Invisibility and perfect reflectivity in waveguides with finite length branches

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Abstract

We study a time-harmonic waves problem in a 2D waveguide. The geometry is symmetric with respect to an axis orthogonal to the direction of propagation of waves. Moreover, the waveguide contains one branch of finite length L. We analyse the behaviour of the complex scattering coefficients $\mathcal{R}$, $\mathcal{T}$ as L goes to $+\infty$ and we exhibit situations where non reflectivity ($\mathcal{R}=0$, $|\mathcal{T}|=1$), perfect reflectivity ($|\mathcal{R}|=1$, $\mathcal{T}=0$) or perfect invisibility ($\mathcal{R}=0$, $\mathcal{T}=1$) hold.

Keywords: waveguides, invisibility, scattering matrix, asymptotic analysis

1 Introduction

In recent articles [1, 2], an approach has been proposed to construct acoustic waveguides different from the reference (straight) geometry where the incident waves produce only exponentially decaying scattered fields. The idea is to perturb the walls of the reference domain in a clever way mimicking the proof of the implicit function theorem. In this work, we wish to obtain a similar result following a different path.

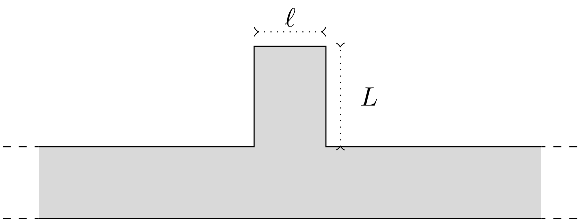

Figure 1: Geometry of Ω_L.

Consider some $\ell > 0$. For $L > 0$, set

$$\Omega_L := \{(x,y) \in \mathbb{R}\times(0;1) \cup (-\frac{\ell}{2};\frac{\ell}{2})\times[1;1+L)\}.$$

Propagation of acoustic waves in the waveguide Ω_L with sound hard walls leads to study the problem

$$\left|\begin{array}{rcll} \Delta v + k^2 v &=& 0 & \text{in } \Omega_L \\ \partial_n v &=& 0 & \text{on } \partial\Omega_L. \end{array}\right. \qquad (1)$$

We assume that $k \in (0;\pi)$ so that only two waves $w^{\pm}(x,y) = e^{\pm ikx}/\sqrt{2k}$ can propagate in Ω_L. The scattering of the wave w^+ coming from the left yields a solution of (1) such that

$$v = \left|\begin{array}{ll} w^+ + \mathcal{R}\,w^- + \dots, & \text{for } x < -\ell \\ \mathcal{T}\,w^+ + \dots, & \text{for } x > \ell. \end{array}\right. \qquad (2)$$

Here the dots correspond to a superposition of modes which are exponentially decaying at $\pm\infty$. In (2), the reflection coefficient $\mathcal{R} \in \mathbb{C}$ and transmission coefficient $\mathcal{T} \in \mathbb{C}$ are uniquely defined. Moreover, energy conservation writes

$$|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1. \qquad (3)$$

In the following, we explain how to find ℓ, L such that $\mathcal{R} = 0$, $|\mathcal{T}| = 1$ (non reflectivity); $|\mathcal{R}| = 1$, $\mathcal{T} = 0$ (perfect reflectivity); or $\mathcal{R} = 0$, $\mathcal{T} = 1$ (perfect invisibility). To get such particular values, we will use the symmetry of the geometry with respect to the (Oy) axis.

2 Half-waveguide problems

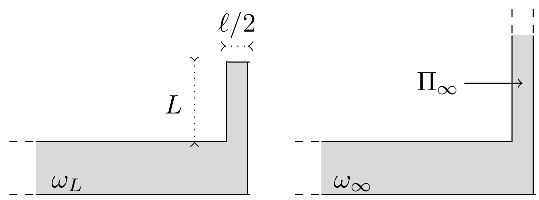

Figure 2: Domains ω_L (left) and ω_∞ (right).

Set $\omega_L := \{(x,y) \in \Omega_L \,|\, x < 0\}$. Introduce the problem with Neumann boundary conditions

$$\left|\begin{array}{rcll} \Delta u + k^2 u &=& 0 & \text{in } \omega_L \\ \partial_n u &=& 0 & \text{on } \partial\omega_L \end{array}\right. \qquad (4)$$

and the one with mixed boundary conditions

$$\left|\begin{array}{rcll} \Delta U + k^2 U &=& 0 & \text{in } \omega_L \\ \partial_n U &=& 0 & \text{on } \partial\omega_L \cap \partial\Omega_L \\ U &=& 0 & \text{on } \{0\}\times(0;L). \end{array}\right. \qquad (5)$$

Problems (4) and (5) respectively admit the solutions $u = w^+ + r\,w^- + \ldots$ and $U = w^+ + R\,w^- + \ldots$ where $r, R \in \mathbb{C}$ are uniquely defined and where the dots stand for terms which are exponentially decaying at $-\infty$. Due to conservation of energy, one has

$$|r| = |R| = 1. \qquad (6)$$

Besides, a simple analysis shows that the coefficients $\mathcal{R}$, $\mathcal{T}$ appearing in (2) are such that

$$\mathcal{R} = \frac{r+R}{2} \qquad \text{and} \qquad \mathcal{T} = \frac{r-R}{2}. \qquad (7)$$

3 Non reflection and perfect reflection

Now, we study the asymptotic behaviour of $\mathcal{R}$, $\mathcal{T}$ as $L \to +\infty$. To proceed, we use (7) and work with r, R. The behaviours of r, R as $L \to +\infty$ depend on the properties of the equivalents of Problems (4), (5) set in the limit geometry ω_∞ obtained from ω_L making formally $L \to +\infty$ (see Figure 2, right). In particular, the number of propagating waves existing in the vertical branch Π_∞ of ω_∞ plays a crucial role.

⋆ Assume that $\ell \in (0; \pi/k)$ ($\ell/2$ is the width of Π_∞). Then for Problem (5) set in ω_∞, propagative modes in Π_∞ do not exist. Due to this property, we can show that $R = R_\infty + \ldots$ where $R_\infty \in \mathbb{S} := \{z \in \mathbb{C}\,|\,|z| = 1\}$ is a constant. Here the dots correspond to a remainder which is exponentially small as $L \to +\infty$. For Problem (4) (with Neumann boundary condition) set in ω_∞, one propagative mode exists in Π_∞. And because of the reflection of this mode on the wall at $y = L$, the coefficient r does not converge as $L \to +\infty$. More precisely, we can prove that it admits the expansion $r = r_{\text{asy}}(L) + \ldots$ where $r_{\text{asy}}(L)$ is a term whose dependence with respect to L can be obtained explicitly and which runs periodically on $\mathbb{S}$ as $L \to +\infty$. Again the dots stand for an exponentially small remainder.

Imagine that we want to have $\mathcal{R} = 0$ (non reflectivity). According to (7), we must impose $r = -R$. Relations (6) guarantee that for all $L > 0$, both r and R are located on the unit circle $\mathbb{S}$. But R tends to a constant $R_\infty \in \mathbb{S}$ while r runs continuously on $\mathbb{S}$ as $L \to +\infty$. This proves the existence of L such that $r = -R$ and so $\mathcal{R} = 0$. This also shows that there are some L such that $r = R$ and so $\mathcal{T} = 0$ (perfect reflectivity). Numerics of Figure 3 confirm these results. To obtain perfect invisibility, *i.e.* $\mathcal{T} = 1$, we must impose both $r = 1$ and $R = -1$. This requires a bit more work but can be achieved.

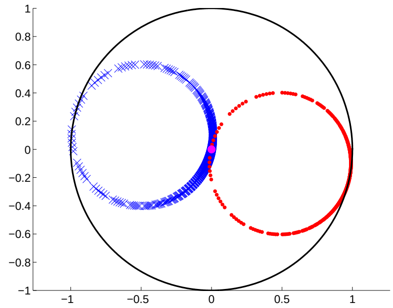

Figure 3: Numerical approximations of $\mathcal{R}$ (×) and $\mathcal{T}$ (•). We take $k = 3$, $\ell = 1$ and $L \in (1; 9)$. As predicted, we obtain circles of radius 1/2 passing through zero.

⋆ When $\ell \in (\pi/k; 2\pi/k)$, both for Problem (5) and (4) set in ω_∞, one propagative mode exists in Π_∞. Then, we can prove that $R = R_{\text{asy}}(L) + \ldots$ and $r = r_{\text{asy}}(L) + \ldots$ where $R_{\text{asy}}(L)$, $r_{\text{asy}}(L)$ are explicitly known coefficients which run periodically on $\mathbb{S}$ with different speeds V, v. This is enough to conclude that $\mathcal{R} = 0$ or $\mathcal{T} = 0$ for an infinite number of L. However, compared to the case $\ell \in (0; \pi/k)$, the behaviour of $\mathcal{R}$ and $\mathcal{T}$ can be much more complex, especially when v/V is not a rational number (see Figure 4).

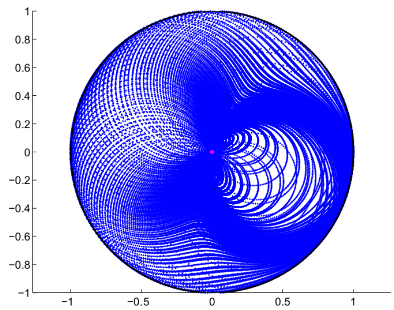

Figure 4: Numerical approximation of $\mathcal{R}$ for $k = 3$, $\ell = 1.7$ and $L \in (1; 99)$.

References

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