A Borg-Levinson theorem for elliptic operators

Lassi Päivärinta
Joint work with Katya Krupchyk

University of Helsinki, Finland

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The big 7-day Lapland hike in 1989: Day 1, 2PM, taxi
Let \((a, b) \subset \mathbb{R}\) be an open bounded interval, and let \(q \in L^\infty((a, b))\) be a real-valued potential.

Consider the Sturm–Liouville operator,

\[-\frac{d^2}{dx^2} + q\quad \text{on}\quad (a, b),\]

with self-adjoint boundary conditions (Dirichlet, Neumann, periodic,...).

**Inverse spectral problem:** given the spectrum of the Sturm–Liouville operator, can we determine \(q\)?
The starting point is the following result of Ambarzumyan, 1929:

**Theorem**

Let \( q \in L^\infty((a, b)) \) be real-valued, and let

\[
P u = -\frac{d^2 u}{dx^2} \quad \text{on} \quad L^2((a, b)),
\]

subject to Neumann boundary conditions,

\[ u'(a) = u'(b) = 0. \]

If the operators \( P \) and \( P + q \) have the same eigenvalues including multiplicities then \( q = 0 \).

**Remark.** Davies, 2013: extension of Ambarzumyan’s result to the case when \( P \) is the Laplace–Beltrami operator on a compact Riemannian manifold with boundary.
Borg, 1946: a single spectrum in general does not suffice to determine the potential uniquely, and the result of Ambarzumyan is an exception.

A positive result is provided by the celebrated Borg–Levinson theorem (Borg, 1946, Levinson, 1949):

**Theorem**

Let $q_1, q_2 \in L^\infty((a, b))$ be real-valued. Assume that the Dirichlet eigenvalues $\lambda_k(q_j), k = 1, 2, \ldots$, of the operator $P + q_j, j = 1, 2$, satisfy

$$\lambda_k(q_1) = \lambda_k(q_2), \quad k = 1, 2, \ldots.$$ 

Let $\varphi_k(x; q_j)$ be the corresponding Dirichlet eigenfunctions of $P + q_j$ such that

$$\partial_x \varphi_k(a; q_1) = \partial_x \varphi_k(a; q_2) = 1, \quad k = 1, 2, \ldots.$$ 

If furthermore,

$$\partial_x \varphi_k(b; q_1) = \partial_x \varphi_k(b; q_2), \quad k = 1, 2, \ldots,$$

then $q_1 = q_2$. 

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Generalization to higher dimensions

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary, and let $q \in L^\infty(\Omega)$ be real-valued.

Let $\lambda_k(q)$, $k = 1, 2, \ldots$, be the Dirichlet eigenvalues of the Schrödinger operator $-\Delta + q$,

$$-\Delta u + qu = \lambda u \quad \text{in} \quad \Omega,$$

$$u|_{\partial\Omega} = 0.$$

Associated to the eigenvalues $\lambda_k(q)$, we have the eigenfunctions $\varphi_k(x; q)$, $k = 1, 2, \ldots$, forming an orthonormal basis in $L^2(\Omega)$.

In what follows we let $\nu$ be the exterior unit normal to the boundary $\partial\Omega$. 

**Theorem**

Let \( q_1, q_2 \in L^\infty(\Omega) \) be real-valued, and let \( \varphi_k(x; q_1) \) be an orthonormal basis in \( L^2(\Omega) \) of the Dirichlet eigenfunctions of the operator \(-\Delta + q_1\). Furthermore, assume that the Dirichlet eigenvalues \( \lambda_k(q_j) \) of \(-\Delta + q_j\) satisfy

\[
\lambda_k(q_1) = \lambda_k(q_2), \quad k = 1, 2, \ldots,
\]

and that there exists an orthonormal basis in \( L^2(\Omega) \) of the Dirichlet eigenfunctions \( \varphi_k(x; q_2) \) of the operator \(-\Delta + q_2\) such that

\[
\partial_\nu \varphi_k(x; q_1) = \partial_\nu \varphi_k(x; q_2), \quad x \in \partial \Omega, \quad k = 1, 2, \ldots,
\]

Then \( q_1 = q_2 \) in \( \Omega \).
Subsequent developments:


P. – Serov, 2002: the case of unbounded potentials $q \in L^p(\Omega)$, $p > n/2$.

Goal of this talk: extend the multidimensional Borg–Levinson theorem to higher order elliptic operators.

Main example: a perturbed polyharmonic operator \((-\Delta)^m + q, m \geq 2\).

Motivation: higher order elliptic operators arise in various problems in physics and geometry such as

- quantum field theory,
- theory of thin elastic plates,
- conformal geometry (the Paneitz-Branson operator).

Higher order elliptic operators have been considered in the context of inverse boundary problems:

- Isakov, 1991: completeness of products of solutions;
- Ikehata, 1991: Faddeev’s Green function for the biharmonic operator;
- Liu 1996: inverse scattering;
**Figure:** Tectonic plates and sediment layers can be modelled by using the biharmonic operator.
In this talk we shall consider higher order elliptic operators in the context of inverse spectral problems.

Let \( P = P(D) \) be an elliptic partial differential operator on \( \mathbb{R}^n, n \geq 2 \), of order \( 2m, m \geq 1 \), with constant real coefficients,

\[
P(D) = \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha, \quad a_\alpha \in \mathbb{R}, \quad D_j = -i \partial_{x_j}, \quad j = 1, \ldots, n.
\]

Without loss of generality, we may assume that its principal symbol satisfies

\[
\sum_{|\alpha| = 2m} a_\alpha \xi^\alpha > 0, \quad 0 \neq \xi \in \mathbb{R}^n.
\]

By Gårding's inequality,

\[
(P \varphi, \varphi)_{L^2(\mathbb{R}^n)} \geq C_1 \| \varphi \|^2_{H^m(\mathbb{R}^n)} - C_2 \| \varphi \|^2_{L^2(\mathbb{R}^n)}, \quad \varphi \in C_0^\infty(\mathbb{R}^n), \quad C_1 > 0, \quad C_2 \in \mathbb{R}.
\]

Here \( H^m(\mathbb{R}^n) \) is the standard Sobolev space on \( \mathbb{R}^n \).
The Friedrichs extension of $P$, defined on $C_0^\infty(\Omega)$, is a self-adjoint operator semi-bounded from below, with the domain

$$\mathcal{D}(P) = \{ u \in H^{2m}(\Omega) : \gamma u = 0 \}.$$ 

Here $\gamma$ is the Dirichlet trace operator, given by

$$\gamma : H^{2m}(\Omega) \rightarrow \mathcal{H}^{0,m-1}(\partial \Omega) := \prod_{j=0}^{m-1} H^{2m-j-1/2}(\partial \Omega),$$

$$\gamma u = (u|_{\partial \Omega}, \partial_{\nu} u|_{\partial \Omega}, \ldots, \partial_{\nu}^{m-1} u|_{\partial \Omega}).$$
Let $q \in L^\infty(\Omega)$ be a real-valued potential.

Then the operator $P + q$ is self-adjoint on the domain $\mathcal{D}(P)$, and the spectrum of $P + q$ is discrete, accumulating at $+\infty$, consisting of eigenvalues of finite multiplicity,

$$-\infty < \lambda_1(q) \leq \lambda_2(q) \leq \cdots \leq \lambda_k(q) \to +\infty, \quad k \to +\infty.$$  

Associated to the eigenvalues $\lambda_k(q)$, we have the eigenfunctions $\varphi_k(x; q) \in \mathcal{D}(P)$, which form an orthonormal basis in $L^2(\Omega)$. 
Let
\[ P(\xi) = \sum_{|\alpha| \leq 2m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^n \]
be the full symbol of the operator $P$.

The holomorphic continuation to $\mathbb{C}^n$:
\[ P(\zeta) = \sum_{|\alpha| \leq 2m} a_\alpha \zeta^\alpha, \quad \zeta \in \mathbb{C}^n \]

Following the Ph.D thesis of Hörmander, 1955, we set
\[ \tilde{P}(\xi) = \left( \sum_{|\alpha| \geq 0} |P^{(\alpha)}(\xi)|^2 \right)^{1/2}, \quad P^{(\alpha)}(\xi) = \partial_\xi^\alpha P(\xi), \quad \alpha \in \mathbb{N}^n. \]
The assumptions on $P(\xi)$:

(A1) There exists a non-empty open subset $U \subset \mathbb{R}^n$ and $\lambda_0 > 0$, such that for any $\xi \in U$ and any $\lambda \leq -\lambda_0$, there are $\zeta_1, \zeta_2 \in \mathbb{C}^n$ such that

$$P(\zeta_j) = \lambda, \quad j = 1, 2, \quad \xi = \zeta_1 - \zeta_2.$$ 

(A2) Let $L_\zeta(\xi) = P(\xi + \zeta) - P(\zeta)$ and let

$$P^{-1}(\lambda) = \{\zeta \in \mathbb{C}^n : P(\zeta) = \lambda\}$$

be the complex characteristic variety of $P - \lambda$. Assume that

$$\sup_{\xi \in \mathbb{R}^n, \zeta \in P^{-1}(\lambda)} \frac{1}{\widetilde{L}_\zeta(\xi)} \to 0, \quad \lambda \to -\infty.$$

Remark. If $\lambda < 0$ with $|\lambda|$ large enough, then $P^{-1}(\lambda) \cap \mathbb{R}^n = \emptyset$. 

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Remark. As we shall see, the assumptions (A1) and (A2) are satisfied for the Laplacian $P = -\Delta$ and for the polyharmonic operator $P = (-\Delta)^m$, $m \geq 2$.

Remark. The assumptions (A1) and (A2) are similar to those, which occur in the work by Isakov, 1991. Such assumptions are introduced there in order to guarantee the completeness of products of solutions of the equation $(P + q)u = 0$ in $\Omega$. 

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To state our results we shall need the Neumann trace operator, which is defined by

$$
\tilde{\gamma} : \mathcal{H}^{2m}(\Omega) \to \mathcal{H}^{m,2m-1}(\partial\Omega) := \prod_{j=m}^{2m-1} \mathcal{H}^{2m-j-1/2}(\partial\Omega),
$$

$$
\tilde{\gamma}u = (\partial^m_{\nu} u|_{\partial\Omega}, \ldots, \partial^{2m-1}_{\nu} u|_{\partial\Omega}).
$$
Theorem (Krupchyk – P., 2012)

Assume that (A1) and (A2) hold. Let \( q_1, q_2 \in L^\infty(\Omega) \) be real-valued and \( \varphi_k(x; q_1) \) be an orthonormal basis in \( L^2(\Omega) \) of the Dirichlet eigenfunctions of \( P + q_1 \). Furthermore, assume that the Dirichlet eigenvalues \( \lambda_k(q_j) \) of \( P + q_j \) satisfy

\[
\lambda_k(q_1) = \lambda_k(q_2), \quad k = 1, 2, \ldots,
\]

and that there exists an orthonormal basis in \( L^2(\Omega) \) of the Dirichlet eigenfunctions \( \varphi_k(x; q_2) \) of \( P + q_2 \) such that

\[
\tilde{\gamma}\varphi_k(x; q_1) = \tilde{\gamma}\varphi_k(x; q_2), \quad k = 1, 2, \ldots.
\]

Then \( q_1 = q_2 \).
Day 2, 3AM. swamp
Idea of the proof

Starting point in the Nachman – Sylvester – Uhlmann proof when $P = -\Delta$: use of scattering solutions of

$$(P + q - \lambda)u = 0,$$

in all of $\mathbb{R}^n$, for $\lambda > 0$ large.

Our starting point: construction of complex geometric optics solutions of

$$(P + q - \lambda)u = 0,$$

in $\Omega$, for $\lambda < 0$, $|\lambda|$ sufficiently large.
Step 1: construction of complex geometric optics solutions:

Proposition

Assume that (A2) holds. Then there exists $\lambda_0 > 0$ such that for any $\lambda < -\lambda_0$ and any $\zeta \in P^{-1}(\lambda)$, there are solutions

$$ u_{\lambda,\zeta} = e^{i\zeta \cdot x} (1 + w_{\lambda,\zeta}) \in L^2(\Omega) $$

to the equation

$$ (P + q - \lambda) u = 0 \quad \text{in} \quad \Omega $$

with $\|w_{\lambda,\zeta}\|_{L^2(\Omega)} \to 0$ as $\lambda \to -\infty$.

How do we construct such solutions?

In order that \((P + q - \lambda)u_{\lambda,\zeta} = 0\), the correction \(w_{\lambda,\zeta}\) should satisfy

\[
L_\zeta(D)w_{\lambda,\zeta} = -q(1 + w_{\lambda,\zeta}),
\]

where

\[
L_\zeta(D) = P(\zeta + D) - P(\zeta).
\]

Using a regular fundamental solution of \(L_\zeta(D)\) (Hörmander, 1983), and a result of Isakov, 1991, we obtain a right inverse \(E_\zeta\) of \(L_\zeta(D)\) on \(L^2(\Omega)\), with the following estimate,

\[
\|E_\zeta\|_{\mathcal{L}(L^2(\Omega))} \leq C \sup_{\xi \in \mathbb{R}^n, \zeta \in P^{-1}(\lambda)} \frac{1}{\tilde{L}_\zeta(\xi)}.
\]
The assumption (A2) implies that $\|E_\zeta\|_{L(L^2(\Omega))} \to 0$, as $\lambda \to -\infty$, uniformly in $\zeta \in P^{-1}(\lambda)$.

Hence, there exists $\lambda_0 > 0$ large enough such that the map

$$F_\zeta : L^2(\Omega) \to L^2(\Omega), \quad f \mapsto E_\zeta(-q(1 + f))$$

is a contraction for any $\zeta \in P^{-1}(\lambda)$ and any $\lambda \leq -\lambda_0$.

The unique fixed point $w_{\lambda, \zeta} \in L^2(\Omega)$ of $F_\zeta$ satisfies

$$L_\zeta(D)w_{\lambda, \zeta} = -q(1 + w_{\lambda, \zeta}).$$

One can easily see that $\|w_{\lambda, \zeta}\|_{L^2(\Omega)} \to 0$ as $\lambda \to -\infty$.

Remark. Constructing our complex geometric optics solution on a slightly larger set, by elliptic regularity, we get $u_{\lambda, \zeta} \in H^{2m}(\Omega)$. 
Step 2. Density of products of solutions.

**Proposition**

*Suppose that the assumptions (A1) and (A2) hold. Then there exists \( \lambda_0 > 0 \) such that the set*

\[
\text{span} \bigcup_{\lambda \leq -\lambda_0} \{ u_{q_1}(\lambda)u_{q_2}(\lambda) : u_{q_j}(\lambda) \in H^{2m}(\Omega), (P + q_j - \lambda)u_{q_j}(\lambda) = 0 \text{ in } \Omega \}
\]

*is dense in \( L^1(\Omega) \).*

This result makes use of the complex geometric optics solutions. The assumption (A1) is crucial as well.
Step 3. From the spectral data to the Dirichlet–to–Neumann map.

Let us recall the Dirichlet–to–Neumann map, associated to the operator $P + q_j - \lambda$. Here $\lambda < 0$, $|\lambda|$ large enough.

Then zero is not an eigenvalue of the operator $P + q_j - \lambda$, $j = 1, 2$, equipped with the domain $\mathcal{D}(P)$.

Therefore, for any $f \in \mathcal{H}^{0,m-1}(\partial\Omega)$, the Dirichlet boundary problem

$$
(P + q_j - \lambda)u = 0, \quad \text{in} \quad \Omega,
$$

$$
\gamma u = f, \quad \text{on} \quad \partial\Omega,
$$

has a unique solution $u_{q_j,f}(\lambda) \in H^{2m}(\Omega)$ and

$$
\| u_{q_j,f}(\lambda) \|_{H^{2m}(\Omega)} \leq C \| f \|_{\mathcal{H}^{0,m-1}(\partial\Omega)}.
$$
When \( \lambda < 0 \), \( |\lambda| \) large enough, we define the Dirichlet–to–Neumann map by

\[
\Lambda_{q_j}(\lambda)(f) = \tilde{\gamma} u_{q_j,f}(\lambda),
\]

which is a bounded map

\[
\Lambda_{q_j}(\lambda) : \mathcal{H}^{0,m-1}(\partial \Omega) \to \mathcal{H}^{m,2m-1}(\partial \Omega).
\]

Recall

\[
\mathcal{H}^{m,2m-1}(\partial \Omega) := \prod_{j=m}^{2m-1} \mathcal{H}^{2m-j-1/2}(\partial \Omega),
\]

and notice that

\[
\mathcal{H}^{m,2m-1}(\partial \Omega) \subset \mathcal{H}^{m,2m-1,\varepsilon}(\partial \Omega) := \prod_{j=m}^{2m-1} \mathcal{H}^{2m-\varepsilon-j-1/2}(\partial \Omega), \quad \varepsilon > 0.
\]
General arguments using **only** the ellipticity and the self-adjointness of $P + q_j$, $j = 1, 2$, yield the following result.

**Proposition**

For any small $\varepsilon > 0$,

$$\|\Lambda_{q_1}(\lambda) - \Lambda_{q_2}(\lambda)\|_{L(H^{0, m-1}(\partial\Omega), H^{m, 2m-1, \varepsilon}(\partial\Omega))} \to 0, \quad \lambda \to -\infty.$$
Using the spectral representation for the resolvent of $P + q_j$,

$$(P + q_j - \lambda)^{-1}h(x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k(q_j) - \lambda} (h, \varphi_k(\cdot, q_j))_{L^2(\Omega)} \varphi_k(x, q_j),$$

when $h \in L^2(\Omega)$, we can conclude **formally** that the equality of the spectral data,

$$\lambda_k(q_1) = \lambda_k(q_2), \quad k = 1, 2, \ldots,$$

$$\tilde{\gamma}\varphi_k(x; q_1) = \tilde{\gamma}\varphi_k(x; q_2), \quad x \in \partial\Omega, \quad k = 1, 2, \ldots,$$

implies the equality of the Dirichlet–to–Neumann maps,

$$\Lambda_{q_1}(\lambda) = \Lambda_{q_2}(\lambda)$$

for all $\lambda < 0$ with $|\lambda|$ large enough.

The argument is only formal: the resolvent expansion only converges in $L^2(\Omega)$ but not in $H^{2m}(\Omega)$. 

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To overcome the difficulty: we shall differentiate the resolvent and hence the Dirichlet–to–Neumann map with respect to $\lambda$ a sufficiently large number of times. We have the following result.

**Proposition**

For all $l \in \mathbb{N}$ satisfying $(l - 1)m > n$ and all $\lambda < 0$, $|\lambda|$ large enough, we have

$$\frac{d^l}{d\lambda^l} (\Lambda_{q_1}(\lambda)f - \Lambda_{q_2}(\lambda)f) = 0, \quad f \in H^{0,m-1}(\partial\Omega).$$

It follows from this proposition that $\Lambda_{q_1}(\lambda) - \Lambda_{q_2}(\lambda)$ is a polynomial in $\lambda$.

The fact that $\|\Lambda_{q_1}(\lambda) - \Lambda_{q_2}(\lambda)\| \to 0$ as $\lambda \to -\infty$, implies that

$$\Lambda_{q_1}(\lambda) = \Lambda_{q_2}(\lambda)$$

for all $\lambda < 0$ with $|\lambda|$ large enough.
Step 4. From the Dirichlet–to–Neumann map to the potential.

**Proposition**

Let

\[ \Lambda_{q_1}(\lambda) = \Lambda_{q_2}(\lambda) \]

for all \( \lambda < 0 \) with \( |\lambda| \) large enough. Then \( q_1 = q_2 \).

This result follows from the density of products of solutions, discussed above.

This concludes the sketch of the proof of the theorem.
The Laplace operator

Let us check that the conditions (A1) and (A2) are satisfied for \( P = -\Delta \) in \( \mathbb{R}^n \), \( n \geq 2 \).

Let \( \xi \in \mathbb{R}^n \). To check the condition (A1), due to the rotational invariance, we may assume that \( \xi = (|\xi|, 0, \ldots, 0) \). Let \( \lambda < 0 \) and consider

\[
\zeta_1 = \left( \frac{|\xi|}{2}, 0, \ldots, 0 \right) + i\left( 0, \sqrt{\frac{|\xi|^2}{4} + |\lambda|}, 0, \ldots, 0 \right) \in \mathbb{C}^n, \quad \zeta_2 = -\zeta_1.
\]

Hence, \( \xi = \zeta_1 - \zeta_2 \), and \( \zeta_j \cdot \zeta_j = \lambda \), for \( \lambda < 0 \), \( j = 1, 2 \).

To check the condition (A2), let

\[
L_\zeta(\xi) = (\xi + \zeta) \cdot (\xi + \zeta) - \zeta \cdot \zeta = \xi \cdot \xi + 2\xi \cdot \zeta, \quad \xi \in \mathbb{R}^n, \quad \zeta \cdot \zeta = \lambda.
\]

Then

\[
\partial_{\xi_i} L_\zeta(\xi) = 2\xi_i + 2\zeta_i, \quad i = 1, \ldots, n.
\]
The condition $\zeta \cdot \zeta = \lambda < 0$ is equivalent to the fact that

$$\text{Re} \zeta \cdot \text{Re} \zeta - \text{Im} \zeta \cdot \text{Im} \zeta = \lambda, \quad \text{Re} \zeta \cdot \text{Im} \zeta = 0.$$  

Hence, $|\text{Re} \zeta|^2 + |\lambda| = |\text{Im} \zeta|^2$, and therefore, $|\text{Im} \zeta| \geq \sqrt{|\lambda|}$.

We get

$$\tilde{L}_\zeta(\xi) \geq \left( \sum_{i=1}^{n} |\partial_{\xi_i} L_\zeta(\xi)|^2 \right)^{1/2} \geq 2 \left( \sum_{i=1}^{n} |\text{Im} \zeta_i|^2 \right)^{1/2} = 2|\text{Im} \zeta| \geq 2\sqrt{|\lambda|},$$

and thus, the assumption (A2) holds.

Hence, applying our theorem to the Laplace operator, we recover the standard multidimensional Borg-Levinson Theorem.
The polyharmonic operator

Consider the polyharmonic operator \( P = (-\Delta)^m, m \geq 2, \) in \( \mathbb{R}^n, n \geq 2. \) Let us show that the conditions (A1) and (A2) are satisfied for this operator.

To check the condition (A1) let us notice that as \( \lambda < 0, \) the fact that \( (\zeta \cdot \zeta)^m = \lambda \) is equivalent to the fact that there is an integer \( k, 0 \leq k \leq m - 1, \) such that

\[
|\text{Re} \zeta|^2 - |\text{Im} \zeta|^2 = |\lambda|^{1/m} \cos \left( \frac{\pi + 2\pi k}{m} \right), \quad (1)
\]

\[
2 \text{Re} \zeta \cdot \text{Im} \zeta = |\lambda|^{1/m} \sin \left( \frac{\pi + 2\pi k}{m} \right). \quad (2)
\]
Corollary (Krupchyk – P., 2012)

Let \( q_1, q_2 \in L^\infty(\Omega) \) be real-valued and \( \varphi_k(x; q_1) \) be an orthonormal basis in \( L^2(\Omega) \) of the Dirichlet eigenfunctions of \( (-\Delta)^m + q_1 \). Furthermore, assume that the Dirichlet eigenvalues \( \lambda_k(q_j) \) of \( (-\Delta)^m + q_j \) satisfy

\[
\lambda_k(q_1) = \lambda_k(q_2), \quad k = 1, 2, \ldots,
\]

and that there exists an orthonormal basis in \( L^2(\Omega) \) of the Dirichlet eigenfunctions \( \varphi_k(x; q_2) \) of \( (-\Delta)^m + q_2 \) such that

\[
\tilde{\gamma}\varphi_k(x; q_1) = \tilde{\gamma}\varphi_k(x; q_2), \quad x \in \partial\Omega, \quad k = 1, 2, \ldots.
\]

Then \( q_1 = q_2 \).
Day 3, 11PM. The river
Pöschel – Trubowitz, 1987: there is a one-to-one correspondence between a potential \( q \in L^2((0, 1)) \) and the pair of all Dirichlet eigenvalues and the normal derivatives of the Dirichlet eigenfunctions for the Sturm-Liouville operator \( -\frac{d^2}{dx^2} + q \) on \( L^2((0, 1)) \).

Novikov, 1988, Isozaki, 1991: this is no longer true for multidimensional Schrödinger operators \( -\Delta + q \).

The Borg-Levinson theorem with incomplete data: the knowledge of all large Dirichlet eigenvalues and the boundary traces of the normal derivatives of the corresponding eigenfunctions for \( -\Delta + q \), still suffices to recover the potential \( q \) uniquely.
The Borg-Levinson theorem with incomplete spectral data in the case of the polyharmonic operator \( P = (-\Delta)^m \).

**Theorem (Krupchyk – P., 2012)**

Let \( q_1, q_2 \in L^\infty(\Omega) \) be real-valued and let \( \varphi_k(x; q_1) \) be an orthonormal basis in \( L^2(\Omega) \) of the Dirichlet eigenfunctions of the operator \( (-\Delta)^m + q_1 \), \( m \geq 2 \). Assume that there exists an integer \( N > 0 \) such that the Dirichlet eigenvalues \( \lambda_k(q_j) \) of \( (-\Delta)^m + q_j \) satisfy

\[
\lambda_k(q_1) = \lambda_k(q_2), \quad k > N,
\]

and there exists an orthonormal basis in \( L^2(\Omega) \) of the Dirichlet eigenfunctions \( \varphi_k(x; q_2) \) of \( (-\Delta)^m + q_2 \) such that

\[
\tilde{\gamma} \varphi_k(x; q_1) = \tilde{\gamma} \varphi_k(x; q_2), \quad x \in \partial \Omega, \quad k > N.
\]

Then \( q_1 = q_2 \).
Idea of proof

Our approach is different from the one, developed in Novikov, 1988, Isozaki, 1991.
It is based on the fact that the linear span of products of solutions to the Schrödinger equation, which satisfy a finite number of linear constraints on the boundary, is dense in $L^1(\Omega)$. Specifically,

**Proposition**

Let $h_{k,i} \in L^2(\partial \Omega)$, $k = 1, \ldots, N$, $i = 0, \ldots, m - 1$, with $N$ being arbitrary but fixed. Then there exists $\lambda_0 > 0$ such that the set

$$
S = \text{span} \bigcup_{\lambda < -\lambda_0, \lambda \in \mathbb{Z}} \{ u_{q_1}(\lambda)u_{q_2}(\lambda) : u_{q_j}(\lambda) \in H^{2m}(\Omega), \ ((-\Delta)^m + q_j - \lambda)u_{q_j}(\lambda) = 0 \text{ in } \Omega, \ j = 1, 2, \\
\int_{\partial \Omega} \partial^i_{\nu} u_{q_1}(\lambda)h_{k,i}dS = 0, \ k = 1, \ldots, N, i = 0, \ldots, m - 1 \}
$$

is dense in $L^1(\Omega)$. 
The linear constraints at the boundary arise in order to compensate for the lack of spectral information corresponding to a finite number of eigenvalues.

For simplicity, let us explain the idea of proof of the density result in the case when $P = -\Delta$. In this case, the result is as follows:

**Proposition**

Let $h_k \in L^2(\partial \Omega)$, $k = 1, \ldots, N$, with $N$ being arbitrary but fixed. Then there exists $\lambda_0 > 0$ such that the space

$$S = \text{span} \bigcup_{\lambda < -\lambda_0, \lambda \in \mathbb{Z}} \{ u_{q_1}(\lambda) \overline{u_{q_2}(\lambda)} : u_{q_j}(\lambda) \in H^2(\Omega), (-\Delta + q_j - \lambda)u_{q_j}(\lambda) = 0 \text{ in } \Omega, \ j = 1, 2, \int_{\partial \Omega} u_{q_1}(\lambda) \overline{h_k} dS = 0, k = 1, \ldots, N \}$$

is dense in $L^1(\Omega)$. 
Idea of proof of the density result.

Let $f \in L^{\infty}(\Omega)$ be such that

$$\int_{\Omega} fg dx = 0, \quad \forall g \in S. \quad (3)$$

We would like to conclude that $f = 0$.

Let $\xi \in \mathbb{R}^n$ and $\lambda < 0$. As before, we may assume that $\xi = (|\xi|, 0, \ldots, 0)$. Consider

$$\zeta_1 = (\frac{|\xi|}{2}, 0, \ldots, 0) + i(0, \sqrt{\frac{|\xi|^2}{4}} + |\lambda|, 0, \ldots, 0) \in \mathbb{C}^n, \quad \zeta_2 = -\zeta_1.$$ 

We have $\xi = \zeta_1 - \overline{\zeta_2}$, and $\zeta_j \cdot \zeta_j = \lambda$, $j = 1, 2$. 
Set

$$\eta_l = \left(-\frac{\lvert \xi \rvert}{2} + l, 0, \ldots, 0\right) + i(0, -\sqrt{\frac{\lvert \xi \rvert^2}{4} + |\lambda|} + \sqrt{l^2 + |\lambda|}, 0, \ldots, 0) \in \mathbb{C}^n,$$

for \( l = 1, \ldots N + 1 \). Then

$$\zeta_1 + \eta_l = (l, 0, \ldots, 0) + i(0, \sqrt{l^2 + |\lambda|}, 0, \ldots, 0),$$

and therefore, \((\zeta_1 + \eta_l) \cdot (\zeta_1 + \eta_l) = \lambda\).

Notice that \(\zeta_1 + \eta_l\) is independent of \(\xi\).

For \(|\lambda|\) large enough, we construct complex geometric optics solutions

$$u_{q_1, \lambda, \zeta_1 + \eta_l} = e^{i(\zeta_1 + \eta_l) \cdot x}(1 + w_{\lambda, \zeta_1 + \eta_l}) \in H^2(\Omega), \quad l = 1, \ldots, N + 1,$$

to the equation

$$(-\Delta + q_1 - \lambda)u = 0 \quad \text{in} \quad \Omega,$$

with \(\|w_{\lambda, \zeta_1 + \eta_l}\|_{L^2(\Omega)} \to 0\) as \(\lambda \to -\infty\).
Also for $|\lambda|$ large enough, we construct complex geometric optics solutions

$$u_{q_2, \lambda, \zeta_2} = e^{i\zeta_2 \cdot x} (1 + w_{\lambda, \zeta_2}) \in H^2(\Omega),$$

to

$$(-\Delta + q_2 - \lambda)u = 0 \quad \text{in} \quad \Omega,$$

with $\|w_{\lambda, \zeta_2}\|_{L^2(\Omega)} \to 0$ as $\lambda \to -\infty$. 
The $N+1$ vectors

$$H_l = \left( \begin{array}{c} \int_{\partial \Omega} u_{q_1, \lambda, \zeta_1 + \eta_l} \overline{h_1} ds \\ \vdots \\ \int_{\partial \Omega} u_{q_1, \lambda, \zeta_1 + \eta_l} \overline{h_N} ds \end{array} \right) \in \mathbb{C}^N, \quad l = 1, \ldots, N + 1,$$

are linearly dependent.

Thus, there are constants $c_l = c_l(\lambda, \zeta_1 + \eta_l) \in \mathbb{C}$, not all equal to zero, such that

$$\sum_{l=1}^{N+1} c_l H_l = 0. \quad (4)$$

Notice that $c_l$ are independent of $\xi$.

We can assume that $\sum_{l=1}^{N+1} |c_l|^2 = 1$, and that $c_l \to \tilde{c}_l$, as $\lambda \to -\infty$, $\lambda \in \mathbb{Z}$, where $\tilde{c}_l \in \mathbb{C}$ are such that

$$\sum_{l=1}^{N+1} |\tilde{c}_l|^2 = 1.$$
Let
\[ u_{q_1}(\lambda) = \sum_{l=1}^{N+1} c_l u_{q_1,\lambda,\zeta_1+\eta_l} \in H^2(\Omega). \]

Then we have
\[ (-\Delta + q_1 - \lambda)u_{q_1}(\lambda) = 0 \text{ in } \Omega. \]

Using (4), we see that \( u_{q_1}(\lambda) \) satisfies the constraints,
\[ \int_{\partial\Omega} u_{q_1}(\lambda) \bar{h}_k dS = 0, \quad k = 1, \ldots, N. \]

Taking
\[ g = u_{q_1}(\lambda) u_{q_2,\lambda,\zeta_2} \]

in the orthogonality condition (3) gives
\[ \int_{\Omega} f \sum_{l=1}^{N+1} c_l e^{i(\xi+\eta_l) \cdot x} (1 + w_{\lambda,\zeta_1+\eta_l})(1 + \overline{w_{\lambda,\zeta_2}}) dx = 0. \]
Letting $\lambda \to -\infty$, we get

$$\int_{\Omega} f \left( \sum_{l=1}^{N+1} \tilde{c}_l e^{i l x_1} \right) e^{i \frac{\xi}{2} \cdot x} dx = 0, \quad \forall \xi \in \mathbb{R}^n.$$ 

Since $\tilde{c}_l$ are independent of $\xi$, we conclude that

$$f \left( \sum_{l=1}^{N+1} \tilde{c}_l e^{i l x_1} \right) = 0.$$ 

The function $\sum_{l=1}^{N+1} \tilde{c}_l e^{i l x_1}$ is real analytic and does not vanish identically.

Thus, $f = 0$ and we are through.
Future work

- The Borg–Levinson problem for complex valued potentials.

  The problem is non-selfadjoint and one has to face the issue of generalized eigenfunctions.

  Lassas, 1995: the second order case.

- The Borg–Levinson problem for higher order perturbations.


  Contrary to the case of a first order perturbation of the Laplacian, a first order perturbation of the polyharmonic operator \((-\Delta)^m, m \geq 2\), can be determined uniquely from the knowledge of the Dirichlet–to–Neumann map.

  Similar phenomena are expected to hold for the inverse spectral problem.
Day 7, 6PM. The rest
Happy Birthday Comrade David!