Waves 2022

Design of a mode converter using thin resonant ligaments

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Coll. with J. Heleine[2], S.A. Nazarov[3].

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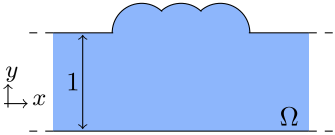
Palaiseau, 27/07/2022

- We consider the propagation of waves in a 2D acoustic waveguide (also relevant in optics, microwaves, water-waves theory,...).



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- We fix $k \in (\pi; 2\pi)$ so that two modes can propagate:

$$\left| \begin{array}{lll} w_1^{\pm}(x,y) = e^{\pm i\beta_1 x}\varphi_1(y), & \varphi_1(y) = \beta_1^{-1/2}, & \beta_1 = k \\ w_2^{\pm}(x,y) = e^{\pm i\beta_2 x}\varphi_2(y), & \varphi_2(y) = \beta_2^{-1/2}\sqrt{2}\cos(\pi y), & \beta_2 = \sqrt{k^2-\pi^2}. \end{array} \right.$$

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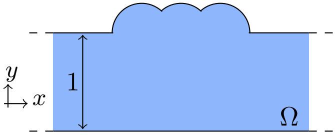

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- The scattering of the incident waves w_1^+, w_2^+ yields the solutions

$$u_1(x,y)=\left|\begin{array}{rl} w_1^+(x,y)+\sum_{j=1}^2 r_{1j}w_j^-(x,y)+\dots & \text{on the left}\\ \sum_{j=1}^2 t_{1j}w_j^+(x,y)+\dots & \text{on the right}\end{array}\right.$$

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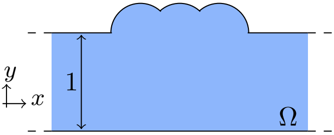

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... are expo. decaying terms

Introduction 2/2

- We define the reflection and transmission matrices

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2} \qquad T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2}.$$

- From conservation of energy, we have, for $i = 1, 2$,

$$\sum_{j=1}^{2} |r_{ij}|^2 + |t_{ij}|^2 = 1.$$

Goal of the talk

We wish to construct a **mode converter**, that is a geometry such that:

1) energy is completely transmitted
2) mode 1/2 is converted into mode 2/1

$$R \approx \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad T \approx \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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Difficulty: the scattering coefficients have a non explicit and non linear dependence wrt the geometry.

→ Due to local minima, we wish to avoid optimization methods (Lunéville et al. 98, Lebbe et al. 19).

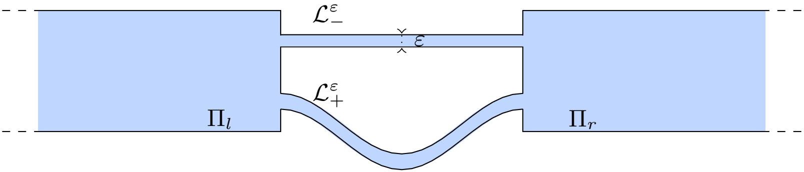
Outline of the talk

1 Choice of geometry

2 Asymptotic analysis in presence of thin resonators

3 Mode converter

- We decide to work in a geometry Ω^ε made of two half-waveguides connected by two thin ligaments of width $0 < \varepsilon \ll 1$.



$$\Omega^\varepsilon := \Pi_l \cup \mathcal{L}_-^\varepsilon \cup \mathcal{L}_+^\varepsilon \cup \Pi_r$$

- This may seem **paradoxical** because in general in this Ω^ε, energy is mostly backscattered:

$$R^\varepsilon \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad T^\varepsilon \approx \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ldots$$

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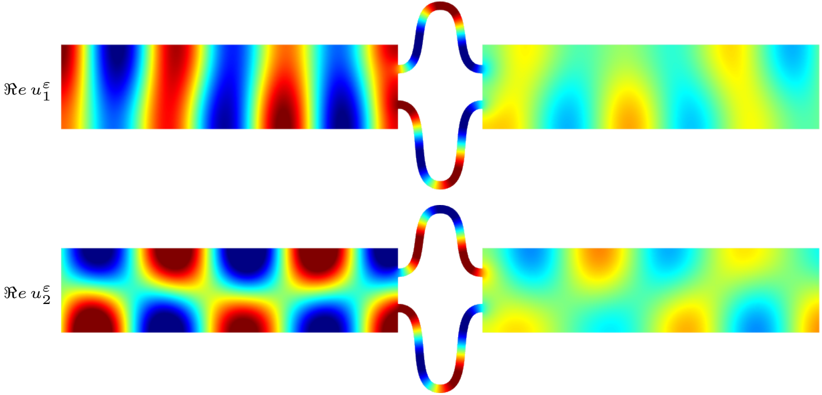

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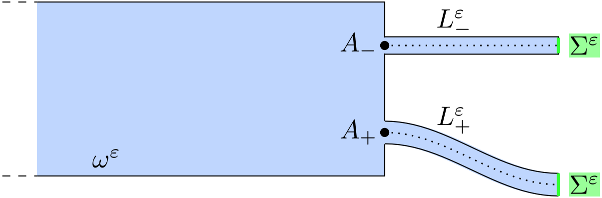
$$R^\varepsilon \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad T^\varepsilon \approx \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \dots$$

- We impose Ω^ε to be symmetric wrt (Oy). Set $\omega^\varepsilon := \{(x,y) \in \Omega^\varepsilon \,|\, x < 0\}$.



- In the half-waveguide ω^ε, consider the two problems with Artificial Boundary Conditions (ABC)

$$(\mathscr{P}^\varepsilon_N)\left|\begin{array}{rl} \Delta u^\varepsilon_N + k^2 u^\varepsilon_N = & 0 \text{ in } \omega^\varepsilon \\ \partial_\nu u^\varepsilon_N = & 0 \text{ on } \partial\omega^\varepsilon \setminus \Sigma^\varepsilon \\ \partial_\nu u^\varepsilon_N = & 0 \text{ on } \Sigma^\varepsilon \end{array}\right. \qquad (\mathscr{P}^\varepsilon_D)\left|\begin{array}{rl} \Delta u^\varepsilon_D + k^2 u^\varepsilon_D = & 0 \text{ in } \omega^\varepsilon \\ \partial_\nu u^\varepsilon_D = & 0 \text{ on } \partial\omega^\varepsilon \setminus \Sigma^\varepsilon \\ u^\varepsilon_D = & 0 \text{ on } \Sigma^\varepsilon \end{array}\right.$$

where $\Sigma^\varepsilon := \partial\omega^\varepsilon \setminus \partial\Omega^\varepsilon$.

Geometry 3/3

- For $(\mathscr{P}_i^\varepsilon)$, $i = N, D$, we have the solutions

$$\left| \begin{array}{lcl} u_{i1}^\varepsilon & = & w_1^+(x,y) + \sum_{j=1}^2 R_{i1j}^\varepsilon w_j^-(x,y) + \dots \\ u_{i2}^\varepsilon & = & w_2^+(x,y) + \sum_{j=1}^2 R_{i2j}^\varepsilon w_j^-(x,y) + \dots \end{array} \right.$$

- This defines two scattering matrices R_N^ε, $R_D^\varepsilon \in \mathbb{C}^{2\times 2}$ and there holds

$$R^\varepsilon = \frac{R_N^\varepsilon + R_D^\varepsilon}{2} \qquad T^\varepsilon = \frac{R_N^\varepsilon - R_D^\varepsilon}{2}.$$

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$$\boxed{R^\varepsilon = \frac{R_N^\varepsilon + R_D^\varepsilon}{2} \qquad\qquad T^\varepsilon = \frac{R_N^\varepsilon - R_D^\varepsilon}{2}.}$$

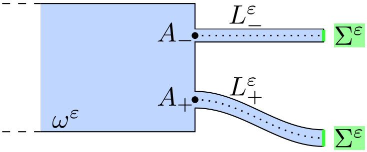
- Therefore our goal is to design ω^ε such that

$$R_N^\varepsilon \approx \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad\qquad R_D^\varepsilon \approx \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Setting

Main ingredient of our approach: outer resonators of width $\varepsilon \ll 1$.

- To set ideas, we work on the problem with Neumann ABC



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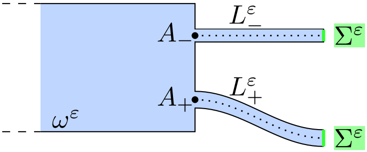

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- To simplify the presentation, we work with only one straight resonator

$$\omega^\varepsilon = \omega \cup L^\varepsilon \qquad \text{with} \qquad \left|\begin{array}{rl} \omega & := (-\infty; 0) \times (0; 1) \\ L^\varepsilon & := [0; \ell) \times (y_A - \varepsilon/2; y_A + \varepsilon/2) \end{array}\right.$$

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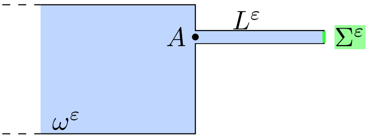

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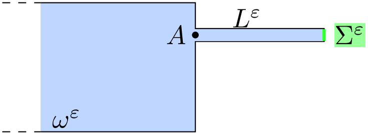

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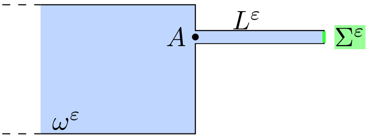

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First observations

- In the limit geometry ω, we have the solution

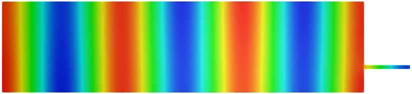
$$u := w_1^+ + \ w_1^- \\ \quad = w_1^+ + 1\, w_1^- + 0\, w_2^- .$$

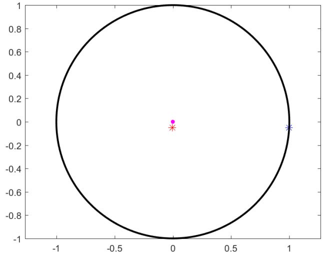
As mentioned above, in general the thin ligament has only a weak influence:

$$u^\varepsilon \approx u \qquad \Rightarrow \qquad R_1^\varepsilon \approx 1 \quad \text{and} \quad R_2^\varepsilon \approx 0.$$

But not always ...

- Below, for a fixed ε, we vary the length ℓ of the ligament:





$*$ $R_1(\ell)$
$*$ $R_2(\ell)$

Asymptotic analysis

To understand the phenomenon, we compute an asymptotic expansion of u^ε, R_1^ε, R_2^ε as $\varepsilon \to 0$.

- To proceed we use techniques of matched asymptotic expansions (see Beale 73, Gadyl'shin 93, Kozlov et al. 94, Nazarov 96, Maz'ya et al. 00, Joly & Tordeux 06, Lin, Shipman & Zhang 20, 18, Holley & Schnitzer 19, ...).

Asymptotic analysis

- We work with the outer expansions

$$\begin{aligned} u^\varepsilon(x,y) &= u^0(x,y) + \dots && \text{in } \omega, \\ u^\varepsilon(x,y) &= \varepsilon^{-1} v^{-1}(x) + v^0(x) + \dots && \text{in the resonator.} \end{aligned}$$

- Considering the restriction of $(\mathscr{P}_N^\varepsilon)$ to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous 1D problem

$$(\mathscr{P}_N^{1\mathrm{D}}) \left| \begin{array}{ll} \partial_x^2 v + k^2 v = 0 & \text{in } (0;\ell) \\ v(0) = \partial_x v(\ell) = 0. & \end{array} \right.$$

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The features of $(\mathscr{P}_N^{1\mathrm{D}})$ play a key role in the physical phenomena and in the asymptotic analysis.

- We denote by ℓ_N^{res} (**resonance lengths**) the values of ℓ, given by

$$\ell_N^{\mathrm{res}} := \pi(m + 1/2)/k, \qquad m \in \mathbb{N},$$

such that $(\mathscr{P}_N^{1\mathrm{D}})$ admits the non zero solution $v(x) = \sin(kx)$.

Asymptotic analysis – Non resonant case

- Assume that $\ell \neq \ell_N^{\mathrm{res}}$. Then we find $v^{-1} = 0$ and when $\varepsilon \to 0$, we get

$$u^\varepsilon(x,y) = u(x,y) + o(1) \qquad \text{in } \omega,$$

$$u^\varepsilon(x,y) = u(A)\, v_0(x) + o(1) \qquad \text{in the resonator,}$$

$$R_1^\varepsilon = 1 + o(1), \qquad R_2^\varepsilon = 0 + o(1).$$

Here $v_0(x) = \cos(kx) + \tan(k\ell)\sin(kx)$.

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The thin resonator has no influence at order ε^0.

→ **Not interesting for our purpose** because we want $\left| \begin{array}{l} R_1^\varepsilon = 0 + \dots \\ R_2^\varepsilon = 1 + \dots \end{array} \right.$

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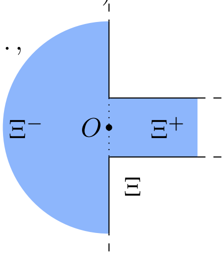
- Inner expansion. Set $\xi = \varepsilon^{-1}(\mathrm{x} - A)$ (stretched coordinates). Since

$$(\Delta_{\mathrm{x}} + k^2) u^{\varepsilon}(\varepsilon^{-1}(\mathrm{x} - A)) = \varepsilon^{-2} \Delta_{\xi} u^{\varepsilon}(\xi) + \dots,$$

when $\varepsilon \to 0$, we are led to study the problem

$$(\star) \left| \begin{array}{rl} -\Delta_{\xi} Y = 0 & \text{in } \Xi \\ \partial_{\nu} Y = 0 & \text{on } \partial\Xi. \end{array} \right.$$



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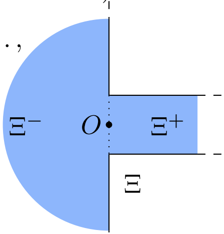

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- Problem $(\star)$ admits a solution Y^1 (up to a constant) with the expansion

$$Y^1(\xi) = \begin{cases} \xi_x + C_\Xi + O(e^{-\pi\xi_x}) & \text{as } \xi_x \to +\infty, \quad \xi \in \Xi^+ \\ \dfrac{1}{\pi}\ln\dfrac{1}{|\xi|} + O\Big(\dfrac{1}{|\xi|}\Big) & \text{as } |\xi| \to +\infty, \quad \xi \in \Xi^-. \end{cases}$$

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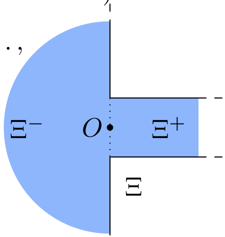

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- Problem $(\star)$ admits a solution Y^1 (up to a constant) with the expansion

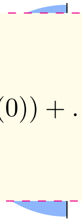
$$Y^1(\xi) = \begin{cases} \xi_x + C_\Xi + O(e^{-\pi\xi_x}) & \text{as } \xi_x \to +\infty, \quad \xi \in \Xi^+ \\ \dfrac{1}{\pi}\ln\dfrac{1}{|\xi|} + O\Big(\dfrac{1}{|\xi|}\Big) & \text{as } |\xi| \to +\infty, \quad \xi \in \Xi^-. \end{cases}$$

- In a neighbourhood of A, we look for u^ε of the form

$$u^\varepsilon(\mathrm{x}) = C^A\,Y^1(\xi) + c^A + \dots \qquad (c^A,\ C^A \text{ constants to determine}).$$

Asymptotic analysis – Resonant case

- Now assume that $\ell = \ell_N^{\text{res}}$. Then we find $v^{-1}(x) = a\sin(kx)$ for some a to determine.
- Inner expansion. Set $\xi = \varepsilon^{-1}(\mathrm{x} - A)$ (stretched coordinates). Since



Since at A, the Taylor formula gives

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- In the ansatz $u^\varepsilon = u^0 + \dots$ in ω, we deduce that we must take

$$u^0 = u + ak\gamma$$

where γ is the outgoing Green function such that $\left|\begin{array}{l}\Delta\gamma + k^2\gamma = 0 \text{ in } \omega\\ \partial_n\gamma = \delta_A \text{ on } \partial\omega.\end{array}\right.$

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- Matching the constant behaviour in the resonator, we obtain

$$v^0(0) = u(A) + ak(\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi).$$

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- This is a Fredholm problem with a non zero kernel. A solution exists iff the **compatibility condition** is satisfied. This sets

$$ak = -\frac{u(A)}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi}$$

and **ends the calculus of the first terms**.

Asymptotic analysis – Resonant case

- Finally for $\ell = \ell_N^{\text{res}}$, when $\varepsilon \to 0$, we obtain

$$u^\varepsilon(x,y) = u(x,y) + ak\gamma(x,y) + o(1) \quad \text{in } \omega,$$

$$u^\varepsilon(x,y) = \varepsilon^{-1} a \sin(kx) + O(1) \quad \text{in the resonator},$$

$$R_1^\varepsilon = 1 + iau(A)/2 + o(1), \qquad R_2^\varepsilon = 0 + iau(A)/2 + o(1).$$

Here γ is the outgoing Green function such that $\left| \begin{array}{l} \Delta\gamma + k^2\gamma = 0 \text{ in } \omega \\ \partial_n \gamma = \delta_A \text{ on } \partial\omega \end{array} \right.$ and

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This time the thin resonator has an influence at order ε^0

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- Similarly for $\ell = \ell_N^{\text{res}} + \varepsilon\eta$ with $\eta \in \mathbb{R}$ fixed, by modifying only the last step with the compatibility relation, when $\varepsilon \to 0$, we obtain

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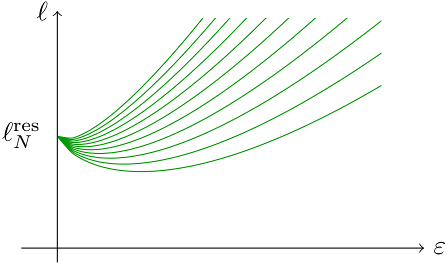
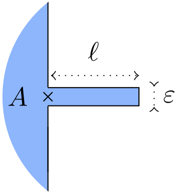
This time the thin resonator has an influence at order ε^0 and it depends on the choice of η!

Asymptotic analysis – Resonant case

- Below, for several $\eta \in \mathbb{R}$, we display the paths

$$\{(\varepsilon, \ell_N^{\mathrm{res}} + \varepsilon(\eta - \pi^{-1}|\ln \varepsilon|)),\ \varepsilon > 0\} \subset \mathbb{R}^2.$$





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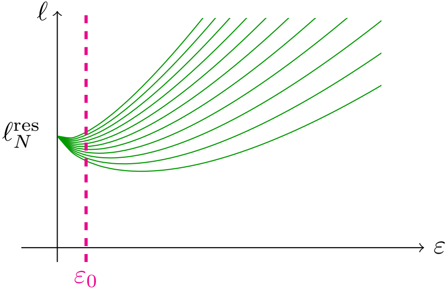

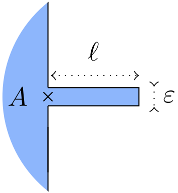

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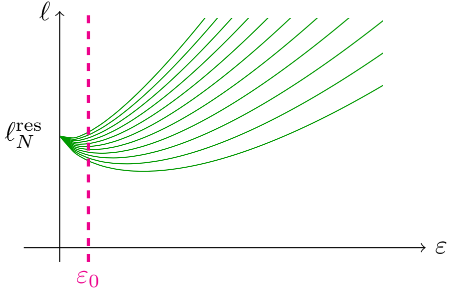

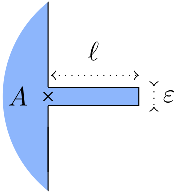

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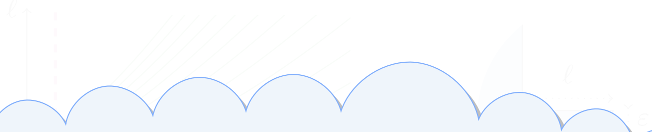

Varying the length of the ligament around the resonant length, we can get a rapid and large variation of the scattering coefficients.

→ How to use that to design the mode converter?

Accor… …ents along the path as $\varepsilon \to 0^+$ is different.

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Asymptotic analysis – Neumann problem

► Using the expressions of $u(A)$, $\Im m\,\Gamma$ and that $C_\Xi \in \mathbb{R}$, we get in particular for

$$\ell = \ell_N^{\mathrm{res}} - \varepsilon(\pi^{-1}|\ln\varepsilon| + C_\Xi + \Re e\,\Gamma),$$

when ε tends to zero,

$$R_1^\varepsilon = \frac{2\beta_1\cos(\pi y_A)^2/\beta_2 - 1}{2\beta_1\cos(\pi y_A)^2/\beta_2 + 1} + \dots, \qquad R_2^\varepsilon = \frac{-\,2\cos(\pi y_A)\sqrt{2\beta_1/\beta_2}}{2\beta_1\cos(\pi y_A)^2/\beta_2 + 1} + \dots$$

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- By choosing $\boldsymbol{y_A}$ such that $\cos(\pi y_A) = -\sqrt{\beta_2/(2\beta_1)}$ (doable), we get

$$R_1^\varepsilon = 0 + \dots, \qquad R_2^\varepsilon = 1 + \dots$$

and so by symmetry and unitarity of R_N^ε,

$$R_N^\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \dots. \qquad \text{(initial goal).}$$

Asymptotic analysis – Dirichlet problem

- The analysis is completely similar for the Dirichlet problem

$$(\mathscr{P}_D^\varepsilon) \left| \begin{array}{r} \Delta u_D^\varepsilon + k^2 u_D^\varepsilon = 0 \text{ in } \omega^\varepsilon \\ \partial_\nu u_D^\varepsilon = 0 \text{ on } \partial\omega^\varepsilon \setminus \Sigma^\varepsilon \\ u_D^\varepsilon = 0 \text{ on } \Sigma^\varepsilon \end{array} \right.$$

except that the corresponding 1D problem is

$$(\mathscr{P}_D^{1\mathrm{D}}) \left| \begin{array}{ll} \partial_x^2 v + k^2 v = 0 & \text{in } (0;\ell) \\ v(0) = v(\ell) = 0. & \end{array} \right.$$

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- For $\ell = \ell_D^{\mathrm{res}} - \varepsilon(\pi^{-1}|\ln\varepsilon| + C_\Xi + \Re e\,\Gamma)$, y_A s.t. $\cos(\pi y_A) = \sqrt{\beta_2/(2\beta_1)}$, we get when ε tends to zero

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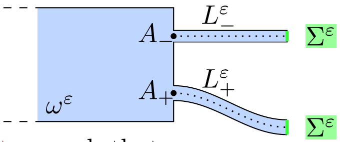
Mode converter

- We come back to the geometry *(note that curving the ligaments does not change the main term in the asymp.)*



- Finally we choose the ligament parameters such that

$$\cos(\pi y_{A_\pm}) = \mp\sqrt{\beta_2/(2\beta_1)}, \qquad \left|\begin{array}{l} \ell_+ = \ell_N^{\text{res}} - \varepsilon(\pi^{-1}|\ln\varepsilon| + C_\Xi + \Re e\,\Gamma) \\ \ell_- = \ell_D^{\text{res}} - \varepsilon(\pi^{-1}|\ln\varepsilon| + C_\Xi + \Re e\,\Gamma). \end{array}\right.$$

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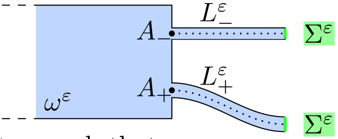

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For the Neumann pb., L_+^ε acts at order ε^0 while L_-^ε acts at higher order.
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$\Rightarrow$ The action of the two ligaments decouple at order ε^0 (**crucial point**).

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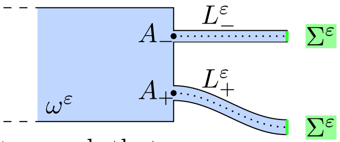

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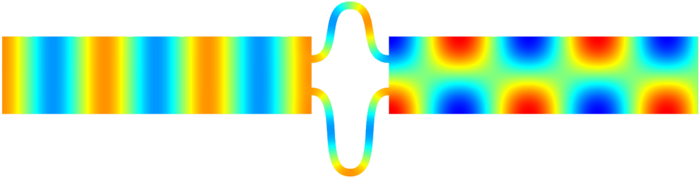
- Then as $\varepsilon \to 0$ we have both

$$R_N^\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + o(1) \qquad \text{and} \qquad R_D^\varepsilon = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + o(1).$$

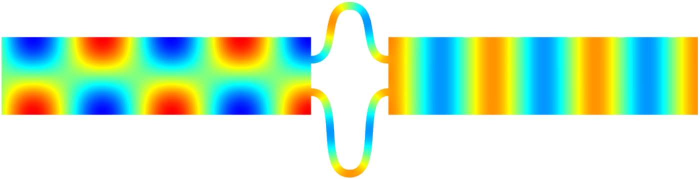
Mode converter - numerical results

► Thus tuning precisely the positions and lengths of the ligaments, we can ensure absence of reflection and mode conversion:

$t \mapsto \Re e\left(u_1^\varepsilon e^{-i\omega t}\right)$



$t \mapsto \Re e\left(u_2^\varepsilon e^{-i\omega t}\right)$



Numerics made with ***Freefem++***.

Remarks

1. $y_{A_\pm}$ are such that $\cos(\pi y_{A_\pm}) = \mp\sqrt{\beta_2/(2\beta_1)}$

The junction points of the ligaments are symmetric wrt the axis $\{1/2\} \times \mathbb{R}$.

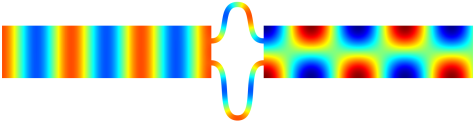
2. What we do is an approximation:

$$R^\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \dots \qquad T^\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \dots$$

Better results for smaller ε. But then the tuning becomes more delicate.

$\rightarrow$ Compromise precision/robustness.

$\Re e\, u_1^\varepsilon$



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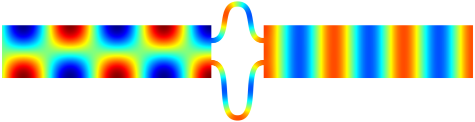

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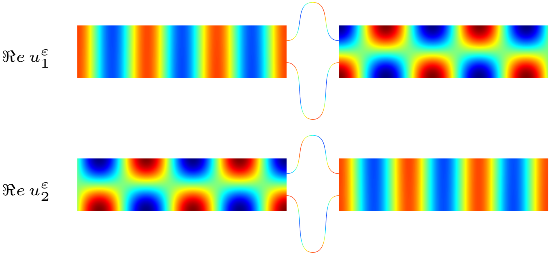

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$\rightarrow$ Compromise precision/robustness.

Remarks

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The junction points of the ligaments are symmetric wrt the axis $\{1/2\} \times \mathbb{R}$.

2. What we do is an approximation:

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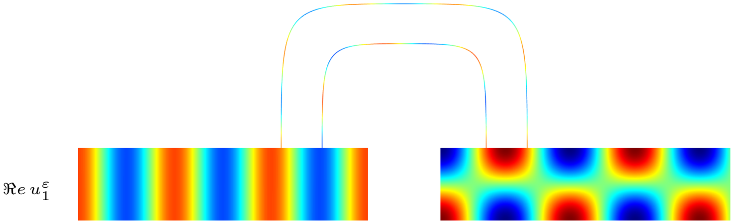

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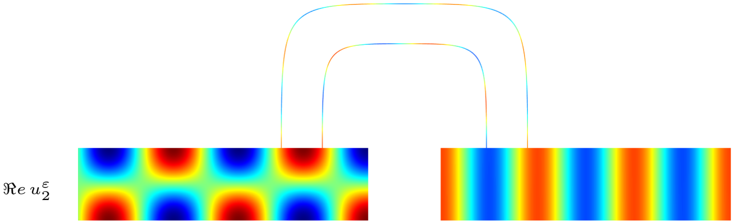

1. Choice of geometry

2. Asymptotic analysis in presence of thin resonators

3. Mode converter

Conclusion

What we did

♠ We explained how to design mode converters using thin resonators. Two main ingredients:

- Around resonant lengths, effects of order ε^0 with perturb. of width ε.
- The 1D limit problems in the resonator provide a rather explicit dependence wrt to the geometry.

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Possible extensions and open questions

1) We could work similarly in 3D.
2) Using close ideas, we can do passive cloaking in waveguides
$\rightarrow$ **see the talk of J. Heleine on Thursday, room red 1, 3pm**.
3) With Dirichlet BCs, other ideas must be found.

Thank you for your attention!

L. Chesnel, J. Heleine and S.A. Nazarov. Design of a mode converter using thin resonant slits. Comm. Math. Sci., vol. 20, 2:425-445, 2022.