Introduction to asymptotic methods for PDEs.
A focus on small obstacle asymptotics.
– Session 1 –

Lucas Chesnel[1] and Xavier Claeys[2]

[1]Idefix team, CMAP, École Polytechnique, France
[2]LJLL, Alpines team, Université Pierre et Marie Curie, France

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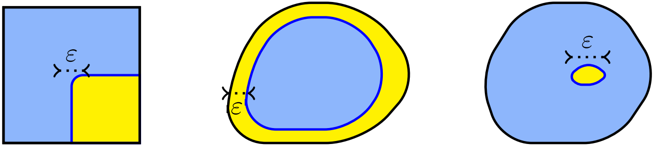
Introduction

- Consider a problem (PDE+BC) depending on a small parameter $\varepsilon > 0$ (coefficient in the PDE, parameter of the geometry,...).



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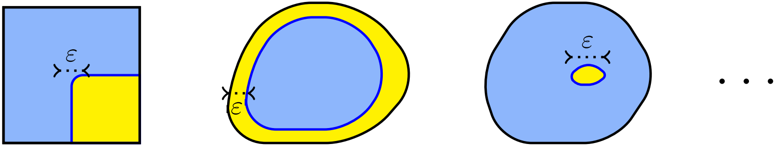

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- The aim is to explicit the behaviour with respect to ε. The expansion (or representation or approximation) should involve functions which are independent of ε and functions with explicit dependence with respect to ε.

Example: $\|u_\varepsilon - \hat{u}_\varepsilon\| \leq C\,\varepsilon^3$ with $\hat{u}_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$.

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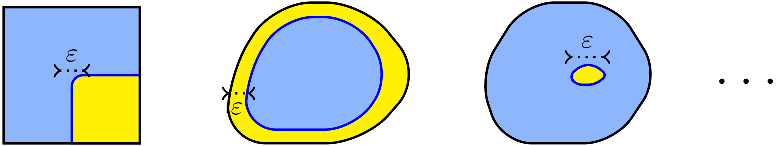

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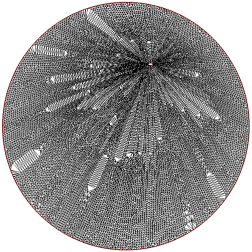

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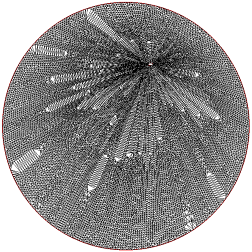

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Goals of the mini course

1) To describe in detail how to treat small obstacle asymptotics.
2) Each problem requires a rather specific treatment. We also wish to give an idea of how to treat different problems of asymptotics and to present a few general techniques.
3) To explain how to establish error estimates, an aspect which is sometimes neglected in literature.
4) To present examples of applications where asymptotic expansions can be useful.

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Structure of the mini course

Session 1. Introduction to asymptotic expansions (smooth perturbations).

Sessions 2 & 3. Small obstacle asymptotics (singular perturbations).

Session 4. Examples of applications.

Outline of session 1

1. Perturbation in the equation

2. Smooth perturbation of the domain

3. Application to invisibility in acoustic waveguides

4. An example of singularly perturbed problem

Perturbation of the Poisson's problem

- We study a first simple example with a perturbation in the equation. For Ω a bounded Lipschitz domain and $f \in \mathrm{L}^2(\Omega)$, consider the problem

$$(\mathscr{P}_\varepsilon) \quad \left| \begin{array}{rcll} -\Delta u_\varepsilon + \varepsilon u_\varepsilon &=& f & \text{in } \Omega \\ u_\varepsilon &=& 0 & \text{on } \partial\Omega. \end{array} \right.$$

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General procedure:

Step I: we propose an expansion (ansatz) and identify the terms of this expansion.

Step II: we prove error estimates.

Step I - ansatz

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- Consider the ansatz

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- Inserting the expansion in $(\mathscr{P}_\varepsilon)$, letting ε tends to zero and identifying the powers in ε, we get

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  $\rightarrow$ This defines the expansion.

Step II - error estimate 1/2

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  1) A stability estimate;

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1) Stability estimate. Green's formula gives

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From the Poincaré inequality

$$\|\varphi\|_{\mathrm{L}^2(\Omega)} \leq C_P \, \|\varphi\|_{\mathrm{H}^1_0(\Omega)} := \|\nabla\varphi\|_{\mathrm{L}^2(\Omega)}, \quad \forall \varphi \in \mathrm{H}^1_0(\Omega),$$

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"The solution of $(\mathscr{P}_\varepsilon)$ is controlled uniformly (C_P is independent of ε, f) by the source term."

Step II - error estimate

2) Consistency results. Set $\hat{u}_\varepsilon := \sum_{n=0}^{N} \varepsilon^n u_n \in \mathrm{H}^1_0(\Omega)$.

Inserting the **error** $u_\varepsilon - \hat{u}_\varepsilon$ in $(\mathscr{P}_\varepsilon)$, we obtain the **discrepancy**

$$(-\Delta + \varepsilon)(u_\varepsilon - \hat{u}_\varepsilon) = f - (-\sum_{n=0}^{N} \varepsilon^n \Delta u_n + \sum_{n=1}^{N+1} \varepsilon^n u_{n-1}) = -\varepsilon^{N+1} u_N.$$

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Using this consistency result in the stability estimate $(*)$, we find

$$\|u_\varepsilon - \hat{u}_\varepsilon\|_{\mathrm{H}_0^1(\Omega)} \leq C_P\, \varepsilon^{N+1} \|u_N\|_{\mathrm{L}^2(\Omega)}.$$

Noting that $\|u_N\|_{\mathrm{L}^2(\Omega)} \leq C_P \|u_N\|_{\mathrm{H}_0^1(\Omega)} \leq C_P^3 \|u_{N-1}\|_{\mathrm{H}_0^1(\Omega)}$, finally we get:

PROPOSITION: We have the error estimate

$$\|u_\varepsilon - \hat{u}_\varepsilon\|_{\mathrm{H}_0^1(\Omega)} \leq C_P^{2N+2}\, \varepsilon^{N+1} \|f\|_{\mathrm{L}^2(\Omega)}.$$

Comments

- Recall the standard scheme

> **Step I**: ansatz and identification of the terms of the ansatz;
> **Step II**: error estimates (stability estimate + consistency result).

What validates the relevance of some ansatz is the error estimate.

- In general, the choice of the ansatz requires experience and knowledge of the problem. The derivation of the stability estimate is the hard part.

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- In our example, the **uniform coercivity property** made things very simple. Direct generalization to the problem:

$$A_\varepsilon u_\varepsilon = f \in \mathrm{X} \qquad \text{with} \qquad A_\varepsilon := A_0 + P(\varepsilon).$$

Here X is a Banach space, $A_0 : \mathrm{X} \to \mathrm{X}$ is an isomorphism and $P(\cdot) : \mathrm{X} \to \mathrm{X}$ is a family of bounded operators that depend analytically on ε s.t. $P(0) = 0$.

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To prove the stability estimate, write

$$A_\varepsilon = A_0 + (A_\varepsilon - A_0) = A_0(\mathrm{Id} + A_0^{-1}(A_\varepsilon - A_0)).$$

This implies $\boxed{\|u_\varepsilon\|_{\mathrm{X}} \le C\,\|f\|_{\mathrm{X}}}$ with $C > 0$ independent of ε for $\varepsilon \in (0; \varepsilon_0]$.

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This applies for example to the problem

$$\text{Find } u \in \mathrm{H}^2_0(\Omega) \text{ such that } \Delta\Delta u_\varepsilon + \frac{i\varepsilon}{1+\sin\varepsilon}\Delta u_\varepsilon = f \in \mathrm{L}^2(\Omega).$$

Smooth perturbation of the domain

- We perturb slightly ($\varepsilon \geq 0$ is small) the geometry

> Locally $\partial\Omega_\varepsilon$ coincides with the graph of $x \mapsto \varepsilon h(x)$,
> where $h \in \mathscr{C}_0^\infty(-1;1)$ is a given profile function.

- We consider the Laplace problem in the perturbed domain

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> What is the dependence of u_ε with respect to ε**?**

$\rightarrow$ This question has been extensively studied in shape optimization.

A first formal approach

- Let $\mathcal{O}$ be a fixed neighbourhood of the perturbation. To simplify, we assume that $f \in \mathrm{L}^2(\Omega_\varepsilon)$ is zero in $\mathcal{O}$. In Ω_0, we consider the ansatz

$$u_\varepsilon = u_0 + \varepsilon u_1 + \dots$$

where the terms u_0, u_1 have to be determined.

- Observing that at the limit $\varepsilon \to 0$, Ω_ε converges to Ω_0, we get

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- For the boundary conditions, for $(x, y) \in I$, we can write

$$\begin{aligned} 0 = u_\varepsilon(x, \varepsilon h(x)) &= u_\varepsilon(x, 0) + \varepsilon h(x) \partial_y u_\varepsilon(x, 0) + \dots \\ &= u_0(x, 0) + \varepsilon u_1(x, 0) + \varepsilon h(x) \partial_y u_0(x, 0) + \dots . \end{aligned}$$

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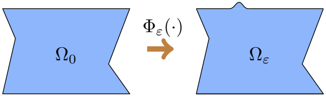
$\rightarrow$ Let us see how to justify this formal calculus.

To establish error estimates, we consider a change of variables to work in a fixed geometry.

- For all $\varepsilon \in [0;\varepsilon_0]$, there is a smooth diffeomorphism

$$\begin{aligned} \Phi_\varepsilon : \quad \Omega_0 \quad &\to \quad \Omega_\varepsilon \\ \mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2) \quad &\mapsto \quad x = \Phi_\varepsilon(\mathrm{x}) = \mathrm{x} + \varepsilon\phi(\mathrm{x}). \end{aligned}$$



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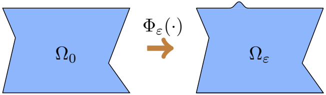
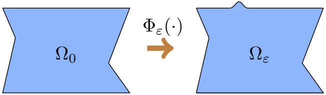


- With this choice, Φ_ε is a small perturbation of the identity.

Error estimates

To establish error estimates, we consider a change of variables to work in a fixed geometry.

- For all $\varepsilon \in [0;\varepsilon_0]$, there is a smooth diffeomorphism



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- We can take ϕ supported in $\mathcal{O}$, of the form

$$\phi(\mathrm{x}) = (\phi_1(\mathrm{x}), \phi_2(\mathrm{x})) = (0, h(\mathrm{x}_1)\rho(\mathrm{x}_2))$$

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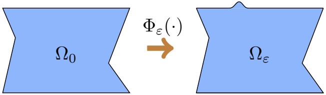

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- Observe that we have $\Phi_\varepsilon|_{\Omega_0 \setminus \overline{\mathcal{O}}} = \mathrm{Id}$.

Error estimates 2/3

- Set $U_\varepsilon = u_\varepsilon \circ \Phi_\varepsilon$, $V = v \circ \Phi_\varepsilon$, $F = f \circ \Phi_\varepsilon$. We have

$$\int_{\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0)} \nabla u_\varepsilon \cdot \nabla v \, dx = \int_{\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0)} f v \, dx$$

$$\Leftrightarrow \boxed{\int_{\Omega_0} (\mathrm{Id} + \varepsilon (D\phi)^\top)^{-1} \nabla U_\varepsilon \cdot (\mathrm{Id} + \varepsilon (D\phi)^\top)^{-1} \nabla V \, J_{\Phi_\varepsilon} \, \mathrm{dx} = \int_{\Omega_0} F V \, J_{\Phi_\varepsilon} \, \mathrm{dx}.}$$

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- Thus we obtain the problem

$$\boxed{\left| \begin{array}{l} \text{Find } U_\varepsilon \in \mathrm{H}^1_0(\Omega_0) \text{ such that} \\ -\mathrm{div}(\sigma_\varepsilon \nabla U_\varepsilon) = F J_{\Phi_\varepsilon} \text{ in } \Omega_0 \end{array} \right.}$$

with

$$\left| \begin{array}{l} \sigma_\varepsilon := J_{\Phi_\varepsilon} (\mathrm{Id} + \varepsilon (D\phi))^{-1} (\mathrm{Id} + \varepsilon (D\phi)^\top)^{-1} = \mathrm{Id} + \varepsilon \sigma_1 + \varepsilon^2 \sigma_2 + \dots \\ F J_{\Phi_\varepsilon} = F + \varepsilon h \partial_{\mathrm{x}_2} \rho F. \end{array} \right.$$

Error estimates 3/3

Now the geometry is fixed and we have a pertubation in the equation.

- Considering the expansion

$$U_\varepsilon = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \dots,$$

we can prove the following error estimate with C independent of $\varepsilon \in (0; \varepsilon_0]$

$$\boxed{\|U_\varepsilon - \sum_{n=0}^{N} \varepsilon^n U_n\|_{\mathrm{H}^1_0(\Omega_0)} \le C\,\varepsilon^{N+1}\|f\|_{\mathrm{L}^2(\Omega_0)}.}$$

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- Using that $\left|\begin{array}{l} U_0 \circ \Phi_\varepsilon^{-1} + \varepsilon U_1 \circ \Phi_\varepsilon^{-1} = U_0 + \varepsilon\,(U_1 - \nabla U_0 \cdot \phi) + \dots \\ U_0 = u_0, \qquad U_1 - \nabla U_0 \cdot \phi = U_1 - h\rho\partial_{\mathrm{x}_2} U_0 = u_1, \end{array}\right.$

finally we obtain $\boxed{\|u_\varepsilon - (u_0 + \varepsilon u_1)\|_{\mathrm{H}^1(\Omega_0\setminus\mathcal{O})} \leq C\,\varepsilon^2\|f\|_{\mathrm{L}^2(\Omega_0)}.}$

Comments

► This is only to give a flavour. Much more refined results exist in the literature concerning shape optimization.

M. Pierre and A. Henrot. Shape Variation and Optimization. A Geometrical Analysis. EMS, 2018.

M.C. Delfour and J.P. Zolésio. Shapes and geometries: metrics, analysis, differential calculus, and optimization. Society for Industrial and Applied Mathematics, 2011.

► In particular:

- For this Dirichlet problem, smoothness assumptions of the geometry can be considerably relaxed and result exist when Ω_0 is only measurable.

- Higher order terms can be computed but then smoothness on f is required.

Eigenvalue problem

- We consider the same perturbation of the geometry as before

> Locally $\partial\Omega_\varepsilon$ coincides with the graph of $x \mapsto \varepsilon h(x)$,
> where $h \in \mathscr{C}_0^{-1;1}(\mathbb{R})$ is a given profile function.

- We study the eigenvalue problem

> Find $(\lambda_\varepsilon, u_\varepsilon) \in \mathbb{R} \times \mathrm{H}_0^1(\Omega_\varepsilon) \setminus \{0\}$ such that
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What is the dependence of $\lambda_\varepsilon^{[n]}$ with respect to ε?

Asymptotic expansion of the eigenvalues

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PROPOSITION: The perturbation of a simple eigenvalue ($\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \dots$), is given by the Hadamard's formula

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J. Hadamard. Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées, 33 (1908) Imprimerie nationale.

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Remark:

If h is non negative, the domain increases and $\lambda_1 \leq 0$.
If h is non positive, the domain decreases and $\lambda_1 \geq 0$.
$\rightarrow$ This is coherent with physics (the smaller Ω, the larger the eigenvalues).

Justification

We consider again the map $\Phi_\varepsilon : \Omega_0 \to \Omega_\varepsilon$ to work in a **fixed geometry**.

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A general theory exists for such problems and we can prove that $\varepsilon \mapsto \lambda_\varepsilon$ and $\varepsilon \mapsto U_\varepsilon$ are analytic near zero.

T. Kato. Perturbation theory for linear operators, Chap. 7, §6.5. 1976.

General setting

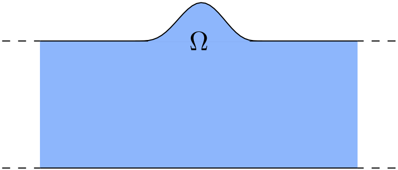
- We wish to study questions of invisibility in acoustic waveguides.

> Can we find situations where waves go through like if there were no defect ?

- One can wish to have good energy transmission through the structure.
- One can wish to hide objects.

Waveguide problem

► Scattering in time-harmonic regime of a plane wave in the acoustic waveguide Ω coinciding with $\{(x,y) \in \mathbb{R} \times (0;1)\}$ outside a compact region.



$$\begin{array}{|rcll} \text{Find } u = u_i + u_s \text{ s. t.} & & & \\ \Delta u + k^2 u & = & 0 & \text{in } \Omega, \\ \partial_n u & = & 0 & \text{on } \partial\Omega, \\ & & u_s \text{ is outgoing.} & \end{array}$$

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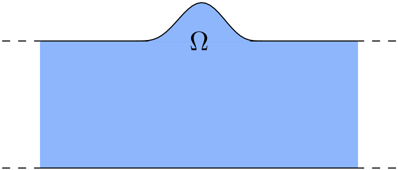

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u_s is outgoing.

- For this problem, the modes are

Propagating	$w_n^{\pm}(x, y) = e^{\pm i\beta_n x} \cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2}$, $n \in [\![0, N-1]\!]$
Evanescent	$w_n^{\pm}(x, y) = e^{\mp \beta_n x} \cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2}$, $n \geq N$.

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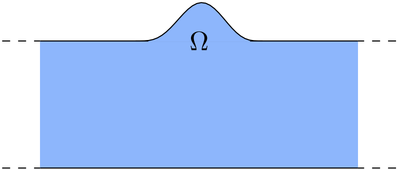

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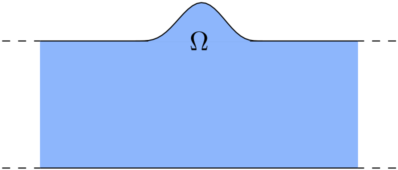

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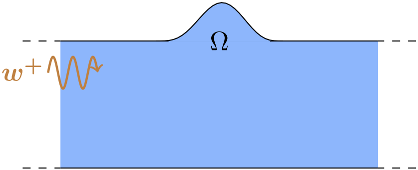

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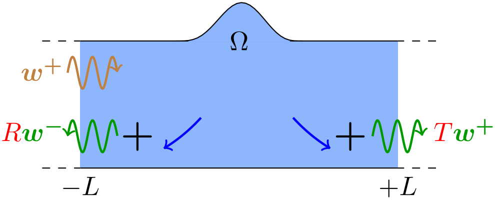

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$$u = \left| \begin{array}{ll} w_+ + R\, w_- + \ldots & \text{for } x \leq -L \\ T\, w_+ + \ldots & \text{for } x \geq +L \end{array} \right.$$

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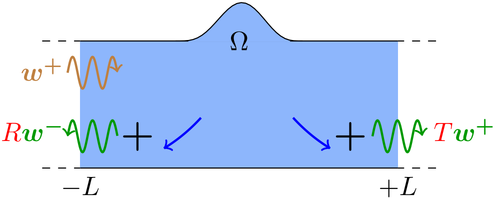

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Definition: R, $T\in\mathbb{C}$ are the reflection and transmission coefficients.

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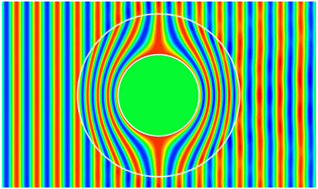

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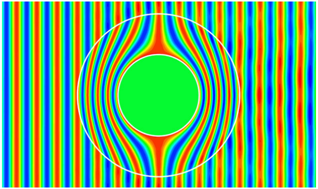
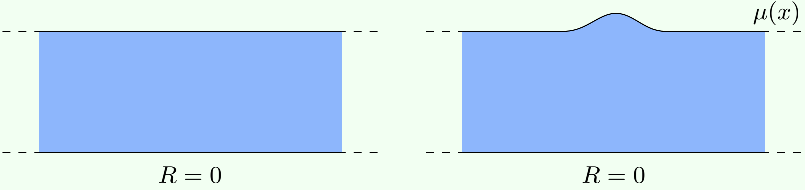


GOAL

We explain how to use perturbative techniques to construct geometries such that $R = 0$ or $T = 1$.

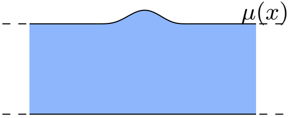
General picture

- Perturbative technique: we construct small non reflecting defects using variants of the implicit functions theorem.



Sketch of the method

- For $\mu \in \mathscr{C}_0^\infty(\mathbb{R})$, set $R = R(\mu) \in \mathbb{C}$.



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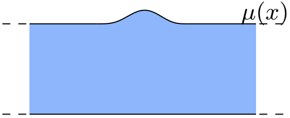

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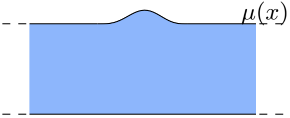

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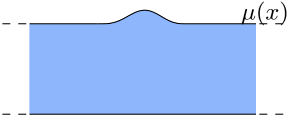

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- We look for small perturbations of the geometry: $\mu = \varepsilon h$ where $\varepsilon > 0$ is a small parameter and where h has be to determined.

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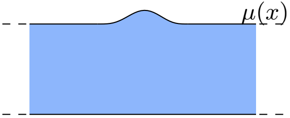

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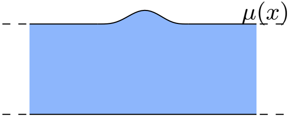

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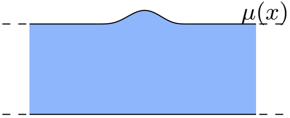

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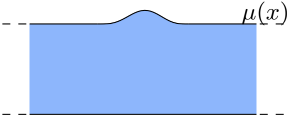

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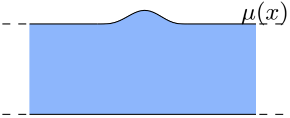

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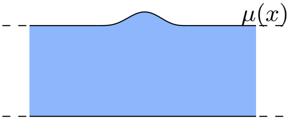

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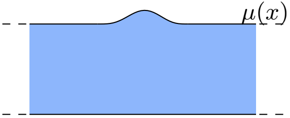

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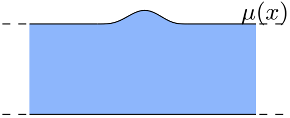

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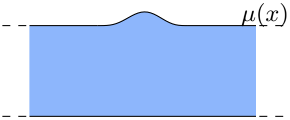

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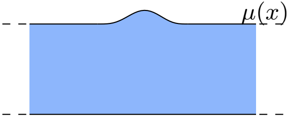

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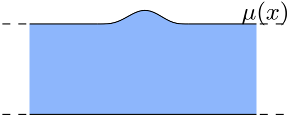

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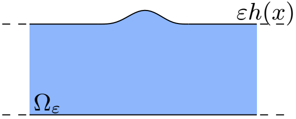
Set $\mu^{\mathrm{sol}} := \varepsilon h^{\mathrm{sol}}$. We have $R(\mu^{\mathrm{sol}}) = 0$ (non reflecting perturbation).

Calculus of the differential 1/2



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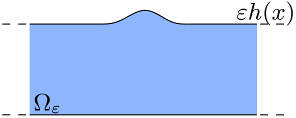

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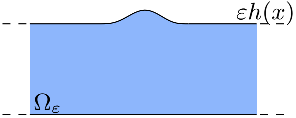

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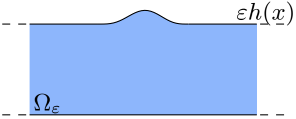

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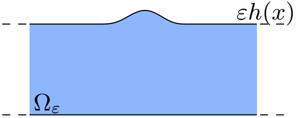
Calculus of the differential 1/2



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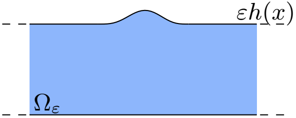

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Calculus of the differential 2/2

- We have $u_0 = w_+$ and u_1 is uniquely defined.

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$$2ikR(\varepsilon h) = \int_{\Sigma_{\pm L}} \partial_n u_\varepsilon w^+ - u_\varepsilon \partial_n w^+ d\sigma, \qquad \text{where } \partial_n = \pm\partial_x \text{ at } x = \pm L,$$

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$\Rightarrow$ Thus we can construct geometries Ω_ε where $R_\varepsilon = 0$.

Comments

- The invisible perturbation coincides with the graph of the function

$$\varepsilon(h_0 + \tau_1^{\mathrm{sol}} h_1 + \tau_2^{\mathrm{sol}} h_1)$$

where $h_0 \in \ker dR(0)$ (remind that $dR(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$).

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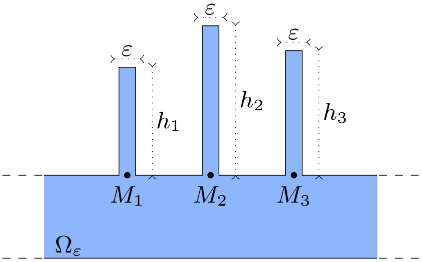
$dT(0)$ is not onto $\Rightarrow$ the approach fails to impose $T = 1$.

A perturbative method to get $T = 1$

- We study the same problem in the geometry Ω_ε



Singular perturbation of the geometry!

- We obtain

$$R_\varepsilon = 0 + \varepsilon \left(ik \sum_{n=1}^{3} (w^+(M_n))^2 \tan(kh_n) \right) + O(\varepsilon^2)$$

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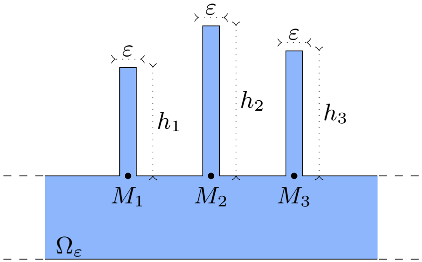

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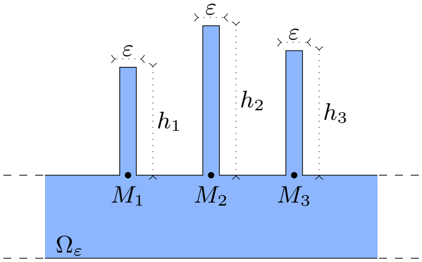

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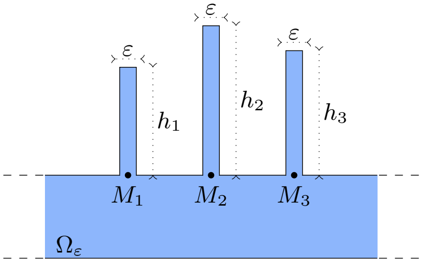

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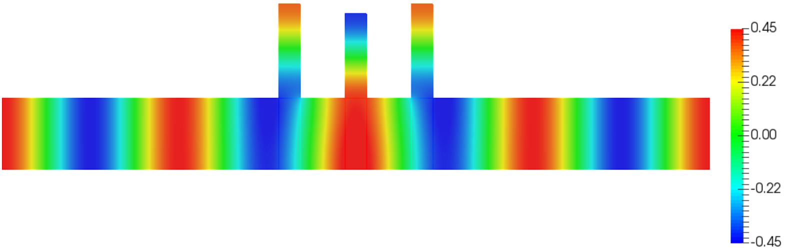
3) Energy conservation + $[T_\varepsilon = 1 + O(\varepsilon)]$ $\Rightarrow$ $T_\varepsilon = 1$.

Numerical results

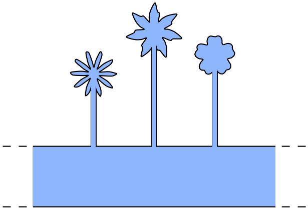
- Perturbed waveguide ($\Re e\,(u_\varepsilon(x,y)e^{-i\omega t})$)



- Reference waveguide ($\Re e\,(u_i(x,y)e^{-i\omega t})$)

Comments

- We could also have hidden gardens of flowers!



- For the second type of perturbations, the asymptotic analysis is quite different (singular perturbed problem).

> For the two problems, we use the first term in the asymptotic whose dependence with respect to the perturbation is explicit and linear to cancel the whole expansion by solving a fixed point problem.

A.-S. Bonnet-Ben Dhia and S. A. Nazarov. Obstacles in acoustic waveguides becoming "invisible" at given frequencies, Acoustical Physics, 59(6), 633-639, 2013.

A.-S. Bonnet-Ben Dhia, L. Chesnel and S. A. Nazarov. Perfect transmission invisibility for waveguides with sound hard walls, J. Math. Pures Appl., vol. 111, 79-105, 2018.

An example of singularly perturbed problem

- For $a > 0$, $a \neq 1$, consider the 1D problem

$$(\mathscr{P}_\varepsilon) \quad \left| \begin{array}{l} \varepsilon u_\varepsilon''(x) + u_\varepsilon'(x) - a = 0 \text{ in } \Omega := (0;1) \\ u_\varepsilon(0) = 0, \qquad u_\varepsilon(1) = 1. \end{array} \right.$$

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But since $\|u_\varepsilon(x) - \hat{u}_0(x)\|_{\mathrm{L}^\infty(\overline{\Omega})} = |1-a|$, (u_ε) does not cv to $\hat{u}_0$ in $\mathrm{H}^1(\Omega)$.

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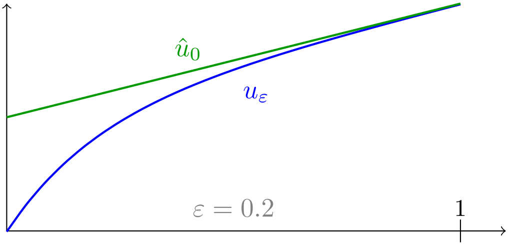
But since $\|u_\varepsilon(x) - \hat{u}_0(x)\|_{\mathrm{L}^\infty(\overline{\Omega})} = |1-a|$, (u_ε) does not cv to $\hat{u}_0$ in $\mathrm{H}^1(\Omega)$.

The expansion $(*)$ does not provide a good representation of u_ε.

An example of singularly perturbed problem

$$u_\varepsilon(x) = ax + (1-a)\,\frac{1-e^{-x/\varepsilon}}{1-e^{-1/\varepsilon}} \qquad\qquad \hat{u}_0(x) = ax + (1-a).$$

- What happens is that the function u_ε has a rapid variation near the origin when $\varepsilon \to 0$:



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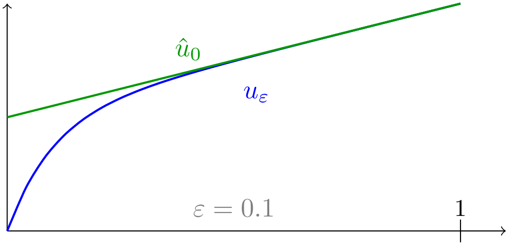

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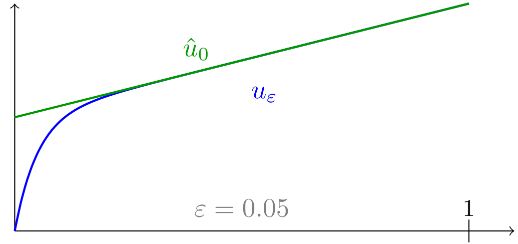

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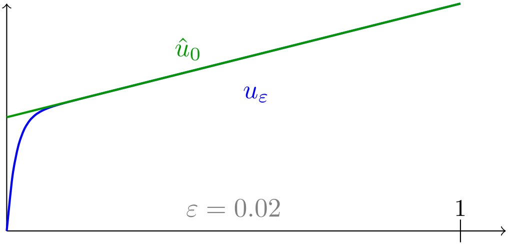

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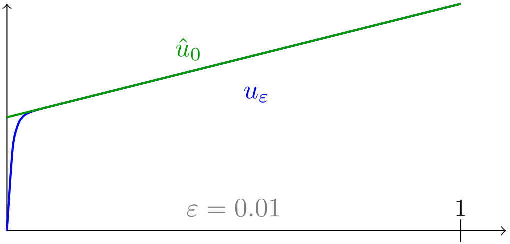

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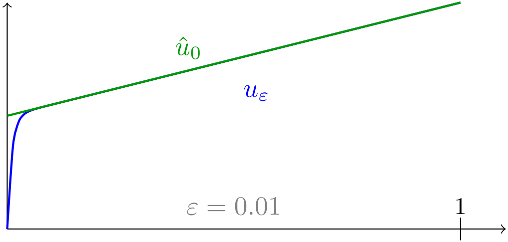

- Our expansion fails to provide a good representation of u_ε due to this boundary layer phenomenon. We say that $(\mathscr{P}_\varepsilon)$ is a **singularly perturbed problem**.

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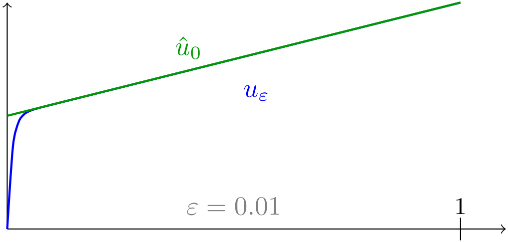

- Our expansion fails to provide a good representation of u_ε due to this boundary layer phenomenon. We say that $(\mathscr{P}_\varepsilon)$ is a **singularly perturbed problem**.

- To approximate correctly u_ε near the origin, we will have to incorporate terms which depend on the rapid variable x/ε.

Conclusion of session 1

What we did

1) Smooth perturbation in the PDE. Recall the standard scheme

Step I: ansatz and identification of the terms of the ansatz;
Step II: error estimates (stability estimate + consistency result).

2) Smooth perturbation of the geometry.

- Use a change of variable to show error estimates in a fixed geometry.
- For the eigenvalue problem, write the compatibility condition to get the corrector term.

3) Application to invisibility in acoustic waveguides.

4) We saw an example of singularly perturbed problem where the expansion $u_\varepsilon = u_0 + \varepsilon u_1 + \dots$ is not adapted.

Next session

♠ We will study in detail a singularly perturbed problem with a PDE set in a domain with a small obstacle.