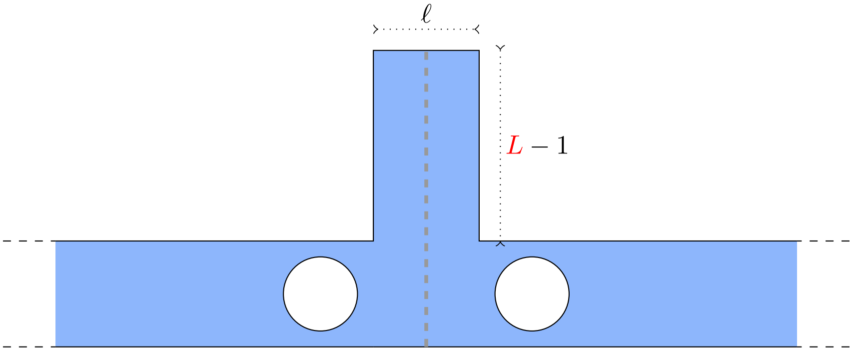

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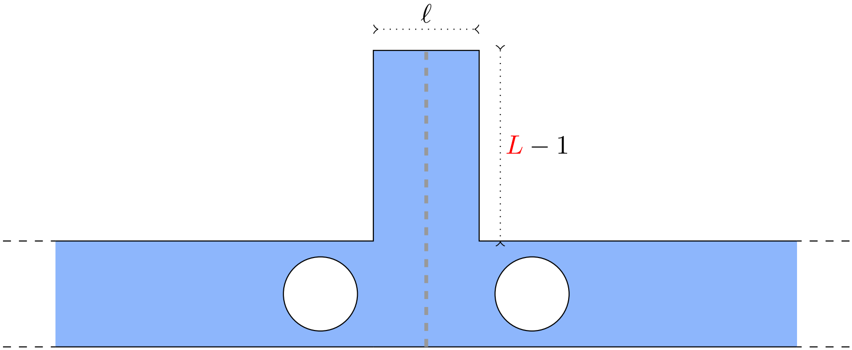

→ We will study the behaviour of the coefficients $R, T \in \mathbb{C}$ as $L \to +\infty$.

Outline of the talk

Half-waveguide problems

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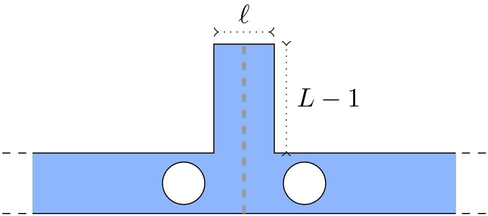

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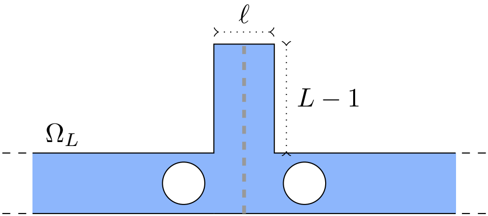

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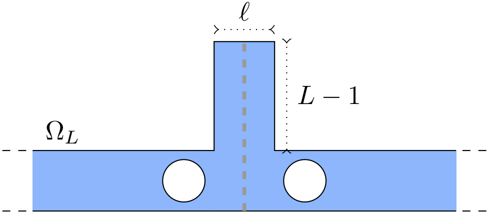

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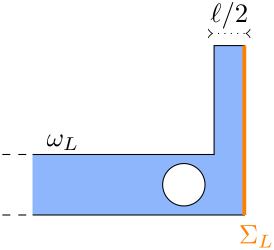

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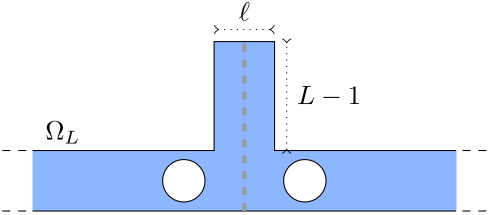

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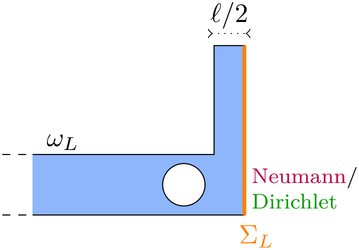

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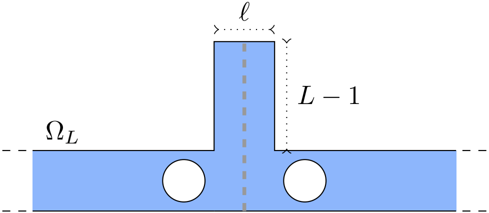

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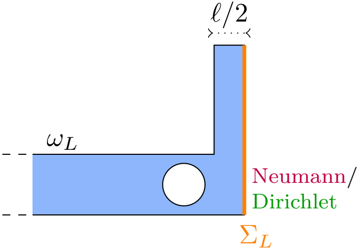

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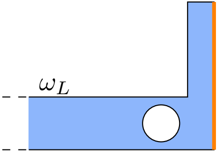

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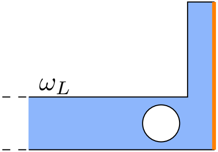

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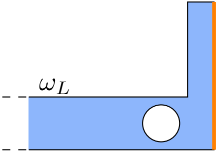

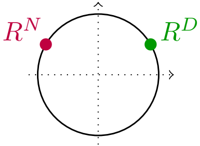

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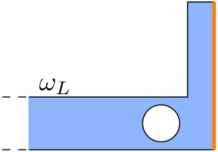

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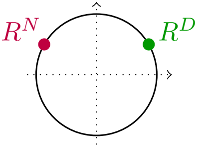

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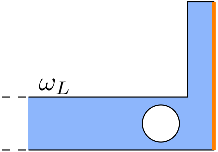

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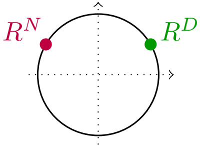

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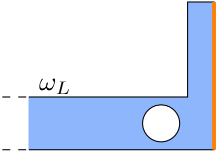

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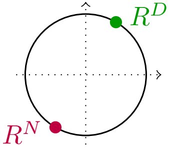

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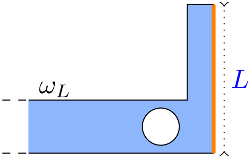

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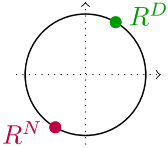

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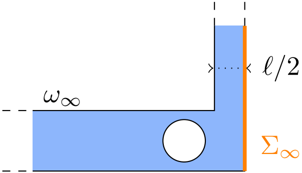
→ Now, we study the behaviour of $R^N = R^N(L)$, $R^D = R^D(L)$ as $L \to +\infty$.

Asymptotics of R^N, R^D

Depend on the nb. of propagative modes in the vertical branch of ω_∞



$$(\mathscr{P}^N) \left| \begin{array}{rcll} -\Delta\varphi &=& k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi &=& 0 & \text{on } \partial\omega_\infty \end{array} \right.$$

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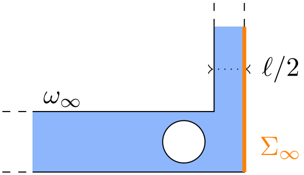
Asymptotics of R^N, R^D 1/2

Depend on the nb. of propagative modes in the vertical branch of ω_∞



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- Analysis for R^D

- For $\boldsymbol{\ell \in (0;\pi/k)}$, no prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

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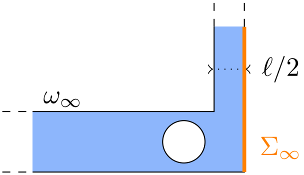

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- Analysis for R^D

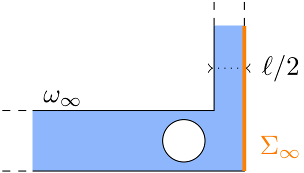
- For $\boldsymbol{\ell \in (0; \pi/k)}$, no prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

- $(\mathscr{P}^D)$ admits the solution

$$U_\infty = w_1^- + R_\infty^D\, w_1^+ + \tilde{U}_\infty, \qquad \text{with } \tilde{U}_\infty \in \mathrm{H}^1(\omega_\infty),\ |R_\infty^D| = 1.$$

Asymptotics of R^N, R^D

Depend on the nb. of propagative modes in the vertical branch of ω_∞



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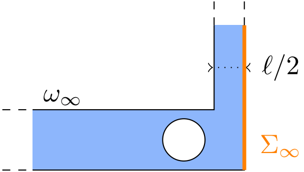
($w_1^\pm = \chi_l\, w^\mp$ where χ_l is a cut-off function s.t. $\chi_l = 1$ for $x \le -2\ell$, $\chi_l = 0$ for $x \ge -\ell$)

Depend on the nb. of propagative modes in the vertical branch of ω_∞



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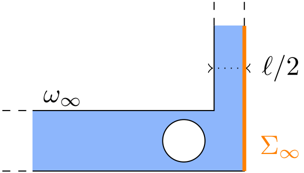
Asymptotics of R^N, R^D

Depend on the nb. of propagative modes in the vertical branch of ω_∞



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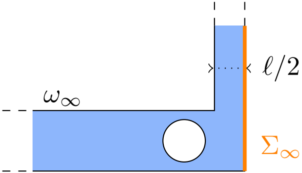
- As $L \to +\infty$, we have $U = U_\infty + \dots$ which implies $|R^D - R_\infty^D| \le C\, e^{-\beta L}.$

Asymptotics of R^N, R^D 1/2

Depend on the nb. of propagative modes in the vertical branch of ω_∞



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- Analysis for R^D

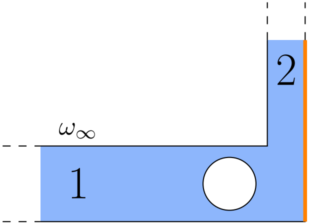
For $\boldsymbol{\ell \in (0;\pi/k)}$, $L \mapsto R^D(L)$ tends to a constant on $\mathscr{C} := \{z \in \mathbb{C},\ |z| = 1\}$.

Asymptotic of R^N, R^D 2/2

▶ Analysis for R^N

- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$$w_2^{\pm} = \chi_t \, e^{\pm iky}/\sqrt{k\ell}$$



Asymptotic of R^N, R^D

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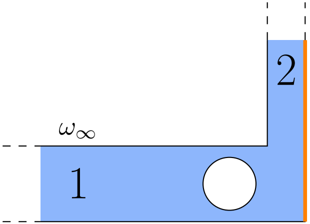
(χ_t is a cut-off function such that $\chi_t = 1$ for $y \geq 2$, $\chi_t = 0$ for $y \leq 1$)



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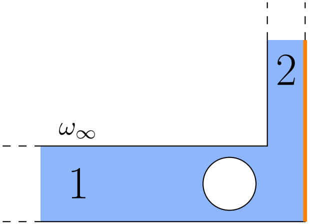

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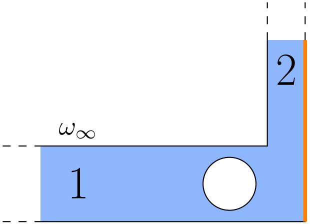
$$\begin{aligned} u_\infty^1 &= w_1^- + s_{11}\, w_1^+ + s_{12}\, w_2^+ + \tilde{u}_\infty^1, \qquad && \text{with } \tilde{u}_\infty^1 \in \mathrm{H}^1(\omega_\infty) \\ u_\infty^2 &= w_2^- + s_{21}\, w_1^+ + s_{22}\, w_2^+ + \tilde{u}_\infty^2, \qquad && \text{with } \tilde{u}_\infty^2 \in \mathrm{H}^1(\omega_\infty). \end{aligned}$$



The scattering matrix

$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ is unitary.

Asymptotic of R^N, R^D

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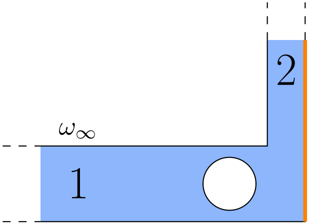
- If $s_{12} \neq 0$, we make the ansatz $u = u_\infty^1 + a(L)\, u_\infty^2 + \dots$.



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Asymptotic of R^N, R^D

- **Analysis for R^N**

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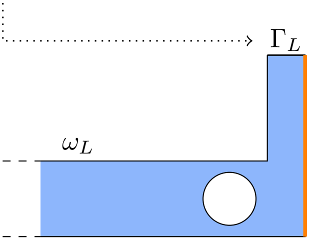

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Asymptotic of R^N, R^D 2/2

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- This gives $a(L)$ and implies, as $L \to +\infty$,

$$|R^N - R^N_{\text{asy}}(L)| \le C\, e^{-\beta L} \quad \text{with} \quad R^N_{\text{asy}}(L) = s_{11} + \frac{s_{12}\, s_{21}}{e^{-2ikL} - s_{22}}.$$

▶ Analysis for R^N

- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$$w_2^{\pm} = \chi_t\, e^{\pm iky}/\sqrt{k\ell}$$

- $(\mathscr{P}^N)$ admits the solutions

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- Unitarity of $\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \Rightarrow$ $L \mapsto R^N_{\text{asy}}(L)$ runs periodically on $\mathscr{C}$.

Asymptotic of R^N, R^D 2/2

- Analysis for R^N

For $\boldsymbol{\ell \in (0; 2\pi/k)}$, $L \mapsto R^N(L)$ runs continuously and almost period. on $\mathscr{C}$.

Conclusions for $\ell \in (0;\pi/k)$, $s_{12} \neq 0$

- Reminder: $R = \dfrac{R^N + R^D}{2}$ and $T = \dfrac{R^N - R^D}{2}$.

PROPOSITION: Asymptotically as $L \to +\infty$, R (resp. T) runs on the circle of radius $1/2$ centered at $R^D_\infty/2$ (resp. $-R^D_\infty/2$).

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PROPOSITION: There is an unbounded sequence (L_n) such that for $L = L_n$, $R^N = -R^D$ and so $R = 0$ (non reflectivity).

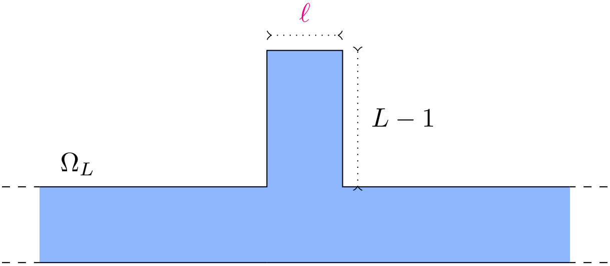
PROPOSITION: There is an unbounded sequence $(\mathcal{L}_n)$ such that for $L = \mathcal{L}_n$, $R^N = R^D$ and so $T = 0$ (complete reflectivity).

- Sequences (L_n) and $(\mathcal{L}_n)$ are almost periodic. As $n \to +\infty$, we have

$$L_{n+1} - L_n = \pi/k + \dots \quad \text{and} \quad \mathcal{L}_{n+1} - \mathcal{L}_n = \pi/k + \dots .$$

Setting

- We compute numerically the scattering coefficients R, T for $L \in (2; 10)$ in the geometry Ω_L

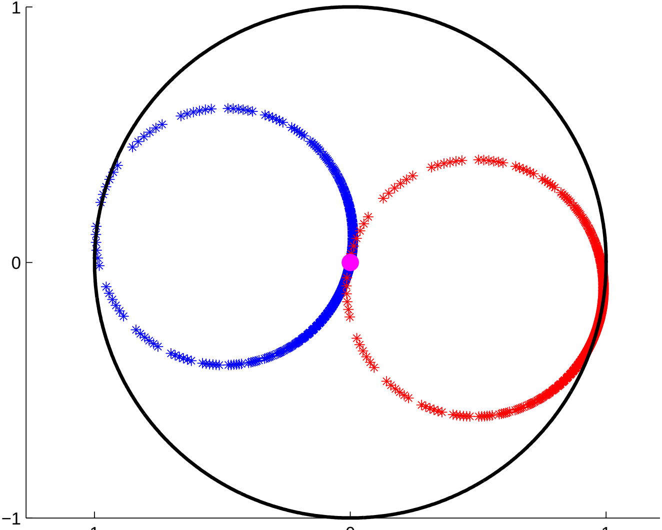


- We use a P2 finite element method with Dirichlet-to-Neumann maps.

- We set $k = 0.8\pi$ and $\ell = 1 \in (0; \pi/k)$.

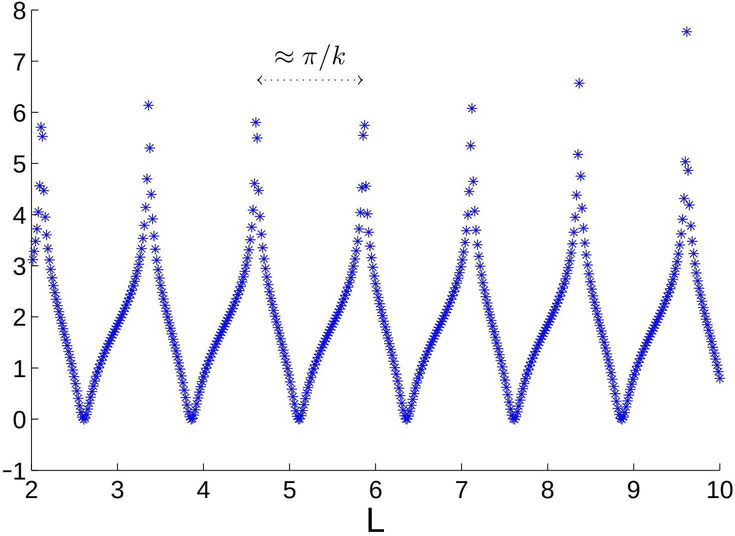
Numerical results

- Reflection coefficient R and transmission coefficient T for $L \in (2; 10)$.
  Due to **conservation of energy**, R and T are inside the **unit disk** of $\mathbb{C}$.



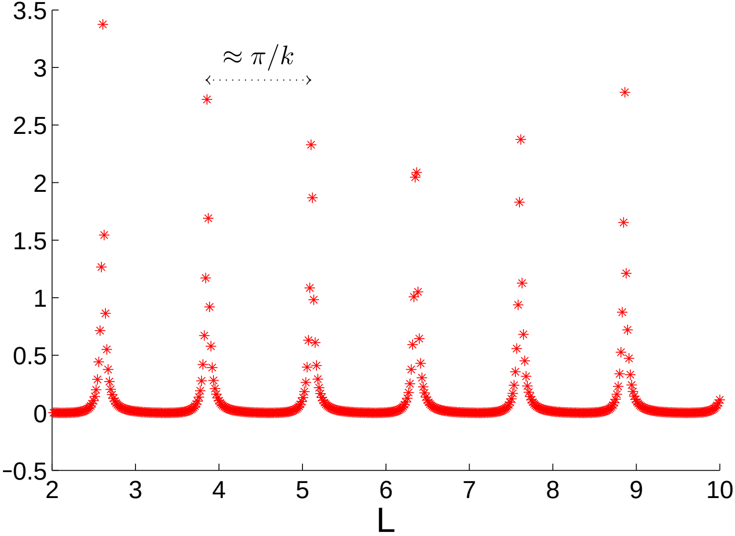
Numerical results

- Curve $L \mapsto -\ln|R|$. Peaks correspond to non reflectivity.



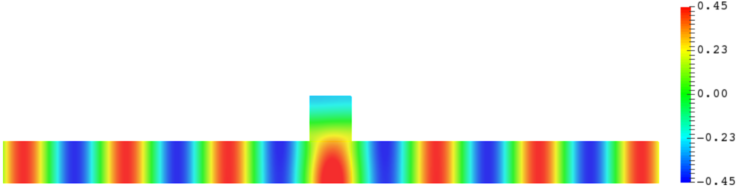
Numerical results

- Curve $L \mapsto -\ln|T|$. Peaks correspond to complete reflectivity.

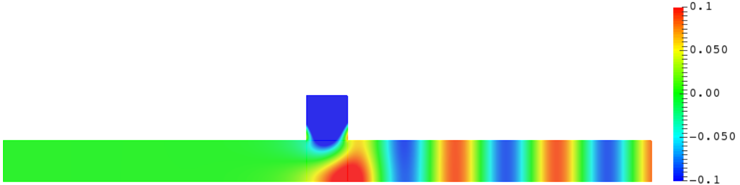


Non reflectivity

- Total field v for L such that $R = 0$.



- Scattered field v_s.



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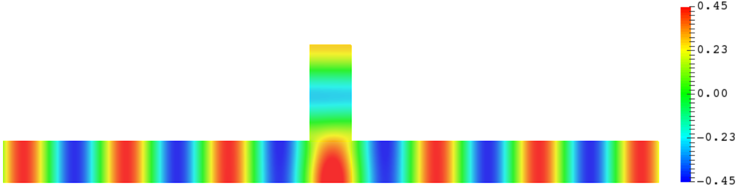

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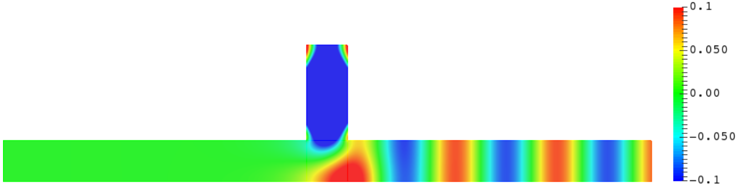

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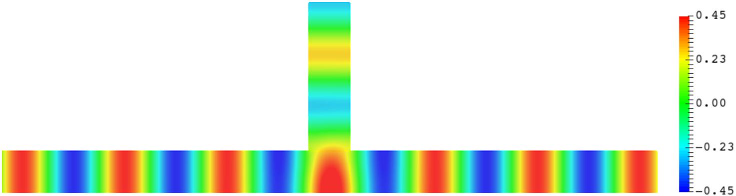

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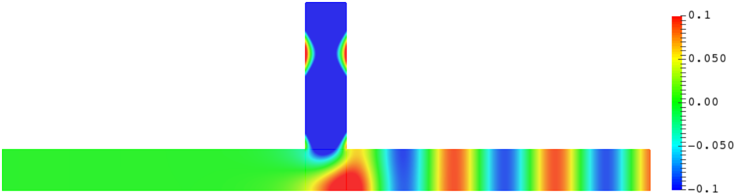
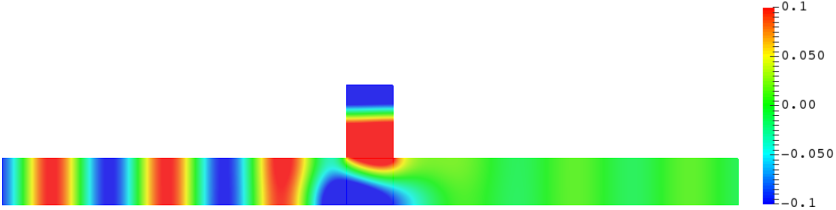


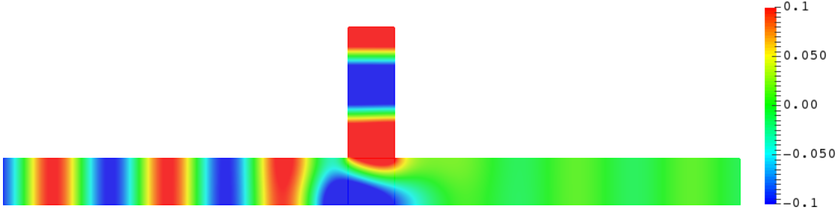
Complete reflectivity

- Total field v for L such that $T = 0$.



Complete reflectivity

- Total field v for L such that $\color{red}{T} = 0$.



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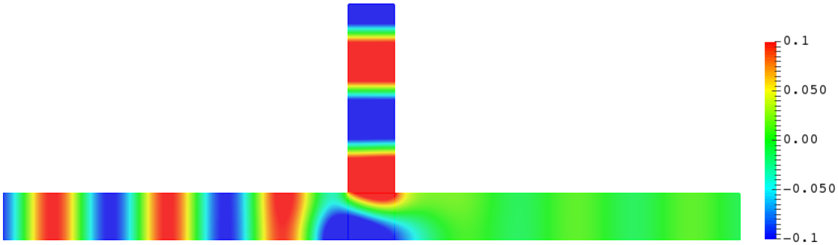
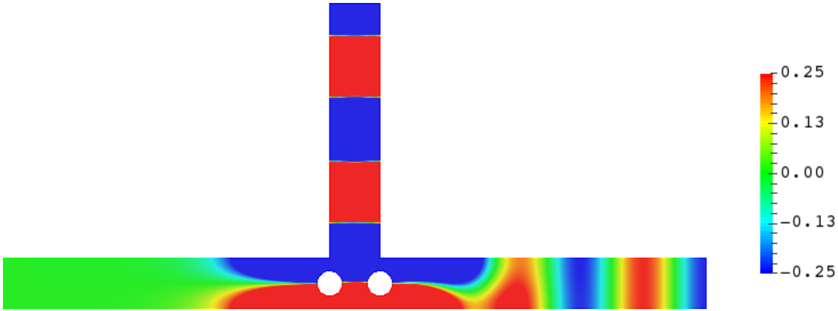


Other non reflective geometry

- Scattered field v_s



Analysis for $\ell \in (\pi/k; 2\pi/k)$

- We still have $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

and analysis for $L \mapsto R^N$ has been done previously.

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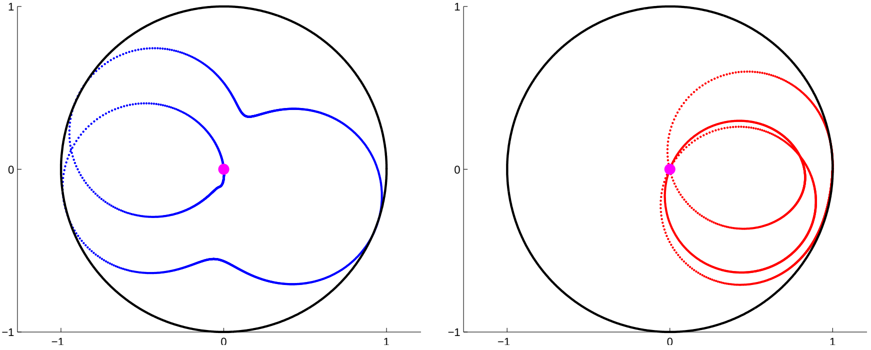
⋆ The curves $L \mapsto R(L)$, $T(L)$ still pass through zero an infinite nb. of times.

⋆ Behaviours of $L \mapsto R(L)$, $T(L)$ can be much more complex than before...

Numerical results for $\ell \in (\pi/k; 2\pi/k)$

- Asympt. curves of $L \mapsto R(L)$, $T(L)$ for $L \in (0; +\infty)$ and ℓ such that

$$k = m\,\alpha, \quad m = 2 \qquad\qquad (\alpha = \sqrt{k^2 - (\pi/\ell)^2}).$$



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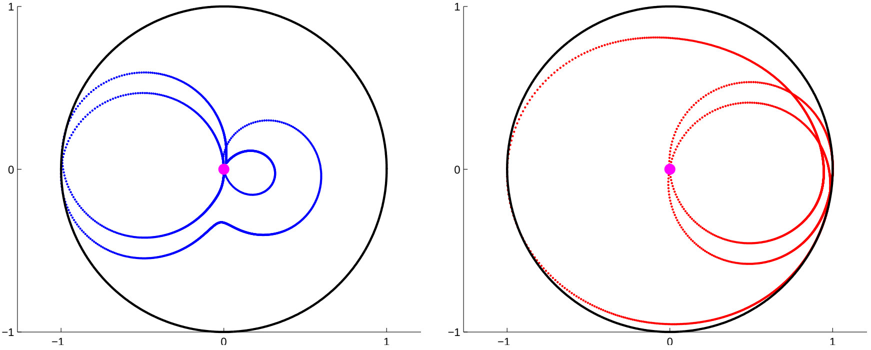

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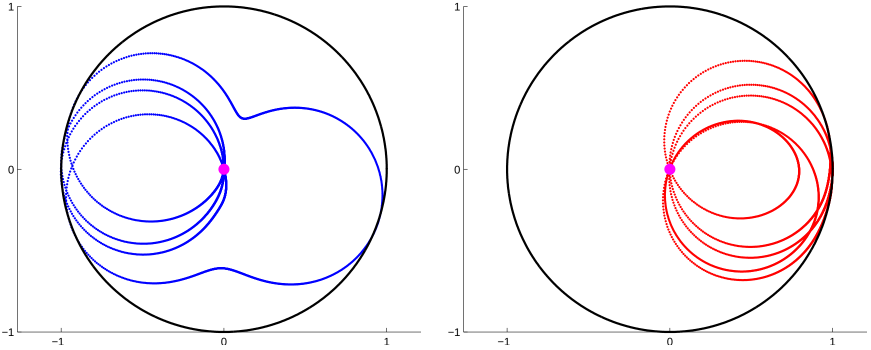

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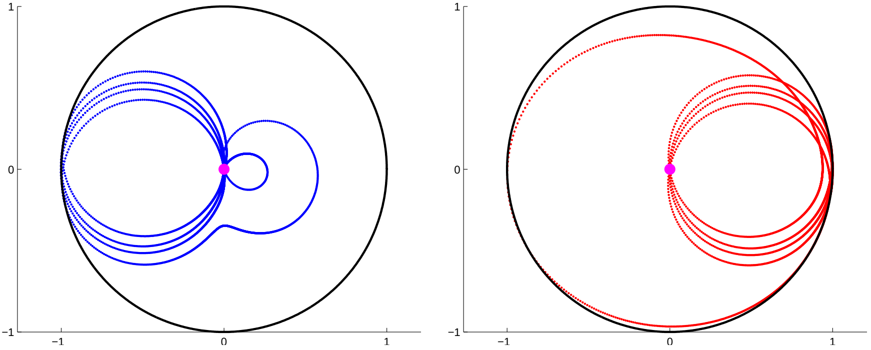
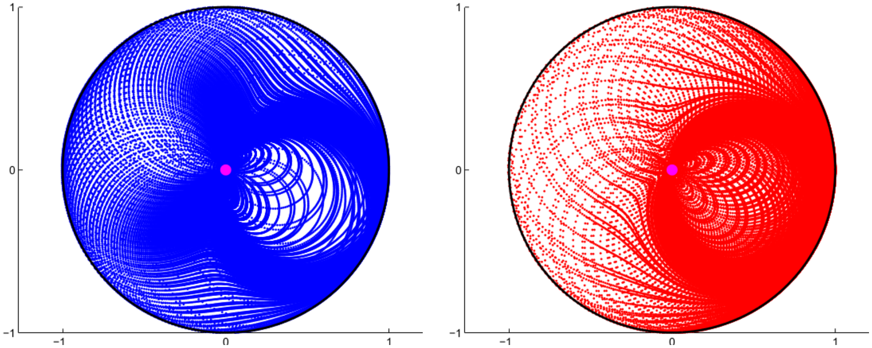


Numerical results for $\ell \in (\pi/k; 2\pi/k)$

- Asympt. curves of $L \mapsto R(L)$, $T(L)$ for $L \in (0; 100)$ and $\ell = 1.7$ ($k/\alpha \notin \mathbb{Q}$).



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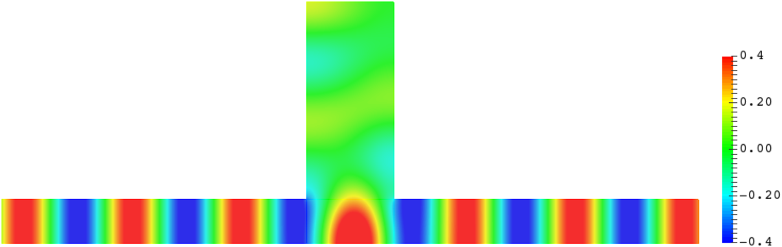
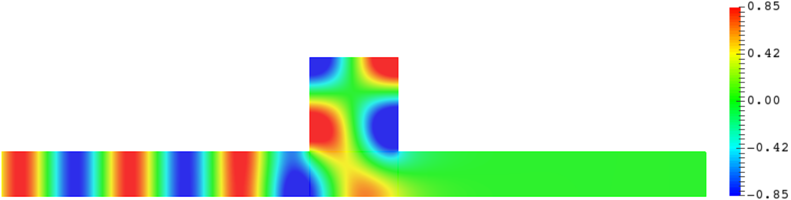
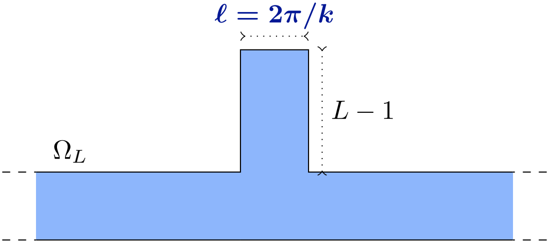


- Completely reflective geometry ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).



The special case $\ell = 2\pi/k$

- We did $\ell \in (0;\pi/k)$, $\ell \in (\pi/k;2\pi/k)$. Now set $\boldsymbol{\ell = 2\pi/k}$ in the geometry



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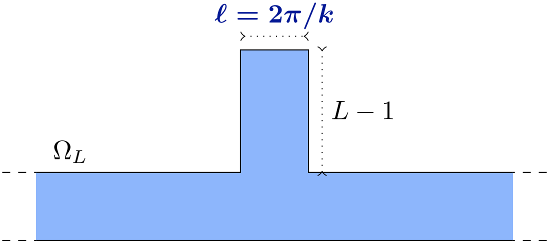

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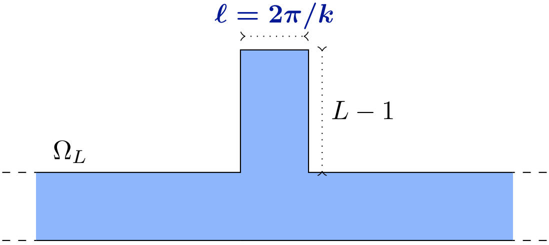

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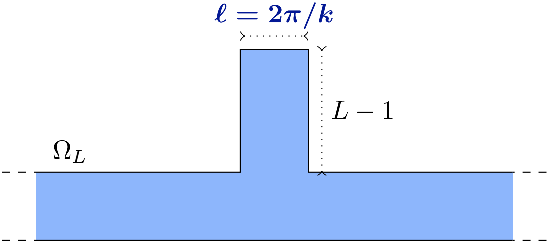

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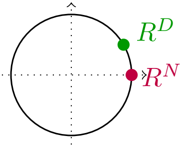

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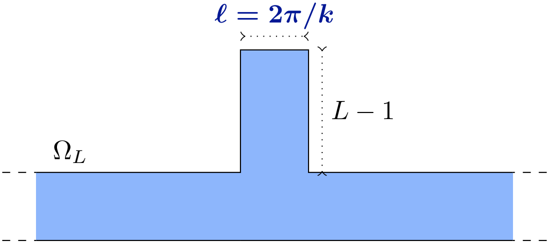

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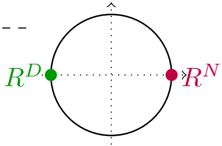

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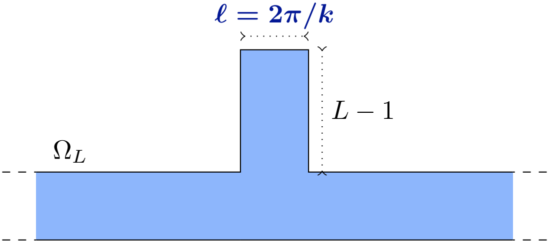

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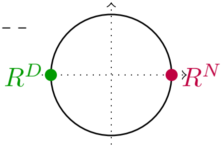

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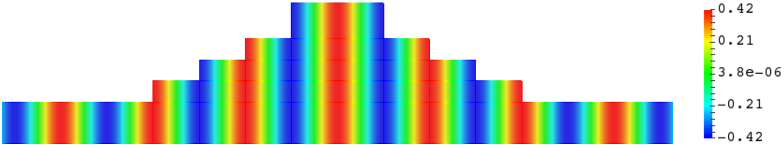
There is a sequence (L_n) such that $T = 1$ (perfect invisibility)

The special case $\ell = 2\pi/k$ - perfect invisibility

- Works also in the geometry below (L is the height of the central branch).

- Perfectly invisible defect ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).



- Reference waveguide ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).

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- Set $\gamma = \sqrt{\pi^2 - k^2}$, $\quad w_1^\pm = \dfrac{e^{\mp ikx}}{\sqrt{2k}}$ $\quad$ and $\quad \boxed{w_2^\pm = \dfrac{e^{-\gamma x} \mp i e^{\gamma x}}{\sqrt{2\gamma}} \cos(\pi y)}$.
- The Neumann problem in ω_L admits the solutions

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Proof: $\mathfrak{s}_{22} = -1 \Rightarrow \mathfrak{s}_{21} = 0$ *($\mathbb{S}$ is unitary) and* $u_2 \in \mathrm{H}^1(\omega_L)$ *is a trapped mode.*

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$$\Rightarrow \mathfrak{s}_{11} = 1 \qquad \Rightarrow |\mathfrak{s}_{22}| = 1, \qquad \forall L > 1.$$

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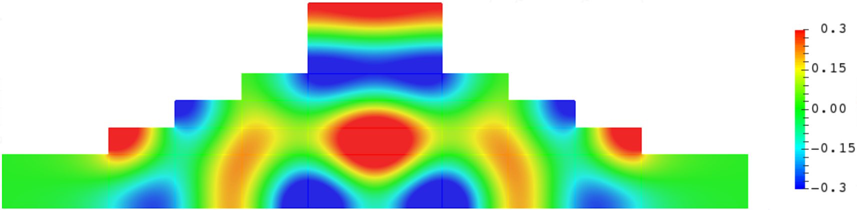
There is a sequence (L_n) such that trapped modes exist in ω_L.

The special case $\ell = 2\pi/k$ - trapped modes

► Symmetry argument w.r.t. $(Oy) \Rightarrow$ existence of trapped modes in Ω_L. It works also in the geometry below (L is the height of the central branch).



Non zero $v \in \mathrm{H}^1(\Omega_L)$ satisfying $\Delta v + k^2 v = 0$ in Ω_L, $\partial_n v = 0$ on $\partial\Omega_L$.

Conclusion

What we did

- ♠ We explained how to construct waveguides such that $R = 0$, $T = 0$ (the method works also for the Dirichlet problem) or $T = 1$.
- ♠ We showed how to construct waveguides supporting trapped modes.

Future work

1) When the symmetry is broken, we can still do things...
2) Can we work at higher frequencies (several propagative modes)?
3) Can we deal with multi-channel waveguides?
4) For a given perturbation, can we study the frequencies such that invisibility holds? $\Rightarrow$ See A.-S. Bonnet-Ben Dhia's talk on Wed..

Thank you for your attention!!!