WAVES 2019

Exact zero transmission during the Fano resonance phenomenon in non symmetric waveguides

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Coll. with S. A. Nazarov[2].

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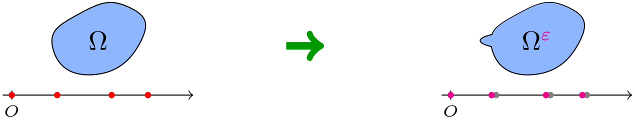
VIENNA, 28/08/2019

Introduction

- Consider the eigenvalue problem

$$\left| \begin{array}{rcll} \Delta v + \lambda v & = & 0 & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

- In bounded domains, small smooth perturbations of the geometry slightly shift the spectrum in $\mathbb{R}$ (eigenvalues remain eigenvalues).



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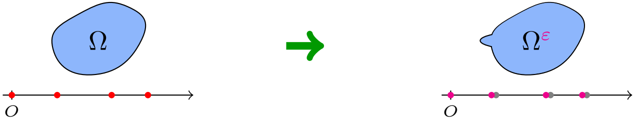

- In unbounded waveguides, small perturbations of the geom. transform eigenvalues embedded in the continuous spectrum into complex resonances.

→ See Aslanyan, Parnovski, Vassiliev, Q. J. Mech. Appl. Math., 00.

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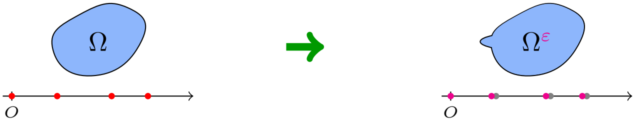

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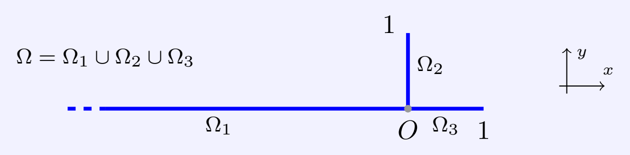
What is the influence of these resonances on the scattering properties?

A 1D toy problem

- First, we consider a simple 1D problem.

A 1D toy problem



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$$\varphi'' + k^2\varphi = 0 \text{ in } \Omega, \quad \left|\begin{array}{l} \varphi_1 = \varphi_2 = \varphi_3 \text{ at } O \\ \varphi_1' = \varphi_2' + \varphi_3' \text{ at } O \\ \varphi_2' = \varphi_3' = 0 \text{ on } \partial\Omega \end{array}\right. \quad \text{with } \underbrace{\varphi_1 = e^{ikx} + R\, e^{-ikx}}_{\text{radiation condition}},\ R \in \mathbb{C}.$$

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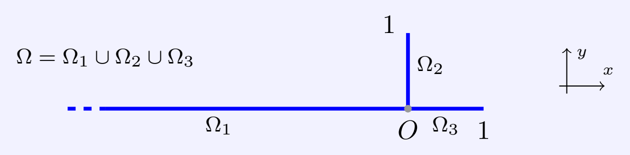

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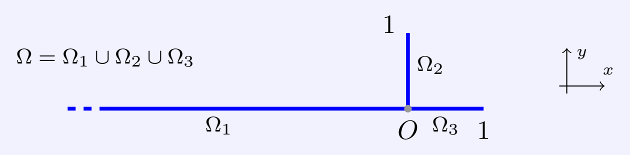

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- Well-posedness $\Leftrightarrow$ invertibility of a 3×3 system $\mathbb{M}\Phi = F$.

- Uniqueness $\Leftrightarrow$ $k \notin (2\mathbb{N}+1)\pi/2$. Existence for all $k \in \mathbb{R}$ ($F \in \ker \mathbb{M}^{\perp}$).

$$R = \frac{\cos(k) + 2i\sin(k)}{\cos(k) - 2i\sin(k)}$$

A 1D toy problem

- We perturb the geometry: $\Omega^\varepsilon = \Omega_1 \cup \Omega_2 \cup \Omega_3^\varepsilon$ with $\Omega_3^\varepsilon = (0; 1+\varepsilon)$.

$$R^\varepsilon = \frac{\cos(k)\cos(k(1+\varepsilon)) + i\sin(k(2+\varepsilon))}{\cos(k)\cos(k(1+\varepsilon)) - i\sin(k(2+\varepsilon))}.$$

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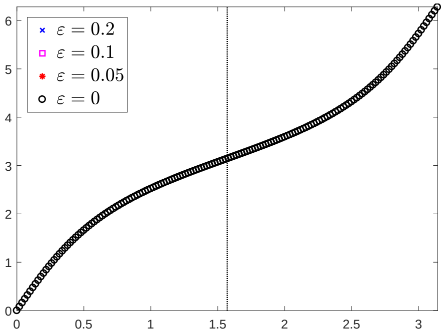

Figure: $k \mapsto \boldsymbol{\theta^\varepsilon(k)}$ for several ε (non uniqueness for $\varepsilon = 0$, $k = \pi/2$).

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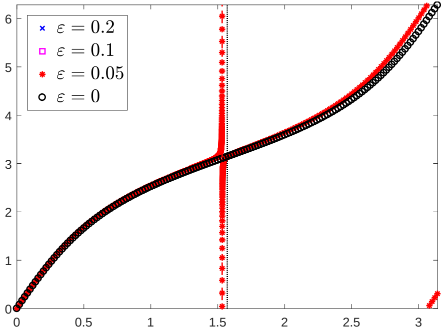

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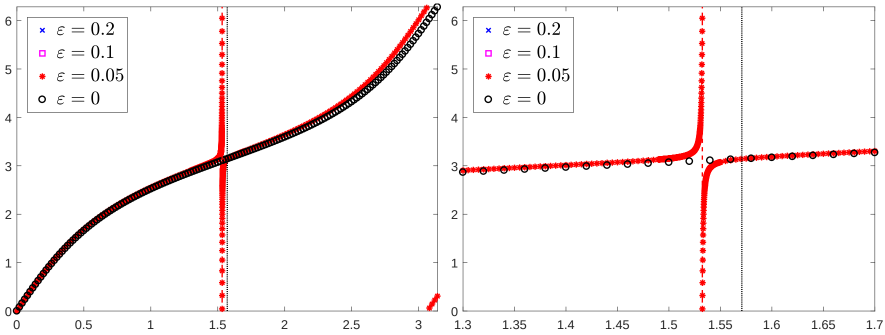

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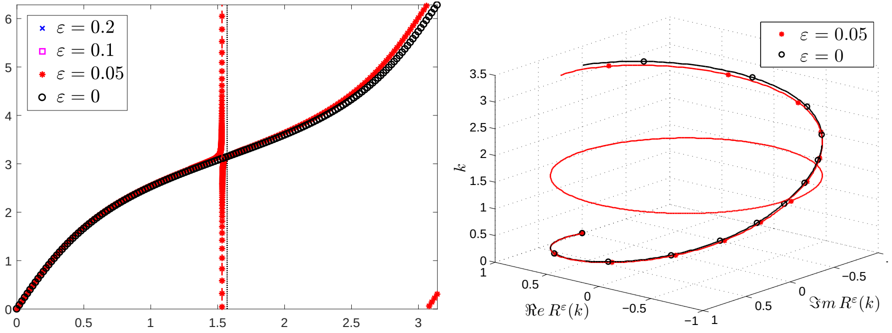

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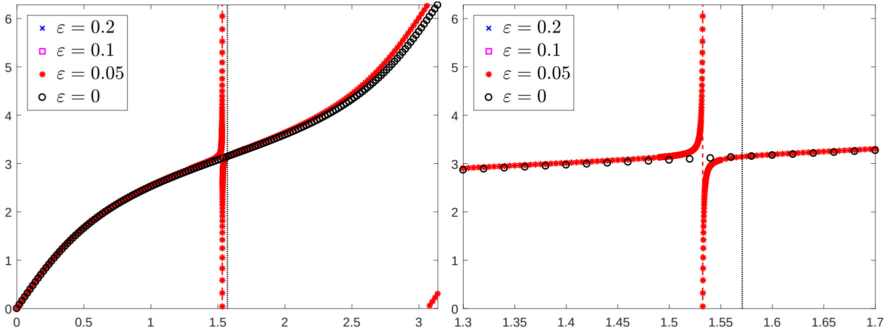

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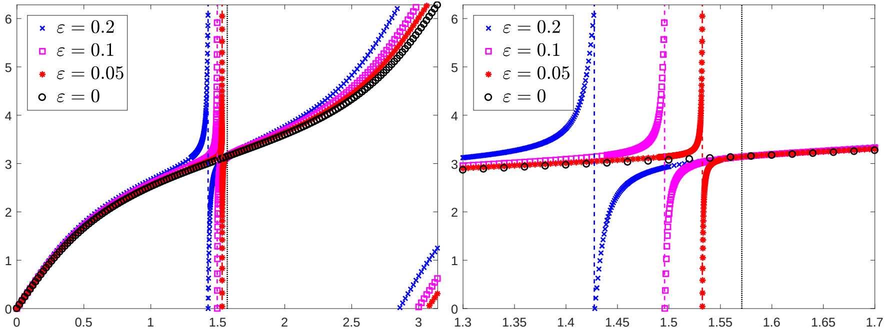

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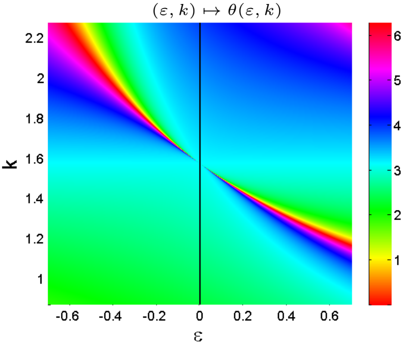

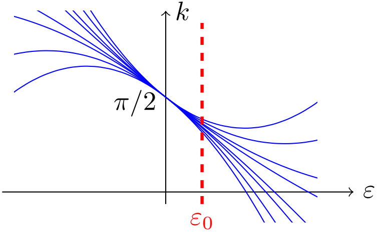

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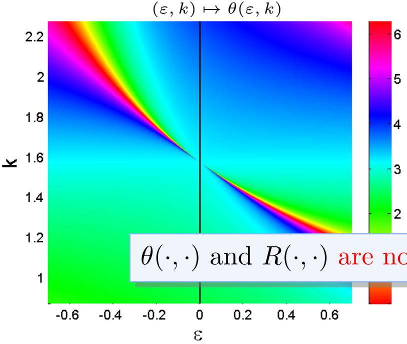

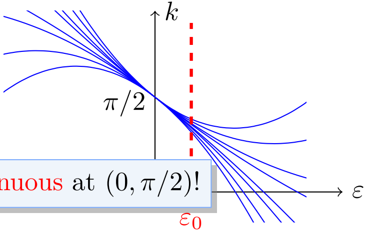

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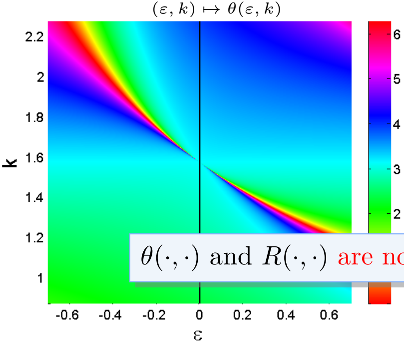

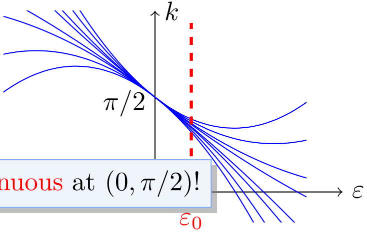

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Goals of the talk

1) Prove a similar Fano resonance phenomenon in waveguides.
2) Show that zero transmission always occurs during the phenomenon.

Outline of the talk

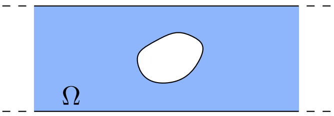
1. The Fano resonance in waveguides

2. Zero transmission

3. Numerical experiments

Setting

- Let Ω be a waveguide which coincides with $\{(x,y) \in \mathbb{R} \times (0;1)\}$ outside a compact region. We consider the problem



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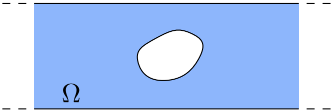

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$$(\mathscr{P})\left|\begin{array}{rcll}\multicolumn{4}{l}{\text{Find } v \text{ s.t. } v-v_i \text{ is outgoing and}}\\ \Delta v+\lambda v&=&0&\text{in }\Omega,\\ \partial_n v&=&0&\text{on }\partial\Omega.\end{array}\right.$$

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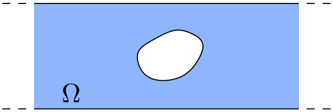

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- For this problem with $k := \sqrt{\lambda} \in (0;\pi)$, the modes are

$$\begin{array}{l|l} \text{Propagating} & w_\pm(x,y) = e^{\pm ikx}, \\ \text{Evanescent} & w_n^\pm(x,y) = e^{\mp\beta_n x}\cos(n\pi y), \quad \beta_n = \sqrt{n^2\pi^2 - \lambda},\ n \geq 1. \end{array}$$

Scattering problem

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- For $\boldsymbol{v_i = w_\pm}$, $(\mathscr{P})$ admits the scattering solutions (existence)

$$v_+ = \left| \begin{array}{l} w_+ + R_+ w_- + \dots \\ \quad\ T\ \ w_+ + \dots \end{array}\right. \qquad v_- = \left| \begin{array}{l} \quad\ T\ \ w_- + \dots \\ w_- + R_- w_+ + \dots \end{array}\right. \qquad \begin{array}{l} \text{for } x < 0 \\ \text{for } x > 0 \end{array}$$

where $R_\pm$, $T \in \mathbb{C}$ and $\dots$ are exponentially decaying terms.

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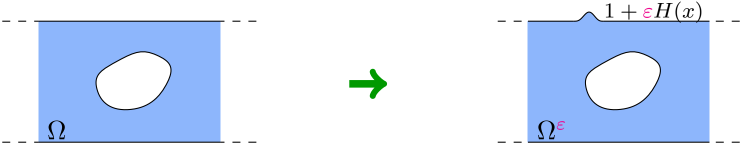
where $R_\pm$, $T \in \mathbb{C}$ and $\dots$ are exponentially decaying terms.

- The scattering matrix

$$\mathbb{S} = \begin{pmatrix} R_+ & T \\ T & R_- \end{pmatrix}$$

is uniquely defined (even for $\lambda = \lambda^0$), unitary ($\mathbb{S}\overline{\mathbb{S}}^\top = \mathrm{Id}$) and symmetric.

- We perturb slightly ($\varepsilon \geq 0$ is small) the geometry



Locally $\partial\Omega^\varepsilon$ coincides with the graph of $x \mapsto 1 + \varepsilon H(x)$,
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Small perturbation of the geometry 1/2

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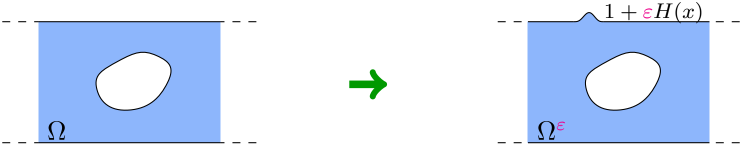

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The following theorem describes the behaviour of $(\varepsilon, \lambda) \mapsto \mathbb{S}(\varepsilon, \lambda)$ in a neighbourhood of $(0, \lambda^0)$ where trapped modes exist.

THEOREM: Set $\mathbb{S}^0 = \mathbb{S}(0, \lambda^0)$. There is $\lambda'_p \in \mathbb{R}$ such that when $\varepsilon \to 0$,

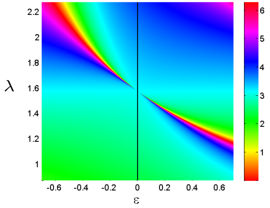
$$\mathbb{S}(\varepsilon, \lambda^0 + \varepsilon\lambda') = \mathbb{S}^{\mathbf{0}} + O(\varepsilon) \qquad \text{for } \lambda' \neq \lambda'_p,$$

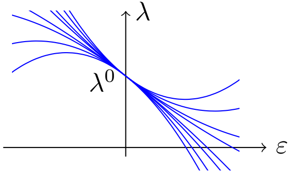
and, for any $\mu \in \mathbb{R}$,

$$\mathbb{S}(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu) = \mathbb{S}^{\mathbf{0}} + \frac{\boldsymbol{\tau}^\top \boldsymbol{\tau}}{i\tilde{\boldsymbol{\mu}} - |\boldsymbol{\tau}|^2/2} + O(\varepsilon).$$

Here $\tau \in \mathbb{C}^2$ depends only on Ω and $\tilde{\mu} = A\mu + B$ for some $A \neq 0$, $B \in \mathbb{R}$.

- Similar to the 1D picture:





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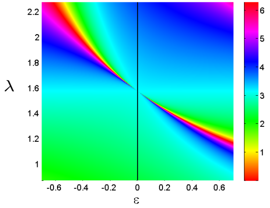

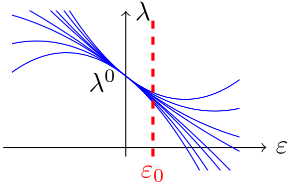

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COMMENTS:

- $\mathbb{S}(\cdot,\cdot)$ is not continuous at $(0,\lambda^0)$.

- For a small given ε_0, the map $\lambda \mapsto \mathbb{S}(\varepsilon_0,\lambda)$ varies quickly at $\lambda^0+\varepsilon^0\lambda'_p$.

- Under certain conditions on H, the variation can be even quicker...

Here $\tau \in \mathbb{C}^2$ depends only on Ω and $\tilde{\mu} = A\mu + B$ for some $A \neq 0$, $B \in \mathbb{R}$.

- Similar to the 1D picture:

Small perturbation of the geometry 2/2

Comments:

- $\mathbb{S}(\cdot,\cdot)$ is not continuous at $(0, \lambda^0)$.

- For a small given ε_0, the map $\lambda \mapsto \mathbb{S}(\varepsilon_0, \lambda)$ varies quickly at $\lambda^0 + \varepsilon^0 \lambda'_p$.

- Under certain conditions on H, the variation can be even quicker...

Ingredients of the proof:

- Use weighted Sobolev spaces with detached asymptotics to define scattering solutions with non standard radiation conditions.

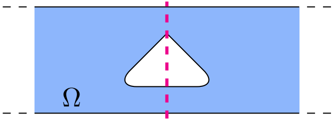
- Define an augmented scattering matrix $\mathfrak{S}$ (`Nazarov, Plamenevsky, 94`).

- Compute an asymptotic expansion of $\mathfrak{S}$ which is smooth at $(0, \lambda^0)$ because uniqueness holds for the problem with non standard radiation conditions.

- Use the connection existing between $\mathbb{S}$ and $\mathfrak{S}$ to get an expansion for $\mathbb{S}$.

Symmetric waveguide 1/2

- We assume that Ω is symmetric with respect to the (Oy) axis.



$$\left|\begin{array}{rcl} \Delta v + \lambda v &=& 0 \quad \text{in } \Omega, \\ \partial_n v &=& 0 \quad \text{on } \partial\Omega. \end{array}\right.$$

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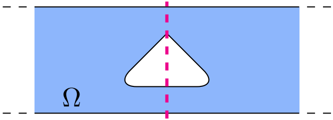
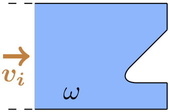


$$\left|\begin{array}{rcll} \Delta v + \lambda v & = & 0 & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega. \end{array}\right.$$

- Introduce the two half-waveguide problems



$$\left|\begin{array}{rl} \Delta u + \lambda u = 0 & \text{in } \omega \\ \partial_n u = 0 & \text{on } \partial\omega \end{array}\right.$$

$$\left|\begin{array}{rl} \Delta U + \lambda U = 0 & \text{in } \omega \\ \partial_n U = 0 & \text{on } \partial\omega \setminus \partial\Omega \\ U = 0 & \text{on } \partial\omega \cap \partial\Omega. \end{array}\right.$$

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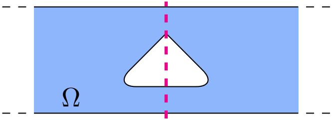

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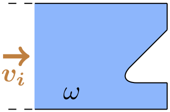

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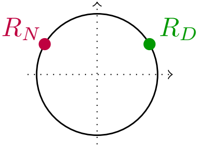
- They admit the solutions

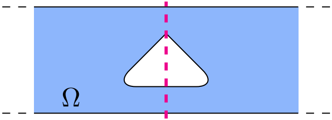


$$\left|\begin{array}{rcl} u & = & e^{ikx} + R_N\, e^{-ikx} + \dots \\ U & = & e^{ikx} + R_D\, e^{-ikx} + \dots \end{array}\right.$$

with $|R_N| = |R_D| = 1$ (conservation of energy).

Symmetric waveguide 1/2

- We assume that Ω is symmetric with respect to the (Oy) axis.



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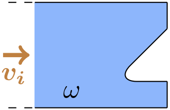

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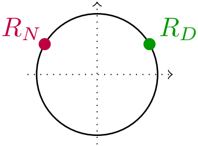

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with $|R_N| = |R_D| = 1$ (conservation of energy).

- One can prove that $\boxed{R_\pm = \dfrac{R_N + R_D}{2}}$ and $\boxed{T = \dfrac{R_N - R_D}{2}.}$

Symmetric waveguide 2/2

$$\boxed{R_{\pm} = \frac{R_N + R_D}{2}} \qquad \boxed{T = \frac{R_N - R_D}{2}}$$

- To set ideas, we assume that eigenfunctions are symmetric w.r.t. (Oy).
  $\Rightarrow$ They are eigenfunctions for the pb with Neumann B.Cs.

i) λ^0 is not an eigenvalue for the pb with Dirichlet condition. This implies

$$|R_D(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu) - R_D(0, \lambda^0)| \le C\,\varepsilon, \qquad \forall \varepsilon \in (0; \varepsilon_0],\ \mu \in [-c\varepsilon^{-1}; c\varepsilon].$$

ii) $\mu \mapsto R_N(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu)$ rushes on the unit circle for $\mu \in [-c\varepsilon^{-1}; c\varepsilon]$.

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Proposition:

$\exists \lambda^\varepsilon$, with $\lambda^\varepsilon - \lambda^0 = O(\varepsilon)$, s.t. for ε small, $R_{\pm}(\varepsilon, \lambda^\varepsilon) = 0$ (zero reflection).

$\exists \tilde{\lambda}^\varepsilon$, with $\tilde{\lambda}^\varepsilon - \lambda^0 = O(\varepsilon)$, s.t. for ε small, $T(\varepsilon, \tilde{\lambda}^\varepsilon) = 0$ (zero transmission).

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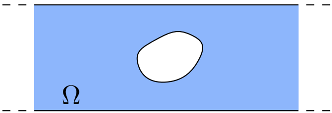
$\rightarrow$ Similar results in `Shipman and Tu, SIAM Appl. Math, 2012`. We use a different approach and consider a perturbation of the geometry.

Non symmetric waveguide 1/2



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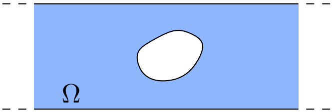

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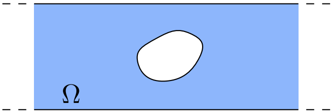
`Proof. 1)` Set $T^\varepsilon(\mu) = T(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu)$. The expansion of $\mathbb{S}$ yields

$$|T^\varepsilon(\mu) - T^{\text{asy}}(\mu)| \le C\,\varepsilon \qquad \text{with} \qquad T^{\text{asy}}(\mu) = T^0 + \frac{ab}{i\tilde{\mu} - (|a|^2 + |b|^2)/2},$$

where a, b are some constants.



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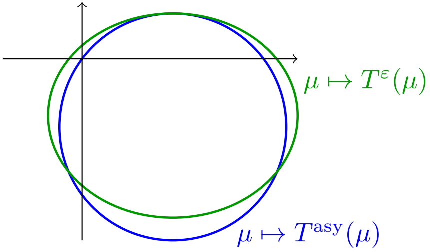
where a, b are some constants.

`2)` One can check that $\{T^{\text{asy}}(\mu)\,|\,\mu \in \mathbb{R}\}$ is a circle passing through zero.

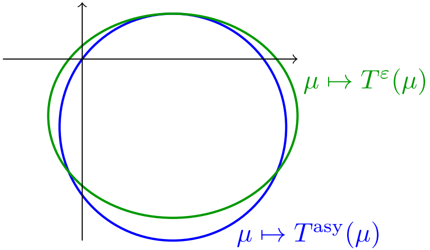


3) If $\mu \mapsto T^{\varepsilon}(\mu)$ does not pass through zero, $\mu \mapsto 2\,\mathrm{phase}(T^{\varepsilon}(\mu))$ varies quickly. One can show that this contradicts the identity

$$T^{\varepsilon}(\mu)/\overline{T^{\varepsilon}(\mu)} = -R_{+}^{\varepsilon}(\mu)/\overline{R_{-}^{\varepsilon}(\mu)}$$

which is a consequence of the unitarity of $\mathbb{S}^{\varepsilon}(\mu)$. $\square$

Non symmetric waveguide



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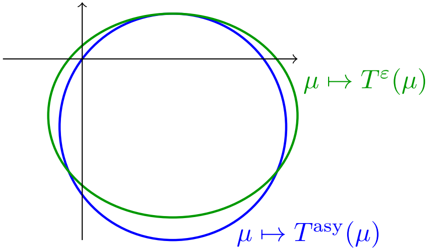
The unitarity structure of $\mathbb{S}$ is the key to conclude.

Non symmetric waveguide



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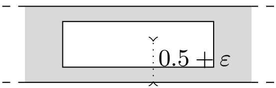
$\rightarrow$ Similar idea in Lee, Phys. Rev. Lett., 99 using a perturbation argument.

Symmetric waveguide

- Numerics using FE methods (`Freefem++`) with DtN maps or PMLs.

Symmetric waveguide

- Left: waveguide. Right: eigenfunction for $\varepsilon = 0$ and $k^0 := \sqrt{\lambda^0} \approx 2.42$.



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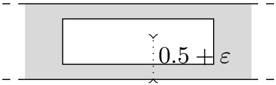
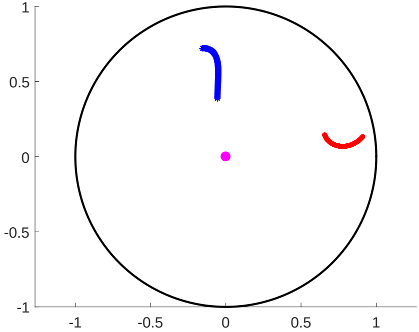


- **Scattering coefficients** for $k \in (2.2; 2.7)$.

No shift ($\boldsymbol{\varepsilon = 0}$)



$k \mapsto R_{\pm}(0,k)$ $k \mapsto T(0,k)$

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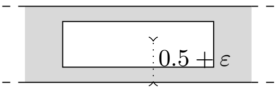

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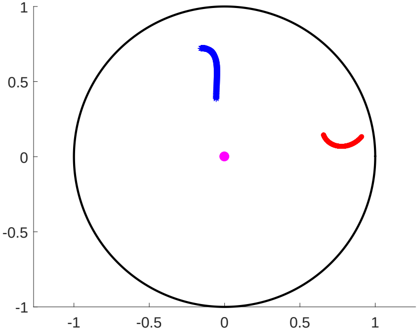
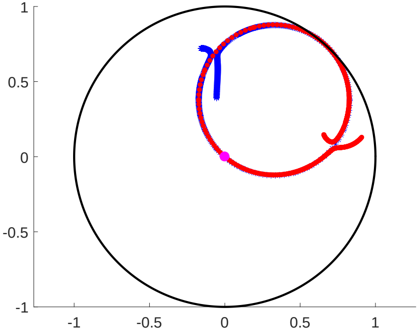


$k \mapsto R_{\pm}(0, k)$ $k \mapsto T(0, k)$

Small shift $(\boldsymbol{\varepsilon > 0})$

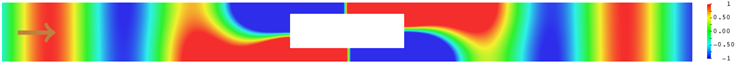


$k \mapsto R_{\pm}(0.02, k)$ $k \mapsto T(0.02, k)$

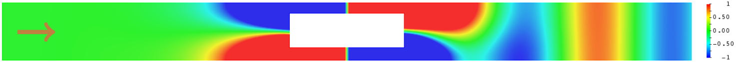
Symmetric waveguide

- Example of setting where $R_{\pm}(\varepsilon, \lambda^{\varepsilon}) = 0$ (zero reflection).

$\Re e\, v_+$



$\Re e\, (v_+ - w_+)$



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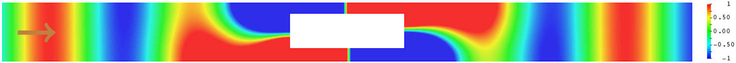

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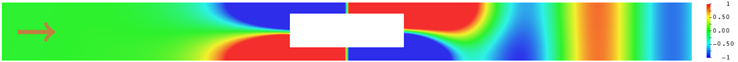
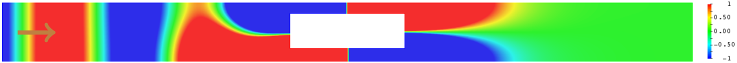


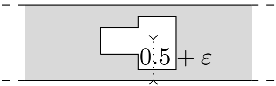
- Example of setting where $T(\varepsilon, \tilde{\lambda}^{\varepsilon}) = 0$ (zero transmission).

$\Re e\, v_+$



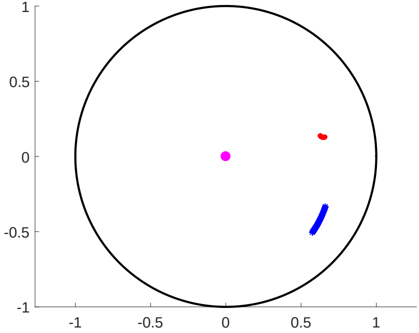
Non symmetric waveguide

- Left: waveguide. Right: eigenfunction for $\varepsilon = 0$ and $k^0 := \sqrt{\lambda^0} \approx 2.03$.



- **Scattering coefficients** for $k \in (1.8; 2.2)$.

No shift $(\boldsymbol{\varepsilon = 0})$



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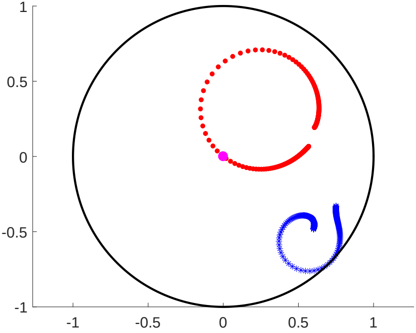


$k \mapsto R_+(0.1, k)$ $k \mapsto T(0.1, k)$

Non symmetric waveguide

- Example of setting where $\boxed{T(\varepsilon, \lambda^{\varepsilon}) = 0}$ (zero transmission).

$\Re e\, v_+$

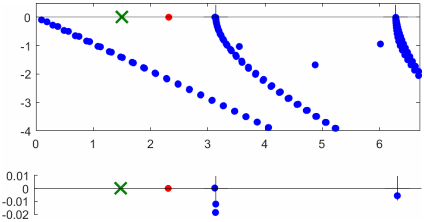
Frequency behaviour

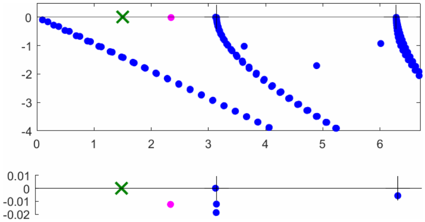
No shift ($\varepsilon = \mathbf{0}$) | Small shift ($\varepsilon > \mathbf{0}$)

- $k \mapsto \Re e\, v_+(k)$

- **Complex spectrum** computed with PMLs (we zoom at the real axis).

  • Trapped mode • Complex resonance





1. The Fano resonance in waveguides
2. Zero transmission
3. Numerical experiments

Conclusion

What we did

- We proved the Fano resonance phenomenon in a 2D waveguide.

  If trapped modes exist for $(\varepsilon, \lambda) = (0, \lambda^0)$*, then for* $\varepsilon > 0$ *small,* $\lambda \mapsto \mathbb{S}(\varepsilon, \lambda)$ *has a quick variation at* λ^0*.* `Symmetry is not needed.`

- If Ω `symmetric w.r.t.` (Oy), zero reflection, zero transmission occur.
  If Ω `not symmetric`, zero transmission occurs.

- The technique works with other B.C. (Dirichlet, ...), other kinds of perturbation (penetrable obstacles, ...), in any dimension.

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Future work

1) Is there zero reflection/zero transmission for $k > \pi$ (monomode regime was essential in the mechanism)?
2) What happens if λ^0 is not a simple eigenvalue?

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- S.P. Shipman and S. Venakides. Resonant transmission near nonrobust periodic slab modes. *Phys. Rev. E*, 71(2):026611, 2005.

Thank you for your attention!