Introduction to asymptotic methods for PDEs. A focus on small obstacle asymptotics.

– Session 4 –

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Organisation

Session 1. Introduction to asymptotic expansions (smooth perturbations).

Sessions 2 & 3. Small obstacle asymptotics (singular perturbations).

Session 4. Examples of applications.

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Outline of session 4

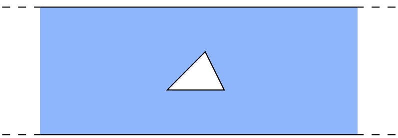
1. Non reflecting small obstacles in waveguide

2. Spectrum in presence of a small negative inclusion

3. Cloaking in acoustic waveguides

Waveguide problem

- Scattering in time-harmonic regime of a wave in a 3D waveguide Ω (Dirichlet BC, e.g. in electromagnetism) coinciding with $\{(x,y) \in \mathbb{R} \times \omega\}$, ω bounded, outside of a compact region.



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$$\begin{array}{rcll} \Delta u + k^2 u & = & 0 & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \\ & & & \end{array}$$

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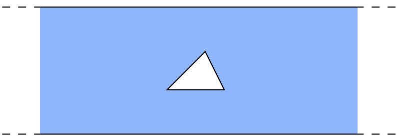

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- For this problem and $\lambda_N < k < \lambda_{N+1}$, the modes are

Propagating	$w_n^\pm(x,y) = e^{\pm i\beta_n x}\varphi_n(y),\ \beta_n = \sqrt{k^2 - \lambda_n^2},\ n \in [\![1, N]\!]$
Evanescent	$w_n^\pm(x,y) = e^{\mp \beta_n x}\varphi_n(y),\ \beta_n = \sqrt{\lambda_n^2 - k^2},\ n \geq N+1$

where the eigenpairs $(\lambda_n, \varphi_n) \in \mathbb{R}_+^* \times \mathrm{H}_0^1(\omega) \setminus \{0\}$ solve the problem

$$-\Delta_y \varphi_n = \lambda_n \varphi_n \text{ in } \omega$$

in the transverse cut.

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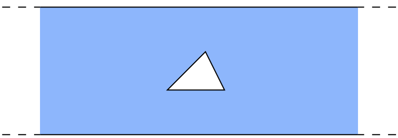

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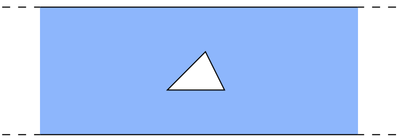

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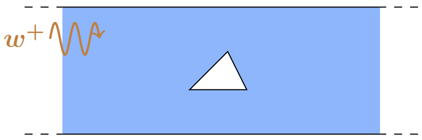

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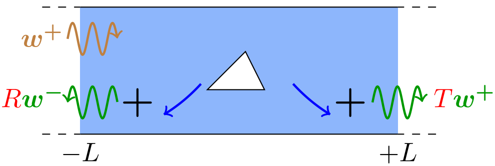

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- We have

$$u = \left| \begin{array}{ll} w_+ + R\, w_- + \dots & \text{for } x \leq -L \\ T\, w_+ + \dots & \text{for } x \geq +L \end{array} \right.$$

The ... are expo. decaying terms.

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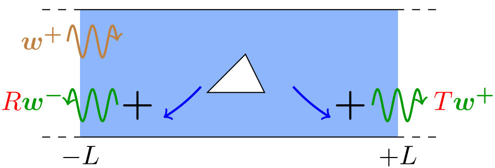

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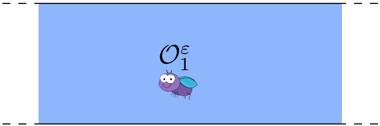
GOAL

We explain how small Dirichlet obstacles can arrange to achieve zero reflection ($R = 0$).

One small obstacle

Can one hide a small Dirichlet obstacle centered at M_1?

- Set $\mathcal{O}_1^\varepsilon := M_1 + \varepsilon\mathcal{O}$ where $M_1 \in \mathbb{R}\times\omega$ and $\mathcal{O}$ is a bounded Lipschitz domain. We consider the problem



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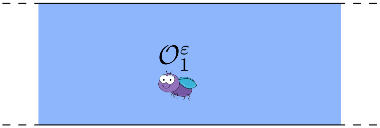

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$$\begin{array}{rcl} R_\varepsilon & = & 0 + \varepsilon\ (4i\pi\,\mathrm{cap}(\mathcal{O})w^+(M_1)^2)\ + O(\varepsilon^2) \\ T_\varepsilon & = & 1 + \varepsilon\ (4i\pi\,\mathrm{cap}(\mathcal{O})|w^+(M_1)|^2)\ + O(\varepsilon^2). \end{array}$$

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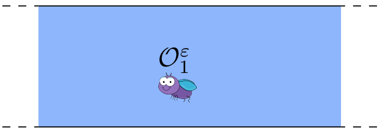

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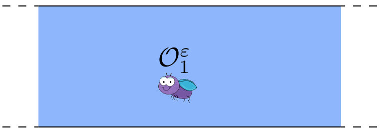

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Non zero terms!
$(\mathrm{cap}(\mathcal{O}) > 0)$

$\Rightarrow$ One single small obstacle cannot be non reflecting.

Derivation of the asymptotic of u_ε 1/3

- To simplify, we remove the index $_1$ of the obstacle. Consider the ansatz

$$u_\varepsilon = u_0 + \zeta(x)\, v_0(\varepsilon^{-1}(\mathrm{x} - M)) + \varepsilon\Big(u_1 + \zeta(x)\, v_1(\varepsilon^{-1}(\mathrm{x} - M))\Big) + \dots$$

where $\zeta \in \mathscr{C}_0^\infty(\Omega_0)$ is equal to one in a neighbourhood of M.

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- Inserting this expansion in $(\mathscr{P}_\varepsilon)$, first we find

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- v_0 serves to impose Dirichlet BC on $\partial\mathcal{O}^\varepsilon$ at order ε^0. For $x \in \partial\mathcal{O}^\varepsilon$,

$u_0(x) = u_0(M) + (\mathrm{x}-M)\cdot\nabla u_0(M) + \dots$ (note that $\mathrm{x}-M$ is of order ε).

Therefore we impose $v_0 = -u_0(M)$ on $\partial\mathcal{O}$.

Derivation of the asymptotic of u_ε 2/3

- Introduce the fast variable $\xi = \varepsilon^{-1}(\mathrm{x} - M)$. In a vicinity of M, we have

$$(\Delta_x + k^2 \mathrm{Id}) \big(v_0(\varepsilon^{-1}(\mathrm{x} - M)) + \varepsilon\, v_1(\varepsilon^{-1}(\mathrm{x} - M)) + \dots \big)$$

$$= \varepsilon^{-2} \boxed{\Delta_\xi v_0(\xi)} + \varepsilon^{-1} \Delta_\xi v_1(\xi) + \dots\ .$$

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- Therefore we impose $\Delta_\xi v_0 = 0$ in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$ and so we take

$$v_0(\xi) = -u_0(M)\,W(\xi).$$

where W is the capacity potential for $\mathcal{O}$ (W is harmonic in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$, vanishes at infinity and verifies $W = 1$ on $\partial\mathcal{O}$).

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- As $|\xi| \to +\infty$, we have

$$W(\xi) = \frac{\mathrm{cap}(\mathcal{O})}{|\xi|} + \vec{q}\cdot\nabla\Phi(\xi) + O(|\xi|^{-3}),$$

where $\Phi := \xi \mapsto -1/(4\pi|\xi|)$ is the Green function of the Laplacian in $\mathbb{R}^3$, $\mathrm{cap}(\mathcal{O}) > 0$, $\vec{q} \in \mathbb{R}^3$.

- Now, we turn to the terms of order ε in the expansion of u^ε

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- By inserting $u_0 + \zeta(x)\, v_0(\varepsilon^{-1}(\mathrm{x} - M))$ into $(\mathscr{P}_\varepsilon)$ and replacing v_0 by its main contribution at infinity, we find that u_1 must solve

$$\left|\begin{array}{rcll} -\Delta u_1 - k^2 u_1 & = & -\left([\Delta_x, \zeta] + k^2\zeta\mathrm{Id}\right)\left(w^+(M)\,\dfrac{\mathrm{cap}(\mathcal{O})}{|\mathrm{x} - M|}\right) & \text{in } \Omega_0 \\ u_1 & = & 0 & \text{on } \partial\Omega_0. \end{array}\right.$$

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$\rightarrow$ This uniquely defines u_1.

Asymptotic of the scattering coefficients

- We consider the ansatz

$$R_\varepsilon = R_0 + \varepsilon R_1 + \dots \qquad T_\varepsilon = T_0 + \varepsilon T_1 + \dots.$$

- Set $\Sigma_{\pm L} = \{\pm L\} \times \omega$ for L large enough. From the known formula

$$2ikR_\varepsilon = \int_{\Sigma_{\pm L}} \partial_n u_\varepsilon w^+ - u_\varepsilon \partial_n w^+ d\sigma, \qquad 2ikT_\varepsilon = \int_{\Sigma_{\pm L}} \partial_n u_\varepsilon w^- - u_\varepsilon \partial_n w^- d\sigma,$$

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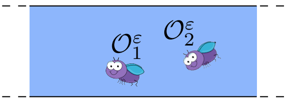
Integrating by parts, finally we get the final result:

Proposition: We have

$$\begin{aligned} R_\varepsilon &= 0 + \varepsilon \boxed{(4i\pi \operatorname{cap}(\mathcal{O}) w^+(M_1)^2)} + O(\varepsilon^2) \\ T_\varepsilon &= 1 + \varepsilon \boxed{(4i\pi \operatorname{cap}(\mathcal{O}) |w^+(M_1)|^2)} + O(\varepsilon^2). \end{aligned}$$

Several small obstacles



- One small obstacle cannot be non reflecting. Let us try with **TWO**, located at M_1, M_2.

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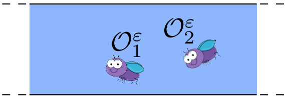

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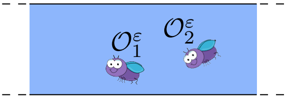

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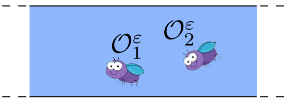
We can find M_1, M_2 such that $R_\varepsilon = O(\varepsilon^2)$.

Several small obstacles



- One small obstacle cannot be non reflecting. Let us try with **TWO**, located at M_1, M_2.

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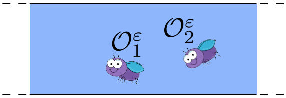

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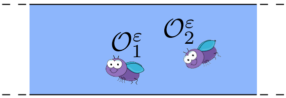

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→ When there are more propagating waves, we need more obstacles.

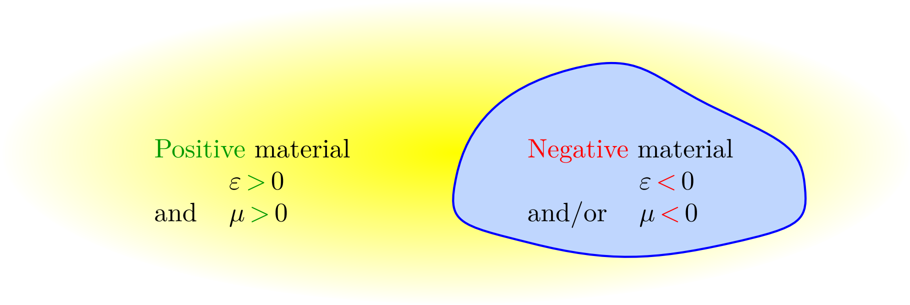
Acting as a team, obstacles can become invisible!

Corresponding reference

L. Chesnel and S. A. Nazarov. Team organization may help swarms of flies to become invisible in closed waveguides, Inverse Problens and Imaging, vol. 10, 4:977-1006, 2016.

Setting

- Scattering by a negative material in electromagnetism in time-harmonic regime (at a given frequency):



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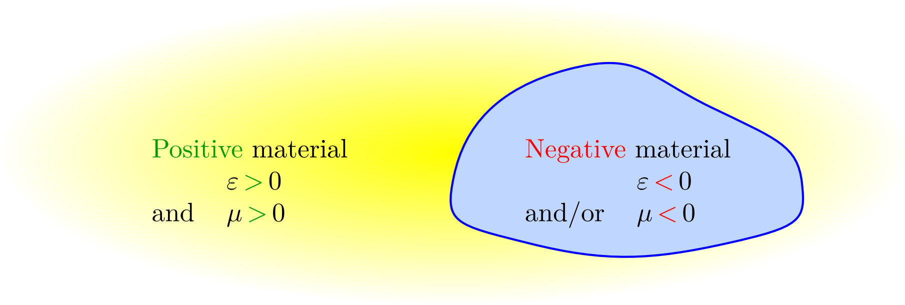

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- Metals at optical frequencies ($\varepsilon < 0$ and $\mu > 0$).

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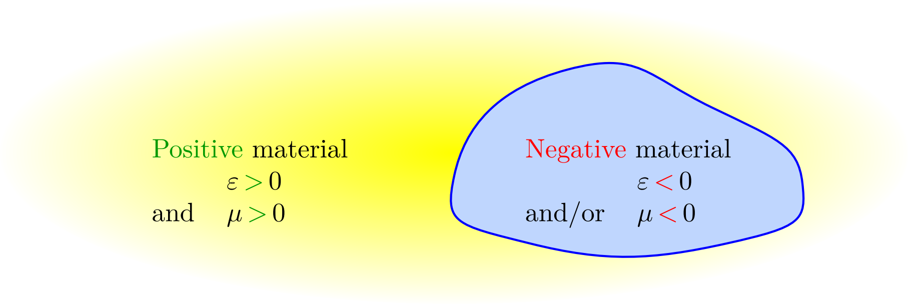

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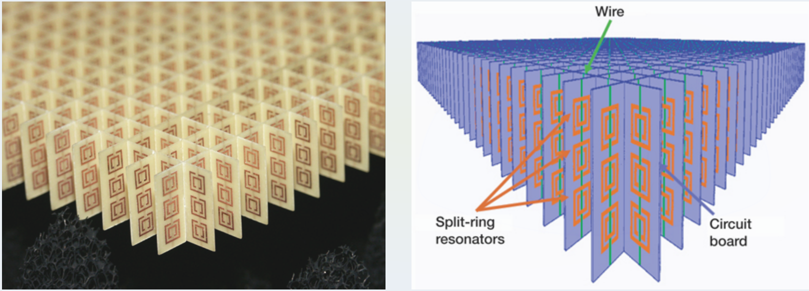
Zoom on a metamaterial: practical realizations of metamaterials are achieved by a periodic assembly of small resonators.



EXAMPLE OF METAMATERIAL (NASA)

Mathematical justification of the homogenized model (`Bouchitté, Bourel, Felbacq 09`).

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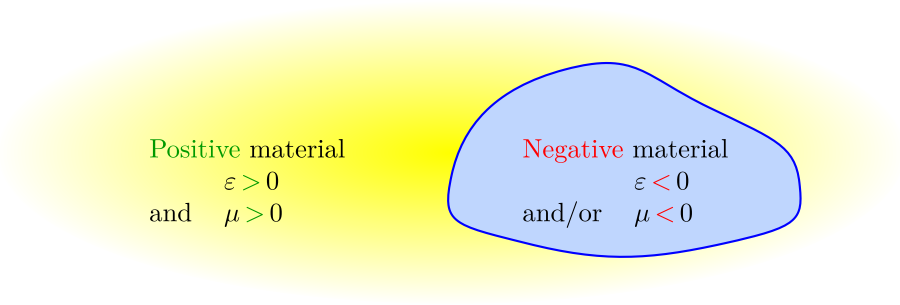

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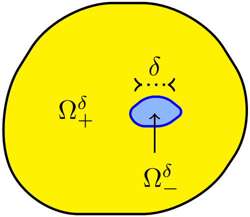
Spectral problem

- We investigate a Dirichlet spectral problem in presence of a small inclusion of negative material in a bounded domain.

- Let Ω, ω be smooth domains of $\mathbb{R}^3$ such that $O \in \omega$, $\overline{\omega} \subset \Omega$. For $\delta \in (0;1]$, we consider the problem

$$\left|\begin{array}{l}\text{Find } (\lambda^\delta, u^\delta) \in \mathbb{C} \times (\mathrm{H}^1_0(\Omega) \setminus \{0\}) \text{ s.t.:}\\ -\mathrm{div}(\sigma^\delta \nabla u^\delta) = \lambda^\delta u^\delta \quad \text{ in } \Omega, \text{ with,}\end{array}\right.$$



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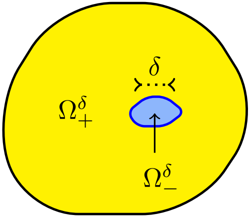

- $\mathrm{H}^1_0(\Omega) := \{u \in \mathrm{H}^1(\Omega) \,|\, u = 0 \text{ on } \partial\Omega\}$
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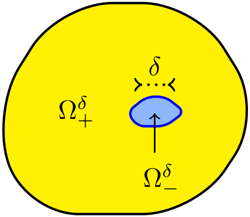

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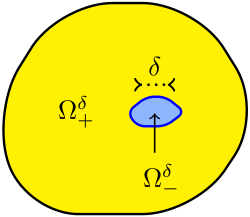

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- We define the operator $\mathrm{A}^\delta : D(\mathrm{A}^\delta) \to \mathrm{L}^2(\Omega)$ such that

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Main question

- Using boundary integral equations (see Costabel and Stephan 85, Dauge and Texier 97) or the T-coercivity approach (see Bonnet-Ben Dhia *et al.* 99,10,12,13), we can prove the :

Proposition. Assume that $\sigma_-/\sigma_+ \neq -1$. For $\delta > 0$, the operator A^δ is selfadjoint and has compact resolvent. Its spectrum $\mathfrak{S}(\mathrm{A}^\delta)$ consists in two sequences of isolated eigenvalues:

$$-\infty \underset{n\to+\infty}{\leftarrow} \dots \lambda^\delta_{-n} \le \dots \le \lambda^\delta_{-1} < 0 \le \lambda^\delta_1 \le \lambda^\delta_2 \le \dots \le \lambda^\delta_n \dots \underset{n\to+\infty}{\rightarrow} +\infty.$$

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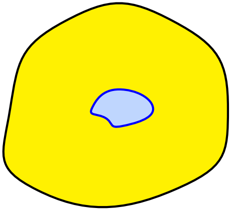
❓ What happens to the negative spectrum when δ tends to zero?

Far field operator

- As $\delta \to 0$, the small inclusion of negative material disappears.



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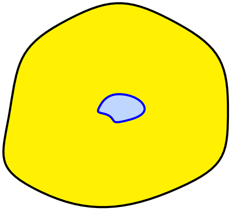

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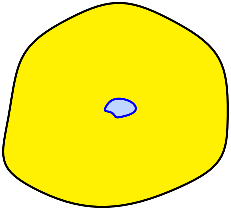

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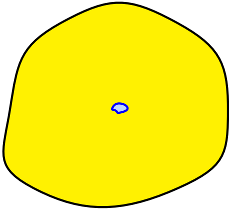

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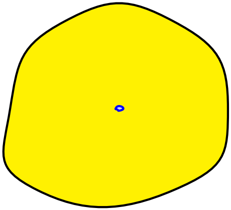

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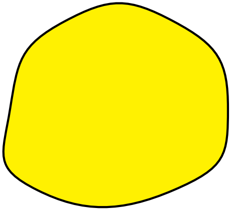

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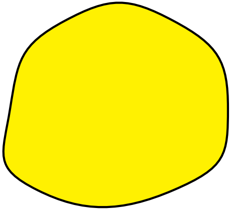


- We introduce the far field operator A^0 such that

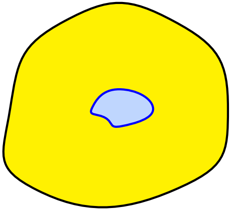
$$\left|\begin{array}{l} D(\mathrm{A}^0) = \{v \in \mathrm{H}^1_0(\Omega)\,|\,\Delta v \in \mathrm{L}^2(\Omega)\} \\ \mathrm{A}^0 v = -\sigma_+ \Delta v. \end{array}\right.$$

Proposition. There holds $\mathfrak{S}(\mathrm{A}^0) = \{\mu_n\}_{n\geq 1}$ with

$$0 < \mu_1 < \mu_2 \leq \dots \leq \mu_n \dots \underset{n\to+\infty}{\to} +\infty.$$

Near field operator

- Introduce the rapid coordinate $\boldsymbol{\xi} := \delta^{-1}\boldsymbol{x}$ and let $\delta \to 0$.



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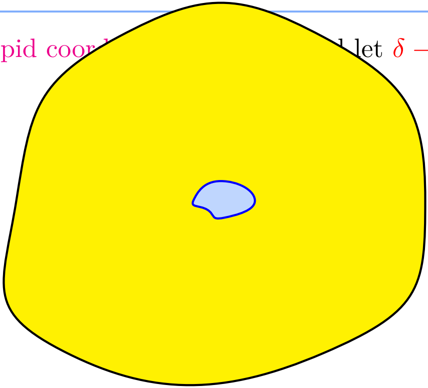

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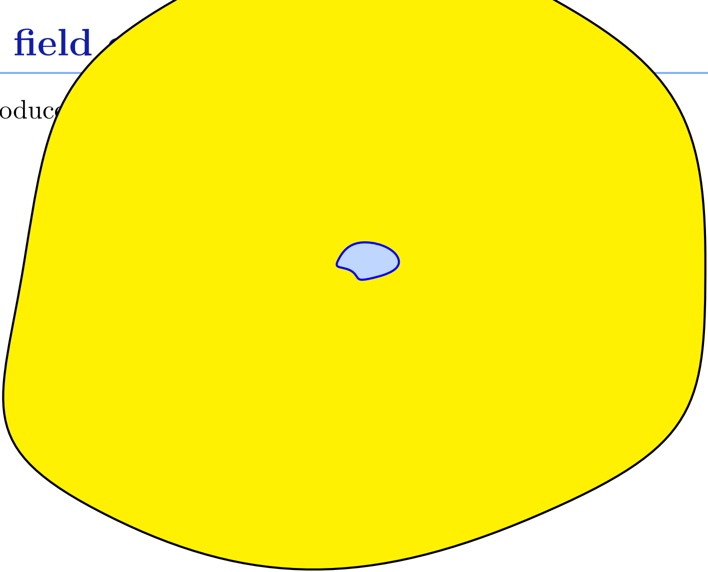

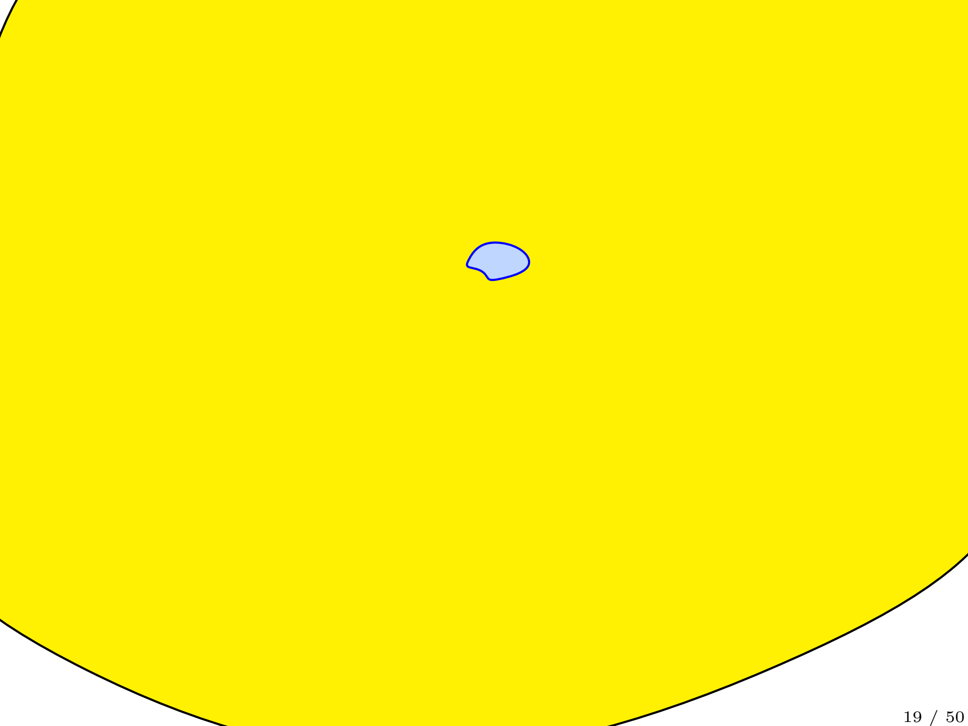

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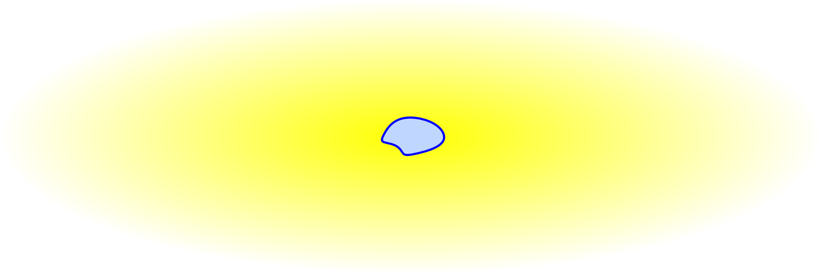

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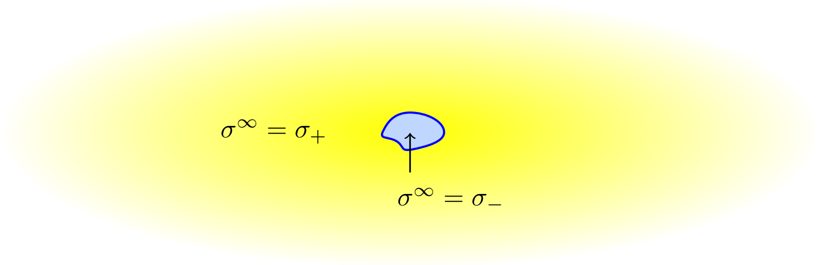

- Define the near field operator B^∞ such that

$$\left| \begin{array}{l} D(\mathrm{B}^\infty) = \{w \in \mathrm{H}^1(\mathbb{R}^3) \mid \operatorname{div}(\sigma^\infty \nabla w) \in \mathrm{L}^2(\mathbb{R}^3)\} \\ \mathrm{B}^\infty w \ = \ -\operatorname{div}(\sigma^\infty \nabla w). \end{array} \right.$$

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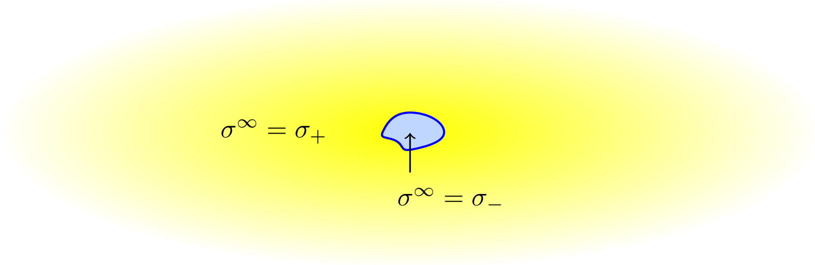

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PROPOSITION. Assume that $\sigma_-/\sigma_+ \neq -1$. The continuous spectrum of B^∞ is equal to $[0; +\infty)$ while its discrete spectrum is a sequence of eigenvalues:

$$\mathfrak{S}(\mathrm{B}^\infty) \setminus \overline{\mathbb{R}_+} = \{\mu_{-n}\}_{n\geq 1} \quad \text{with} \quad 0 > \mu_{-1} \geq \cdots \geq \mu_{-n} \ldots \underset{n\to+\infty}{\to} -\infty.$$

Results

Assume that $\sigma_-/\sigma_+ \neq -1$ and that B^∞ is injective. For $n \in \mathbb{N}^*$, we denote $\lambda^\delta_{\pm n}$, μ^δ_n, μ^δ_{-n} the eigenvalues of A^δ, A^0, B^∞ as in the previous slides.

Theorem. (Positive spectrum) For all $n \in \mathbb{N}^*$, $\varepsilon \in (0;1)$, there exist constants $C, \delta_0 > 0$ depending on n, ε but independent of δ, such that

$$|\lambda^\delta_n - \mu_n| \leq C\,\delta^{3/2-\varepsilon}, \qquad \forall \delta \in (0;\delta_0].$$

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$$|\lambda_n^\delta - \mu_n| \leq C\,\delta^{3/2-\varepsilon}, \qquad \forall \delta \in (0;\delta_0].$$

Idea of the proof:

1. We prove the *a priori* estimate $\|u^\delta\|_{\mathrm{H}_0^1(\Omega)} \leq c\,\|\mathrm{A}^\delta u^\delta\|_\Omega$ for δ small enough (♠ hard part of the proof: weighted Sobolev spaces+overlapping cut-off functions +construction of almost inverse).

2. If (μ_n, v_n) is an eigenpair of A^0, we construct u such that

$$\|\mathrm{A}^\delta u - \mu_n u\|_\Omega \leq c\,\delta^\beta \|u\|_\Omega, \qquad \text{for some } \beta > 0.$$

3. If $(\lambda_n^\delta, u_n^\delta)$ is an eigenpair of A^δ, we construct v such that

$$\|\mathrm{A}^0 v - \lambda_n^\delta v\|_\Omega \leq c\,\delta^\beta \|v\|_\Omega, \qquad \text{for some } \beta > 0.$$

4. We conclude with a classical lemma on quasi eigenvalues.

Results

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THEOREM. (POSITIVE SPECTRUM) For all $n \in \mathbb{N}^*$, $\varepsilon \in (0;1)$, there exist constants $C, \delta_0 > 0$ depending on n, ε but independent of δ, such that

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Theorem. (Negative spectrum) For all $n \in \mathbb{N}^*$, there exist constants $C, \gamma, \delta_0 > 0$, depending on n but independent of δ, such that

$$|\lambda^\delta_{-n} - \delta^{-2}\mu_{-n}| \leq C\exp(-\gamma/\delta), \qquad \forall \delta \in (0;\delta_0].$$

Results

THEOREM. (NEGATIVE SPECTRUM) For all $n \in \mathbb{N}^*$, there exist constants $C, \gamma, \delta_0 > 0$, depending on n but independent of δ, such that

$$|\lambda^\delta_{-n} - \delta^{-2}\mu_{-n}| \leq C \exp(-\gamma/\delta), \qquad \forall \delta \in (0; \delta_0].$$

Why is it a δ^{-2}?

- If $(\lambda^\delta_{-n}, u^\delta_{-n})$ is an eigenpair of A^δ, there holds

$$\int_\Omega \sigma^\delta \nabla_x u^\delta \cdot \nabla_x v \, dx = \lambda^\delta \int_\Omega u^\delta v \, dx, \quad \forall v \in \mathrm{H}^1_0(\Omega).$$

- $x = \delta\xi \Rightarrow \nabla_x = \delta^{-1}\nabla_\xi$. Denoting $U^\delta(\xi) = u^\delta(\delta\xi)$, we deduce

$$\int_{\delta^{-1}\Omega} \sigma^\infty \nabla_\xi U^\delta \cdot \nabla_\xi V \, d\xi = \delta^2\lambda^\delta \int_{\delta^{-1}\Omega} U^\delta V \, d\xi, \quad \forall V \in \mathrm{H}^1_0(\delta^{-1}\Omega).$$

Why the convergence is exponential?

- If (μ_{-n}, v_{-n}) is an eigenpair of B^∞, v_{-n} is exponentially decaying at ∞.

Results

Assume that $\sigma_-/\sigma_+ \neq -1$ and that B^∞ is injective. For $n \in \mathbb{N}^*$, we denote $\lambda^\delta_{\pm n}$, μ^δ_n, μ^δ_{-n} the eigenvalues of A^δ, A^0, B^∞ as in the previous slides.

THEOREM. (POSITIVE SPECTRUM) For all $n \in \mathbb{N}^*$, $\varepsilon \in (0;1)$, there exist constants $C, \delta_0 > 0$ depending on n, ε but independent of δ, such that

$$|\lambda^\delta_n - \mu_n| \leq C\,\delta^{3/2-\varepsilon}, \qquad \forall \delta \in (0;\delta_0].$$

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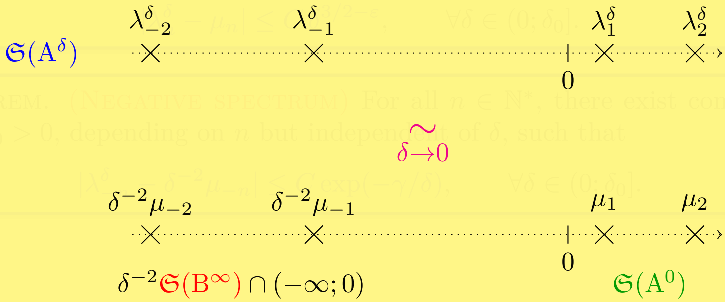
Schematically, we have:



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PROPOSITION. (LOCALIZATION EFFECT) For all $n \in \mathbb{N}^*$, let u^δ_{-n} be an eigenfunction corresponding to the negative eigenvalue λ^δ_{-n}. There exist constants $C, \gamma, \delta_0 > 0$, depending on n but independent of δ, such that

$$\int_\Omega (|u^\delta_{-n}|^2 + |\nabla u^\delta_{-n}|^2) e^{\gamma x/\delta} d\boldsymbol{x} \leq C\,\|u^\delta_{-n}\|_\Omega, \qquad \forall \delta \in (0;\delta_0].$$

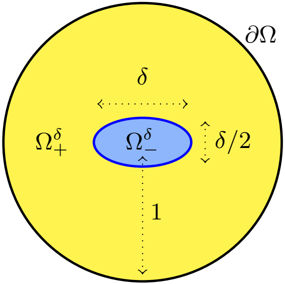
Numerical experiments

- Using *FreeFem++*, we approximate numerically the spectrum of A^δ using a usual P1 Finite Element Method. We solve the problem

$$\boxed{\begin{array}{|l} \text{Find } (\lambda_h^\delta, u_h^\delta) \in \mathbb{C} \times (\mathrm{V}_h \setminus \{0\}) \text{ s.t.:} \\ \displaystyle\int_\Omega \sigma_h^\delta \nabla u_h^\delta \cdot \nabla v_h = \lambda_h^\delta \int_\Omega u_h^\delta\, v_h, \quad \forall v_h \in \mathrm{V}_h, \end{array}}$$

where V_h approximates $\mathrm{H}_0^1(\Omega)$ as $h \to 0$ (h is the mesh size).

- We consider the following 2D geometry:



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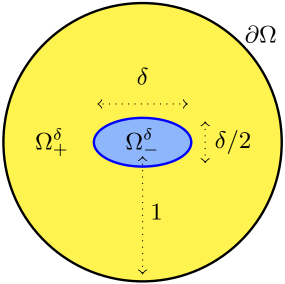
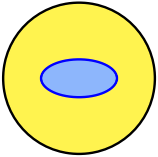
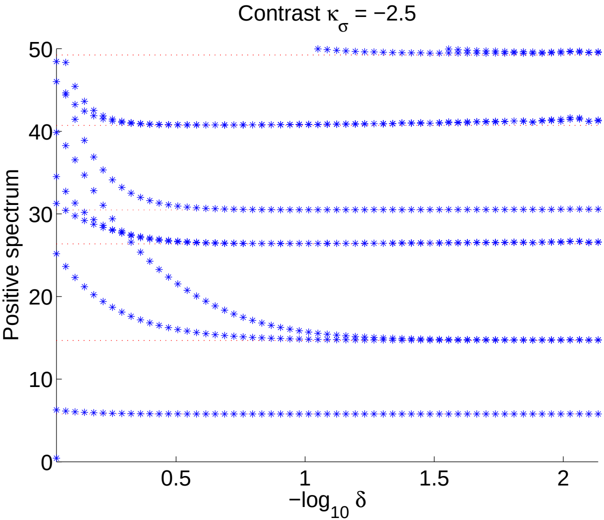
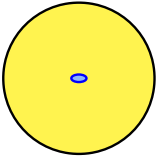


We display the spectrum as $\delta \to 0$ (h is more or less fixed).

Numerical experiments

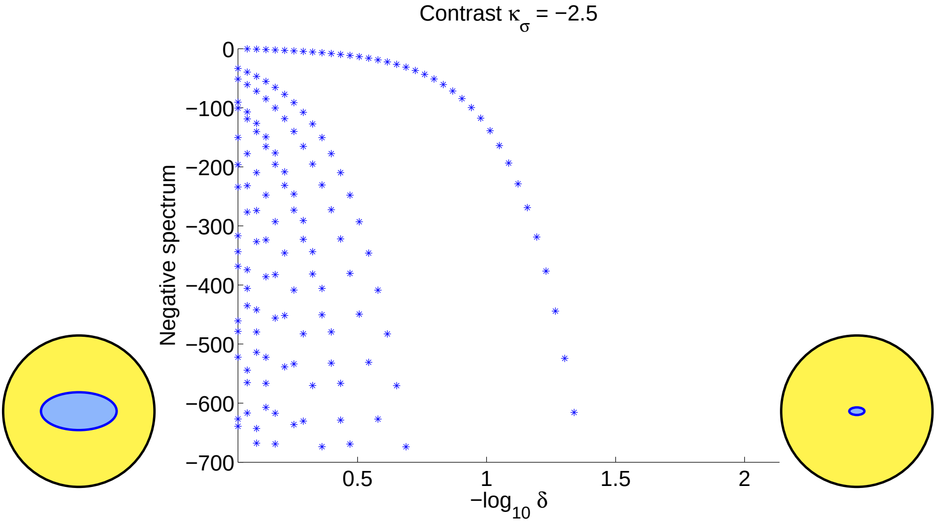






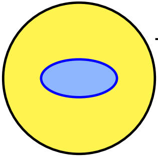
- The positive part of $\mathfrak{S}(A^\delta)$ converges to $\mathfrak{S}(A^0)$ when $\delta \to 0$.

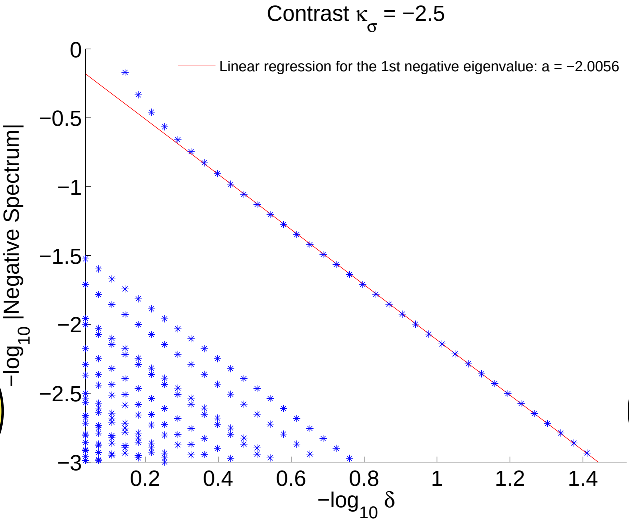
Numerical experiments

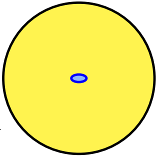


- The negative part of $\mathfrak{S}(\mathrm{A}^{\delta})$ is asymptotically equivalent to the negative part of $\delta^{-2}\mathfrak{S}(\mathrm{B}^{\infty})$ when $\delta \to 0$.

Numerical experiments







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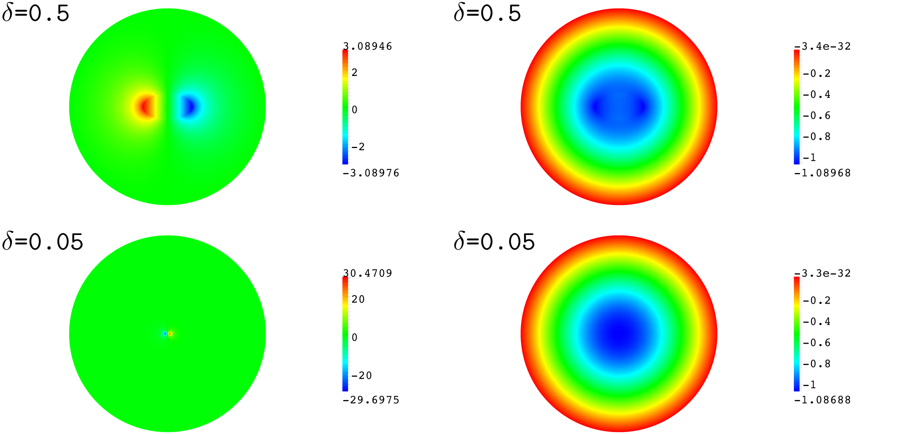
Localization effect

Eigenfunction associated to the first negative eigenvalue	Eigenfunction associated to the first positive eigenvalue



- The eigenfunctions corresponding to the negative eigenvalues are localized around the small inclusion. Here, $\sigma_-/\sigma_+ = -2.5$.

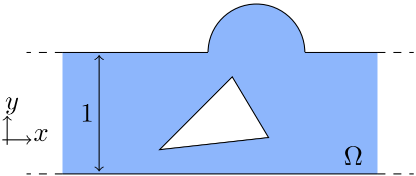
Corresponding references

A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., *T-coercivity for scalar interface problems between dielectrics and metamaterials*, M2AN, 46, 1363–1387, 2012.

L. Chesnel, X. Claeys, S.A. Nazarov, *Spectrum for a small inclusion of negative material*, Math. Mod. Num. Anal., vol. 52, 4:1285-1313, 2018.

Setting

- We consider the propagation of waves in a 2D acoustic waveguide with an obstacle (also relevant in optics, microwaves, water-waves theory,...).



$$\boxed{(\mathscr{P})\left|\begin{array}{rcll}\Delta u + k^2 u & = & 0 & \text{in } \Omega,\\ \partial_n u & = & 0 & \text{on } \partial\Omega\end{array}\right.}$$

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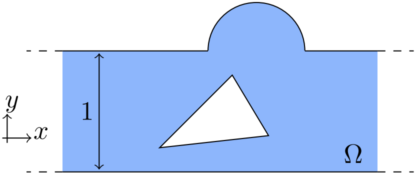

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► The scattering of these waves leads us to consider the solutions of $(\mathscr{P})$ with the decomposition

$$u_+ = \left| \begin{array}{l} e^{ikx} + R_+\, e^{-ikx} + \dots \\ \qquad T \quad e^{+ikx} + \dots \end{array} \right. \qquad u_- = \left| \begin{array}{ll} \qquad T \quad e^{-ikx} + \dots & x \to -\infty \\ e^{-ikx} + R_-\, e^{+ikx} + \dots & x \to +\infty \end{array} \right.$$

$R_\pm$, $T \in \mathbb{C}$ are the scattering coefficients , the $\dots$ are expon. decaying terms.

Goal

We wish to slightly perturb the walls of the guide to obtain $R_{\pm} = 0$, $T = 1$ in the new geometry (as if there were no obstacle) $\Rightarrow$ cloaking at "infinity".

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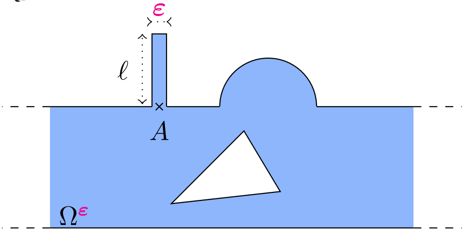
Difficulty: the scattering coefficients have a not explicit and not linear dependence wrt the geometry.

Difference with what we did previously: we wish to cloak big obstacles and not only small perturbations.

Setting

Main ingredient of our approach: outer resonators of width $\varepsilon \ll 1$.



$$(\mathscr{P}^{\varepsilon}) \left| \begin{array}{rl} \Delta u + k^2 u = 0 & \text{in } \Omega^{\varepsilon}, \\ \partial_n u = 0 & \text{on } \partial\Omega^{\varepsilon} \end{array} \right.$$

- In this geometry, we have the scattering solutions

$$u_+^{\varepsilon} = \left| \begin{array}{r} e^{ikx} + R_+^{\varepsilon}\, e^{-ikx} + \dots \\ T^{\varepsilon}\, e^{+ikx} + \dots \end{array} \right. \qquad u_-^{\varepsilon} = \left| \begin{array}{r} T^{\varepsilon}\, e^{-ikx} + \dots \\ e^{-ikx} + R_-^{\varepsilon}\, e^{+ikx} + \dots \end{array} \right. \qquad \begin{array}{l} x \to -\infty \\ x \to +\infty \end{array}$$

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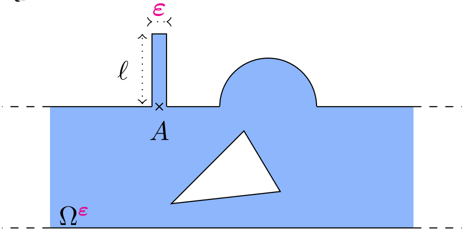

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Next we compute an asymptotic expansion of $u_\pm^\varepsilon$, $R_\pm^\varepsilon$, T^ε as $\varepsilon \to 0$.
(see Beale 73, Gadyl'shin 93, Kozlov et al. 94, Nazarov 96, Maz'ya et al. 00, Joly & Tordeux 06, Lin & Zhang 17, 18, ...).

Asymptotic analysis

- We work with the outer expansions

$$\begin{aligned} u_+^\varepsilon(x,y) &= u^0(x,y) + \dots && \text{in } \Omega, \\ u_+^\varepsilon(x,y) &= \varepsilon^{-1} v^{-1}(y) + v^0(y) + \dots && \text{in the resonator.} \end{aligned}$$

- Considering the restriction of $(\mathscr{P}^\varepsilon)$ to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous 1D problem

$$\boxed{(\mathscr{P}_{1\mathrm{D}}) \left| \begin{array}{ll} \partial_y^2 v + k^2 v = 0 & \text{in } (1; 1+\ell) \\ v(1) = \partial_y v(1+\ell) = 0. & \end{array} \right.}$$

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The features of $(\mathscr{P}_{1\mathrm{D}})$ play a key role in the physical phenomena and in the asymptotic analysis.

- We denote by ℓ_{res} (**resonance lengths**) the values of ℓ, given by

$$
\ell_{\mathrm{res}} := \pi(m + 1/2)/k, \qquad m \in \mathbb{N},
$$

such that $(\mathscr{P}_{1\mathrm{D}})$ admits the non zero solution $v(y) = \sin(k(y-1))$.

Asymptotic analysis – Non resonant case

- Assume that $\ell \neq \ell_{\text{res}}$. Then we find $v^{-1} = 0$ and when $\varepsilon \to 0$, we get

$$
\boxed{
\begin{array}{ll}
u^\varepsilon_\pm(x,y) = u_\pm + o(1) & \text{in } \Omega, \\[2ex]
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The thin resonator has no influence at order ε^0.

$\to$ **Not interesting for our purpose** because we want $\left|\begin{array}{l} R^\varepsilon_\pm = 0 + \dots \\ T^\varepsilon = 1 + \dots \end{array}\right.$

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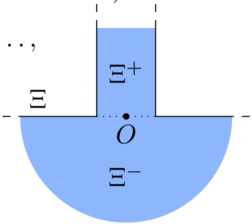
$$(\Delta_{\mathrm{x}} + k^2)u_+^\varepsilon(\varepsilon^{-1}(\mathrm{x}-A)) = \varepsilon^{-2}\Delta_\xi u^\varepsilon(\xi) + \dots,$$

when $\varepsilon \to 0$, we are led to study the problem

$$(\star)\left|\begin{array}{rl} -\Delta_\xi Y = 0 & \text{in } \Xi \\ \partial_\nu Y = 0 & \text{on } \partial\Xi. \end{array}\right.$$



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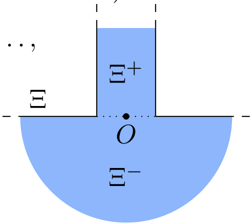

- Problem $(\star)$ admits a solution Y^1 (up to a constant) with the expansion

$$Y^1(\xi) = \begin{cases} \xi_y + C_\Xi + O(e^{-\pi\xi_y}) & \text{as } \xi_y \to +\infty, \quad \xi \in \Xi^+ \\ \dfrac{1}{\pi}\ln\dfrac{1}{|\xi|} + O\left(\dfrac{1}{|\xi|}\right) & \text{as } |\xi| \to +\infty, \quad \xi \in \Xi^-. \end{cases}$$

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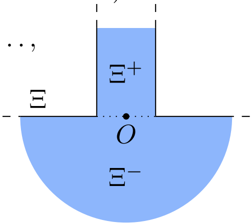

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Asymptotic analysis – Resonant case

- In the ansatz $u_+^\varepsilon = u^0 + \dots$ in Ω, we deduce that we must take

$$u_0 = u_+ + ak\gamma$$

where γ is the outgoing Green function such that $\left|\begin{array}{l}\Delta\gamma + k^2\gamma = 0 \text{ in } \Omega\\ \partial_n\gamma = \delta_A \text{ on } \partial\Omega.\end{array}\right.$

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- Matching the constant behaviour in the resonator, we obtain

$$v^0(1) = u_+(A) + ak(\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi).$$

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$$\left|\begin{array}{ll} \partial_y^2 v^0 + k^2 v^0 = 0 & \text{in } (1; 1+\ell) \\ v^0(1) = u_+(A) + ak(\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi), & \partial_y v^0(1+\ell) = 0. \end{array}\right.$$

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- This is a Fredholm problem with a non zero kernel. A solution exists iff the **compatibility condition** is satisfied. This sets

$$ak = -\frac{u_+(A)}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi}$$

and **ends the calculus of the first terms**.

Asymptotic analysis – Resonant case

- Finally for $\ell = \ell_{\text{res}}$, when $\varepsilon \to 0$, we obtain

$$u_+^\varepsilon(x,y) = u_+(x,y) + ak\gamma(x,y) + o(1) \quad \text{in } \Omega,$$

$$u_+^\varepsilon(x,y) = \varepsilon^{-1} a \sin(k(y-1)) + O(1) \quad \text{in the resonator},$$

$$R_+^\varepsilon = R_+ + i a u_+(A)/2 + o(1), \qquad T^\varepsilon = T + i a u_-(A)/2 + o(1).$$

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This time the thin resonator has an influence at order ε^0

Asymptotic analysis – Resonant case

- Similarly for $\ell = \ell_{\text{res}} + \varepsilon\eta$ with $\eta \in \mathbb{R}$ fixed, by modifying only the last step with the compatibility relation, when $\varepsilon \to 0$, we obtain

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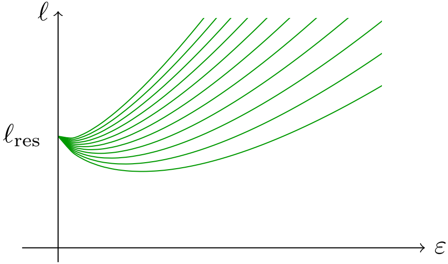
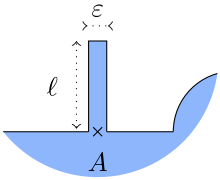
This time the thin resonator has an influence at order ε^0 and it depends on the choice of η!

Asymptotic analysis – Resonant case

- Below, for several $\eta \in \mathbb{R}$, we display the paths

$$\{(\varepsilon, \ell_{\text{res}} + \varepsilon(\eta - \pi^{-1}|\ln \varepsilon|)),\ \varepsilon > 0\} \subset \mathbb{R}^2.$$





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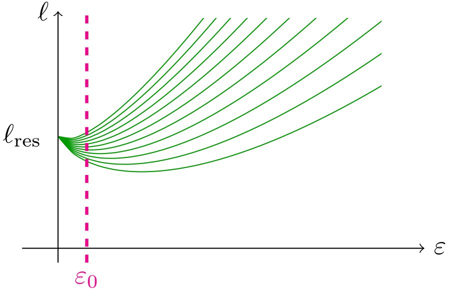
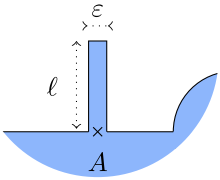




According to η, the limit of the scattering coefficients along the path as $\varepsilon \to 0^+$ is different.

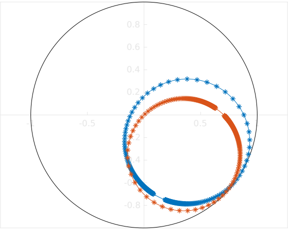
- For a fixed small ε_0, the scattering coefficients have a rapid variation for ℓ varying in a neighbourhood of the resonance length.

Almost zero reflection

- We got $\left| \begin{array}{rcl} R_+^\varepsilon & = & R_+^0(\eta) + o(1) \\ T^\varepsilon & = & T^0(\eta) + o(1) \end{array} \right.$ with $\left| \begin{array}{rcl} R_+^0(\eta) & := & R_+ + ia(\eta)\, u_\pm(A)\,/2 \\ T^0(\eta) & := & T + ia(\eta)\, u_\pm(A)\,/2. \end{array} \right.$

- One can show that $\{R_+^0(\eta)\,|\,\eta \in \mathbb{R}\}$, $\{T^0(\eta)\,|\,\eta \in \mathbb{R}\}$ are circles in $\mathbb{C}$.

Asymptotically, when the length of the resonator is perturbed **around the resonance length**, R_+^ε, T^ε run on circles.



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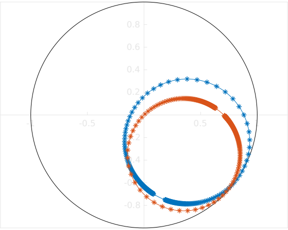

- Using the expansions of $u_\pm(A)$ far from the obstacle, one shows:

PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta)\,|\,\eta \in \mathbb{R}\}$ passes **through zero**.

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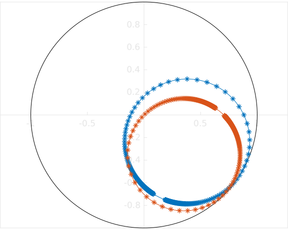
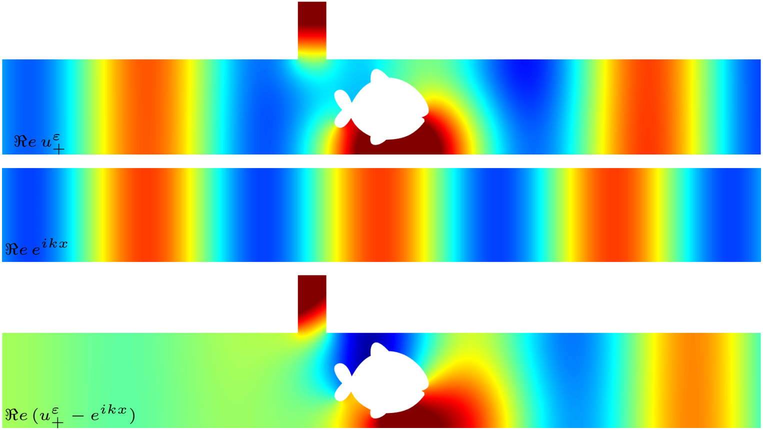


- Using the expansions of $u_\pm(A)$ far from the obstacle, one shows:

PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta)\,|\,\eta\in\mathbb{R}\}$ passes **through zero**. $\Rightarrow \exists$ situations s.t. $R_+^\varepsilon = 0 + o(1)$.

Almost zero reflection

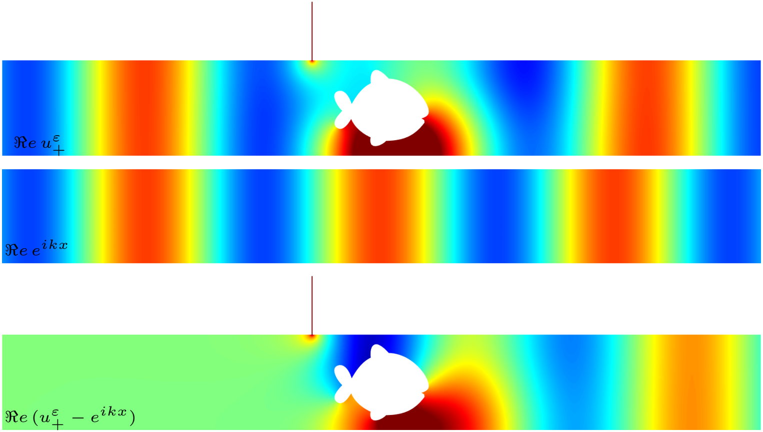
- Example of situation where we have almost zero reflection ($\varepsilon = \mathbf{0.3}$).



$\rightarrow$ Simulations realized with the `Freefem++` library.

Almost zero reflection

- Example of situation where we have almost zero reflection ($\varepsilon = \mathbf{0.01}$).



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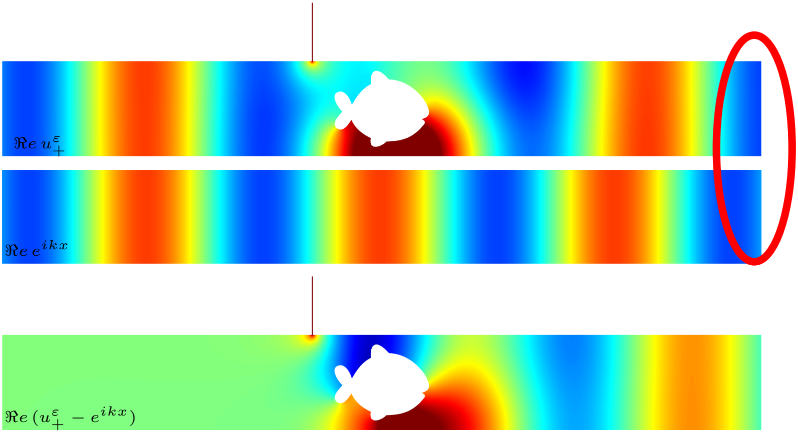
Almost zero reflection

- Example of situation where we have almost zero reflection ($\boldsymbol{\varepsilon = 0.01}$).

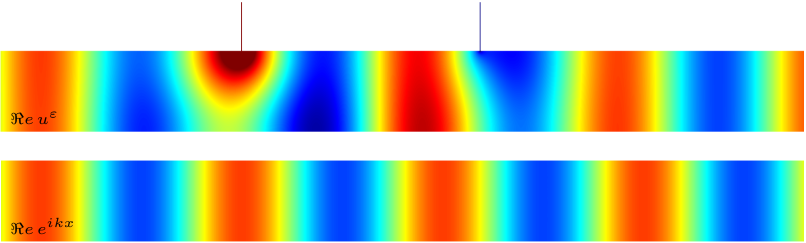


$\rightarrow$ Simulations realized with the `Freefem++` library.

To cloak the object, it remains to compensate the phase shift!

Phase shifter

- Working with two resonators, we can create phase shifters, that is devices with almost zero reflection and any desired phase.

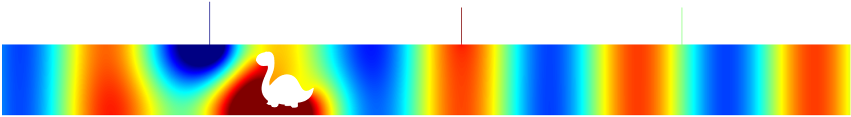


- Here the device is designed to obtain a phase shift approx. equal to $\pi/4$.

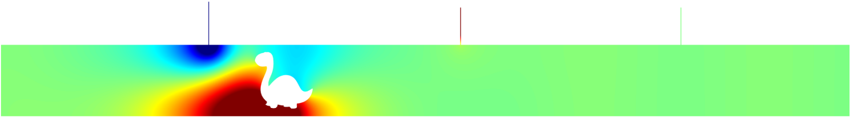
Cloaking with three resonators

- Gathering the two previous results, we can cloak any object with three resonators.

$\Re e\, u_+$



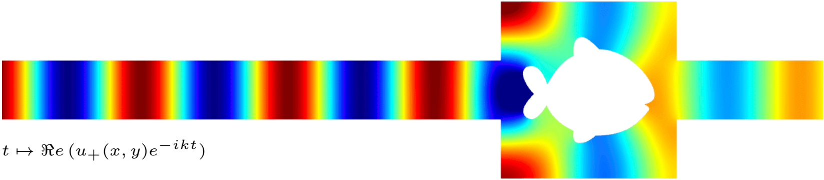
$\Re e\, u_+^\varepsilon$



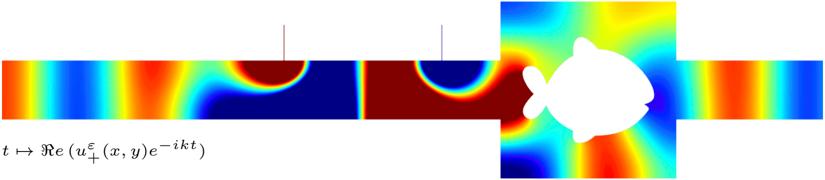
$\Re e\, (u_+^\varepsilon - e^{ikx})$

Cloaking with two resonators

- Working a bit more, one can show that two resonators are enough to cloak any object.



$t \mapsto \Re e\,(u_+(x,y)e^{-ikt})$

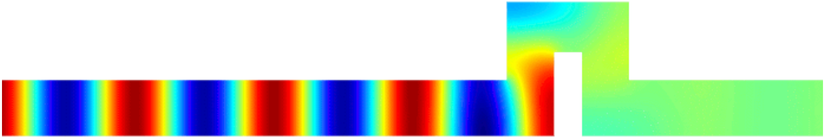


$t \mapsto \Re e\,(u_+^\varepsilon(x,y)e^{-ikt})$

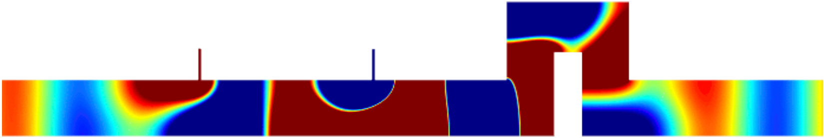
$t \mapsto \Re e\,(e^{ik(x-t)})$

Cloaking with two resonators

- Another example



$t \mapsto \Re e\left(u_+(x,y)e^{-ikt}\right)$



$t \mapsto \Re e\left(u_+^{\varepsilon}(x,y)e^{-ikt}\right)$

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Recap of the cloaking strategy

What we did

♠ We explained how to approximately cloak any object in monomode regime using thin resonators. Two main ingredients:

- Around resonant lengths, effects of order ε^0 with perturb. of width ε.
- The 1D limit problems in the resonator provide a rather explicit dependence wrt to the geometry.

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- Around resonant lengths, effects of order ε^0 with perturb. of width ε.
- The 1D limit problems in the resonator provide a rather explicit dependence wrt to the geometry.

Possible extensions and open questions

1) We can similarly hide penetrable obstacles or work in 3D.
2) We can do cloaking at a finite number of wavenumbers (thin structures are resonant at one wavenumber otherwise act at order ε).
3) With Dirichlet BCs, other ideas must be found.
4) Can we realize exact cloaking ($T = 1$ exactly)? This question is also related to robustness of the device.

Corresponding reference

L. Chesnel, J. Heleine and S.A. Nazarov. Acoustic passive cloaking using thin outer resonators. submitted, arXiv:2105.00922, 2021.

1. Non reflecting small obstacles in waveguide
2. Spectrum in presence of a small negative inclusion
3. Cloaking in acoustic waveguides

Conclusion of session 4

What we did

1) We explained how small obstacles can be arranged to get zero reflection in waveguides.
2) We studied the spectrum of a diffusion operator in presence of a small inclusion of negative material.
3) We showed how to approximately cloak defects in acoustic waveguides using thin resonators.

Conclusion of the course

What we did

1) We gave on certain examples of smooth perturbations a few general ideas of asymptotic analysis.

2) We detailed how to address small obstacle asymptotics.

3) We explained how to establish error estimates in certain situations.

4) We presented examples of applications of asymptotic analysis.

It is important to mention however that each problem requires a rather specific treatment. There is no real systematic approach and non trivial questions appear very often.

$\rightarrow$ **To be continued...**