A curious instability phenomenon for a rounded corner in presence of a negative material

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Abstract

We study a 2D transmission problem between a positive material and a negative material. In electromagnetism, this negative material can be a metal at optical frequencies or a negative metamaterial. We highlight an unusual instability phenomenon for this problem in some configurations: when the interface between the two materials presents a rounded corner, it can happen that the solution depends critically on the value of the rounding parameter.

1 Numerical observations

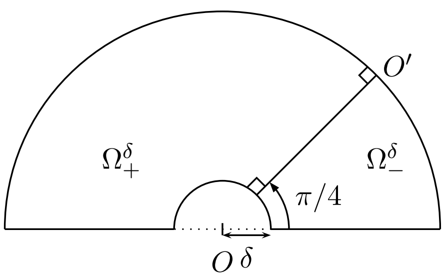

Figure 1: Domain Ω^δ.

Let us denote (r,θ) the polar coordinates centered at the origin O. Consider $\delta \in (0;1)$ and define (see Figure 1) the domains:

$$\begin{aligned}\Omega^\delta_+ &:= \{(r\cos\theta, r\sin\theta)\,|\,\delta < r < 1,\ \pi/4 < \theta < \pi\};\\ \Omega^\delta_- &:= \{(r\cos\theta, r\sin\theta)\,|\,\delta < r < 1,\ 0 < \theta < \pi/4\};\\ \Omega^\delta &:= \{(r\cos\theta, r\sin\theta)\,|\,\delta < r < 1,\ 0 < \theta < \pi\}.\end{aligned}$$

We define the function $\sigma^\delta : \Omega^\delta \to \mathbb{R}$ by $\sigma^\delta = \sigma_\pm$ in $\Omega^\delta_\pm$, where $\sigma_+ > 0$ and $\sigma_- < 0$ are constants. We shall focus on the problem:

$$\left|\begin{array}{l}\text{Find } u^\delta \in \mathrm{H}^1_0(\Omega^\delta) \quad \text{such that}\\ -\mathrm{div}(\sigma^\delta \nabla u^\delta) = f,\end{array}\right. \qquad (1)$$

where $\mathrm{H}^1_0(\Omega^\delta) := \{v \in \mathrm{H}^1(\Omega^\delta) \text{ s.t. } v|_{\partial\Omega^\delta} = 0\}$. We choose a source term $f \in \mathrm{L}^2(\Omega^\delta)$ whose support does not meet O and we try to approximate the solution of problem (1), assuming it is uniquely defined, by a classical finite element method. We call u^δ_h the numerical solution and we make δ tends to zero. The results are displayed on Figure 2. For a contrast $\kappa_\sigma := \sigma_-/\sigma_+ = -1.0001$, the sequence $(\|u^\delta_h\|_{\mathrm{H}^1_0(\Omega^\delta)})_\delta$ is relatively stable with respect to δ, for δ small enough. For $\kappa_\sigma := \sigma_-/\sigma_+ = -0.9999$, it looks that there exists of sequence of values of δ, which accumulates in zero, such that problem (1) is not well-posed. In other words, it seems that the solution of problem (1) is not stable with respect to δ when δ tends to zero. The goal of the present document is to understand these two observations.

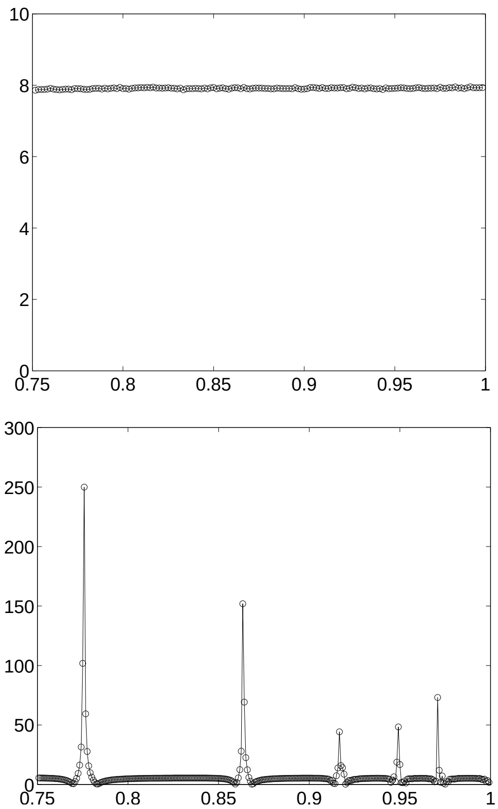

Figure 2: Evolution of $\|u^\delta_h\|_{\mathrm{H}^1_0(\Omega^\delta)}$ w.r.t. $1-\delta$. Above, we take $\sigma_+ = 1$ and $\sigma_- = -1.0001$. Below, we take $\sigma_+ = 1$ and $\sigma_- = -0.9999$.

2 Properties of the problem for $\delta = 0$

We associate with problem (1) the continuous linear operator $\mathcal{A}^\delta : \mathrm{H}^1_0(\Omega^\delta) \to \mathrm{H}^{-1}(\Omega^\delta)$ defined by $\langle \mathcal{A}^\delta u, v\rangle_{\Omega^\delta} = (\sigma^\delta \nabla u, \nabla v)_{\Omega^\delta}$, $\forall u, v \in \mathrm{H}^1_0(\Omega^\delta)$. As it is known from [1], $\mathcal{A}^\delta$ is a Fredholm operator of index 0 if and only if $\kappa_\sigma := \sigma_-/\sigma_+ \neq -1$, as the interface $\Sigma^\delta := \overline{\Omega}_+^\delta \cap \overline{\Omega}_-^\delta$ is smooth and meets $\partial\Omega^\delta$ orthogonally.

Now, for $\delta = 0$, the interface no longer meets $\partial\Omega^\delta$ perpendicularly. In the sequel, we denote $\mathcal{A}$, Ω and σ instead of $\mathcal{A}^0$, Ω^0 and σ^0. As shown in [1], there exist values of the contrasts $\kappa_\sigma = \sigma_-/\sigma_+$ for which the operator $\mathcal{A}$ fails to be of Fredholm type. More precisely, for the chosen configuration, $\mathcal{A}$ is a Fredholm operator (and actually, an isomorphism) if and only if, $\kappa_\sigma < 0$ does not belong to the *critical interval* $[-1;-1/3]$. Here, the value 3 comes from the ratio of the two apertures: $3 = (\pi - \pi/4)/(\pi/4)$.

$\star$ When $\kappa_\sigma = -1.0001 \notin [-1;-1/3]$, $\mathcal{A}$ is an isomorphism (c.f. [1]). In this case, we can prove that $\mathcal{A}^\delta$ is an isomorphism for δ small enough. Moreover, defining $u^\delta = (\mathcal{A}^\delta)^{-1}f$ and $u = \mathcal{A}^{-1}f$, we can show that the sequence (u^δ) converges to u for the H^1 norm. This explains the first curve of Figure 2.

$\star$ When $\kappa_\sigma = -0.9999 \in [-1;-1/3]$, $\mathcal{A}$ is not of Fredholm type (c.f. [1]). In this configuration, there is a qualitative difference between problem (1) for $\delta > 0$, and problem (1) for $\delta = 0$. In [2], we define a new functional framework to restore Fredholmness for the limit problem. More precisely, we prove that, for $\kappa_\sigma \in (-1;-1/3)$ the operator $\mathcal{A}^+ : \mathrm{V}_\beta^+ \to \mathrm{V}_\beta^1(\Omega)^*$ defined by $\langle \mathcal{A}^+u, v\rangle_\Omega = (\sigma\nabla u, \nabla v)_\Omega$, $\forall u \in \mathrm{V}_\beta^+$, $v \in \mathscr{C}_0^\infty(\Omega)$, is an isomorphism for all $\beta \in (0;2)$. In this notation, $\mathrm{V}_\beta^+ := \mathrm{span}\{s^+\} \oplus \mathrm{V}_{-\beta}^1(\Omega)$, where $s^+ \in \mathrm{L}^2(\Omega) \setminus \mathrm{H}^1(\Omega)$ is a singular function at O and $\mathrm{V}_{-\beta}^1(\Omega)$ is the completion of $\mathscr{C}_0^\infty(\Omega)$ for the weighted norm $\|\cdot\|_{\mathrm{V}_{-\beta}^1(\Omega)} = (\|r^{-\beta}\nabla\cdot\|^2_{\mathrm{L}^2(\Omega)} + \|r^{-\beta-1}\cdot\|^2_{\mathrm{L}^2(\Omega)})^{1/2}$.

3 Asymptotic expansion of the solution inside the critical interval

For a contrast inside the critical interval, the exotic functional framework introduced for the limit problem leads to two surprising phenomena in the asymptotic expansion of the solution of problem (1). First, when we proceed to a usual matched asymptotic expansion method, we observe that we can define an asymptotic expansion of the solution u^δ only for

$$\delta \in \mathscr{S}_{\mathrm{adm}} := (0;1) \setminus \mathscr{S}_{\mathrm{forb}} \quad \text{with } \mathscr{S}_{\mathrm{forb}} := \bigcup_{k\in\mathbb{N}} \delta_\star^k \delta_0,$$

$\delta_\star, \delta_0$ being two numbers of $(0;1)$. Notice that 0 is an accumulation point for $\mathscr{S}_{\mathrm{forb}}$. For $\alpha \in (0;1/2)$, we define $I(\alpha) := \bigcup_{k\in\mathbb{N}}[\delta_\star^{k+1-\alpha}\delta_0; \delta_\star^{k+\alpha}\delta_0] \subset \mathscr{S}_{\mathrm{adm}}$. In [3], we prove the following result:

Proposition 1. *Let $\beta \in (0;2)$ and $f \in \mathrm{V}_\beta^1(\Omega)^*$. There exists δ_0 such that problem (1) is uniquely solvable for all $\delta \in (0;\delta_0) \cap \mathrm{I}(\alpha)$, with $\alpha \in (0;1/2)$. Moreover, we can build an approximation $\hat{u}^\delta \in \mathrm{H}_0^1(\Omega^\delta)$ of u^δ such that, for all ε in $(0;\beta)$, $\forall \delta \in (0;\delta_0) \cap \mathrm{I}(\alpha)$, there holds*

$$\|u^\delta - \hat{u}^\delta\|_{\mathrm{H}_0^1(\Omega^\delta)} \leq c\, \delta^{\beta-\varepsilon} \|f\|_{\mathrm{V}_\beta^1(\Omega)^*},$$

where $c > 0$ is a constant independent of δ and f.

The second original phenomenon in this asymptotic expansion concerns the approximation $\hat{u}^\delta$ introduced in Proposition 1. The function $\hat{u}^\delta$ depends on δ and its far field does not converge to the far field of $(\mathcal{A}^+)^{-1}f$ when $\delta \to 0$, even for the L^2 norm . This proves that the solution of problem (1), when it is well-defined, is unstable with respect to δ.

4 Discussion

In this document, we have considered a special geometry for Ω^δ because it simplifies the numerical calculations of the first paragraph. However, we observe exactly the same curiosities for a rounded corner: when the contrast lies inside the critical interval, the solution of problem (1), which is defined except for a sequence of values of δ which tends to 0, depends critically on the rounding parameter. From a physical point of view, one may wonder what happens in a neighbourhood of the corner...

References

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