Non-scattering wavenumbers and far field invisibility for a finite set of incident/scattering directions

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Abstract

We investigate a time harmonic acoustic scattering problem by a penetrable inclusion with compact support embedded in the free space. We consider cases where an observer can produce incident plane waves and measure the far field pattern of the resulting scattered field only in a finite set of directions. In this context, we say that a wavenumber is a non-scattering wavenumber if the associated relative scattering matrix has a non trivial kernel. Under certain assumptions on the physical coefficients of the inclusion, we show that the non-scattering wavenumbers form a (possibly empty) discrete set. Then, in a second step, for a given real wavenumber, we present a constructive technique (which provides a numerical algorithm) to prove that there exist inclusions for which the corresponding relative scattering matrix is null. These inclusions have the important property to be impossible to detect from far field measurements.

Keywords: non-scattering wavenumbers, invisibility.

1 Setting

Consider an inclusion supported in $\overline{\mathcal{D}}$, where $\mathcal{D} \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with Lipschitz boundary. We assume that the scattering of the incident plane wave $u_{\mathrm{i}} := e^{ik\boldsymbol{\theta}_{\mathrm{i}}\cdot\boldsymbol{x}}$, of direction of propagation $\boldsymbol{\theta}_{\mathrm{i}} \in \mathbb{S}^{d-1}$, by $\mathcal{D}$, is described by the problem

$$\left|\begin{array}{l}\text{Find } u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d) \text{ such that}\\ \begin{array}{rcll} -\Delta u &=& k^2\rho\, u & \text{in } \mathbb{R}^d,\\ u &=& u_{\mathrm{i}} + u_{\mathrm{s}} & \text{in } \mathbb{R}^d,\end{array}\\ \displaystyle\lim_{r\to+\infty} r^{\frac{d-1}{2}}\left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}}\right) = 0.\end{array}\right. \qquad (1)$$

In (1), the real valued function ρ models the properties of the inclusion and is such that $\rho - 1$ is supported in $\overline{\mathcal{D}}$. It is known that the scattered field $u_{\mathrm{s}}(\cdot, \boldsymbol{\theta}_{\mathrm{i}})$ admits the expansion

$$u_{\mathrm{s}}(\boldsymbol{x}, \boldsymbol{\theta}_{\mathrm{i}}) = e^{ikr} r^{-\frac{d-1}{2}} \left(u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_{\mathrm{s}}, \boldsymbol{\theta}_{\mathrm{i}}) + O(1/r) \right),$$

as $r \to +\infty$, uniformly in $\boldsymbol{\theta}_{\mathrm{s}} \in \mathbb{S}^{d-1}$. Here $\boldsymbol{\theta}_{\mathrm{s}}$ is the direction of observation. We shall assume we have a finite set of emitters and receivers located at the same positions so that we can produce incident plane waves in some given directions $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N \in \mathbb{S}^{d-1}$ and measure the far field pattern of the resulting scattered field only in the directions $-\boldsymbol{\theta}_1, \dots, -\boldsymbol{\theta}_N$ (backscattering directions). This corresponds to knowing all elements of the relative scattering matrix $\mathscr{A}(k) \in \mathbb{C}^{N\times N}$ such that

$$\mathscr{A}_{mn}(k) = u_{\mathrm{s}}^{\infty}(-\boldsymbol{\theta}_m, \boldsymbol{\theta}_n). \qquad (2)$$

2 Discreteness of non-scattering wavenumbers

We say that $k > 0$ is a non-scattering wavenumber if $\mathscr{A}_{mn}(k)$ has a non trivial kernel. In this case, there is an incident field, combination of the plane waves of directions $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N$ whose scattered field vanishes at infinity in the directions $-\boldsymbol{\theta}_1, \dots, -\boldsymbol{\theta}_N$. To prove that non-scattering wavenumbers form an empty or discrete set, we use the following strategy.

i) We show that $k \mapsto \mathscr{A}(k)$ can be meromorphically continued to the complex plane.

ii) For $k = i\kappa$, with $\kappa > 0$, we establish energy identities allowing to infer that $\mathscr{A}(k)$ is injective under certain assumptions on ρ.

iii) We conclude using the principle of isolated zeros.

3 Construction of invisible inclusions

Now, assume that $k > 0$, $\mathcal{D}$ and $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N$ are given. We develop a technique (introduced in [1, 3]) to build real valued functions ρ supported in $\overline{D}$ such that $\mathscr{A}(k)$ is the null matrix. For such inclusions, for all incident fields

combinations of the plane waves of directions $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N$, the scattered field vanishes at infinity in the directions $-\boldsymbol{\theta}_1, \dots, -\boldsymbol{\theta}_N$. Below, we explain the general procedure (for details, see [2]).
Because of the reciprocity relation

$$u_{\mathrm{s}}^{\infty}(-\boldsymbol{\theta}_m, \boldsymbol{\theta}_n) = u_{\mathrm{s}}^{\infty}(-\boldsymbol{\theta}_n, \boldsymbol{\theta}_m),$$

the matrix $\mathscr{A}(k)$ is symmetric. Let us look for ρ under the form $\rho = 1 + \varepsilon\mu$ with $\varepsilon > 0$, $\mu \in \mathrm{L}^{\infty}_{\mathbb{R}}(\mathcal{D})$ (the set of real valued L^{∞} functions). Define the map $F : \mathrm{L}^{\infty}_{\mathbb{R}}(\mathcal{D}) \to \mathbb{R}^{N(N+1)}$ by

$$F(\varepsilon\mu) = \Big(\Re e(u_{\mathrm{s}}^{\infty}(-\boldsymbol{\theta}_m, \boldsymbol{\theta}_n)), \\ \Im m(u_{\mathrm{s}}^{\infty}(-\boldsymbol{\theta}_m, \boldsymbol{\theta}_n))\Big)_{1 \le m \le n \le N}.$$

Our goal is to find $\varepsilon\mu \not\equiv 0$ such that $F(\varepsilon\mu) = 0$. Since $F(0) = 0$, for ε small enough, we obtain the following Taylor expansion

$$F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}(\varepsilon, \mu).$$

Assume that there are $\mu_1, \dots, \mu_{N(N+1)} \in \mathrm{L}^{\infty}_{\mathbb{R}}(\mathcal{D})$ such that $dF(0)(\mu_1), \dots, dF(0)(\mu_{N(N+1)})$ is a basis of $\mathbb{R}^{N(N+1)}$. Decompose μ as

$$\mu = \mu_0 + \sum_{i=1}^{N(N+1)} \tau_i \, \mu_i, \tag{3}$$

where the τ_i are real parameters to tune and $\mu_0 \in \ker dF(0)$. There holds $F(\varepsilon\mu) = 0$ iff $\vec{\tau} = (\tau_1, \dots, \tau_{N(N+1)})^{\top} \in \mathbb{R}^{N(N+1)}$ verifies

$$\mathbb{D}\,\vec{\tau} = \hat{F}^{\varepsilon}(\vec{\tau}), \tag{4}$$

where $\mathbb{D}$ is an invertible matrix and where $\hat{F}^{\varepsilon}(\vec{\tau}) = -\varepsilon\tilde{F}(\varepsilon, \mu)$. For any $\gamma > 0$, we can show that $\hat{F}^{\varepsilon}$ is a contraction of $\mathbb{B}_{\gamma} := \{\vec{\tau} \in \mathbb{R}^{N(N+1)} \,|\, |\vec{\tau}| \le \gamma\}$ for ε small enough. Therefore, the Banach fixed-point theorem guarantees the existence of some $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0; \varepsilon_0]$, (4) has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$ in $\mathbb{B}_{\gamma}$. Define $\rho = 1 + \varepsilon\mu$ with μ as in (3) and $\vec{\tau} = \vec{\tau}^{\,\mathrm{sol}}$. Then, for this inclusion there holds $\mathscr{A}(k) = 0$.
$\star$ When the vectors of the family $\{\boldsymbol{\theta}_m + \boldsymbol{\theta}_n\}_{1 \le m \le n \le N}$ are all non null and all different, we can prove that the functions $\mu_1, \dots, \mu_{N(N+1)}$ mentioned above exist. This allows to construct invisible inclusions in this situation.
$\star$ When there holds $\boldsymbol{\theta}_m + \boldsymbol{\theta}_n = 0$ for some incident directions $\boldsymbol{\theta}_m$, $\boldsymbol{\theta}_n$, the previous procedure fails. Actually, for any given $\boldsymbol{\theta}_{\mathrm{i}}$, we can show that imposing $u_{\mathrm{s}}^{\infty}(-\boldsymbol{\theta}_{\mathrm{i}}, \boldsymbol{\theta}_{\mathrm{i}}) = 0$ requires to impose $u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}, \boldsymbol{\theta}_{\mathrm{i}}) = 0$ for all $\boldsymbol{\theta}$. As a consequence of the Rellich lemma, this means that our task consists in finding an inclusion such that the incident plane wave $e^{ik\boldsymbol{\theta}_{\mathrm{i}}\cdot\boldsymbol{x}}$ produces no scattered field outside $\mathcal{D}$. This is a much more constrained problem and we do not know if it has a solution.

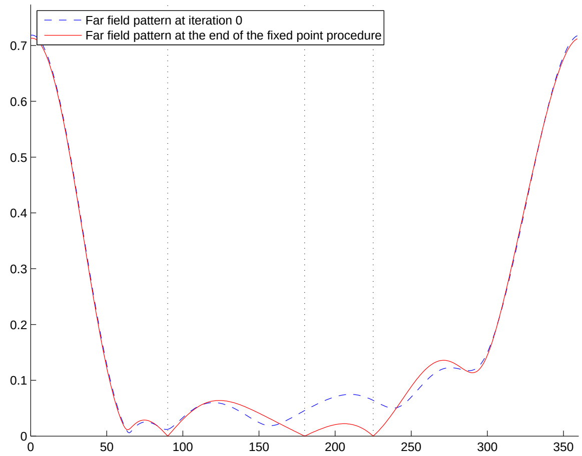

Figure 1: The fixed point problem (4) can be solved numerically. Here, we have constructed an inclusion which is invisible at infinity in the three directions indicated by the dotted lines (the solid curve represents the far field pattern at the end of the fixed point procedure).

References

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