Régularité de frontières libres et de formes optimales

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1 Introduction

Tous les problèmes à frontière libre ne sont pas de nature variationnelle, mais beaucoup le sont, en particulier ceux issus de l'étude des formes optimales. L'analyse de leur régularité emprunte d'ailleurs beaucoup aux méthodes spécifiques aux frontières libres auxquelles s'ajoutent des informations issues de l'optimalité. Le but de ce mini-cours est de décrire des méthodes et des résultats assez typiques dans ce contexte.

Rappelons que l'existence de formes optimales est souvent établie par des arguments topologiques (compacité/semi-continuité) dans des espaces fonctionnels adéquats. La plupart du temps, ces espaces fonctionnels sont très vastes pour justement assurer l'existence d'optima. En conséquence, les formes optimales trouvées arrivent avec extrêmement peu de régularité. Ce sont par exemple tout juste des ensembles ouverts de $\mathbb{R}^d$, voire même des *quasi-ouverts* -voir plus loin- ou même des ensembles seulement mesurables. Et pourtant, dans beaucoup de situations, on s'attend à ce qu'elles soient régulières, ceci pouvant aller jusqu'à être des ouverts à bord C^∞ ou même analytique.

Beaucoup d'outils ont été développés permettant de prouver que des formes optimales sont (très) régulières à partir du moment où on sait déjà qu'elles ont un peu de régularité. Par exemple, si on sait que la forme optimale est un ouvert dont la frontière peut être représentée comme le graphe d'une fonction lipschitzienne, alors il est souvent possible d'écrire *les équations d'Euler-Lagrange du problème de minimisation* sur la fonction définissant ce graphe. Elles se traduisent généralement par un système d'équations aux dérivées partielles caractérisant *la frontière libre* de la forme optimale. On peut alors pêcher dans la riche collection de résultats de régularité (classiques mais non triviaux) pour les solutions de ces systèmes d'EDP pour montrer qu'en fait, elles sont plus régulières qu'on ne le pense initialement. Des arguments de "bootstrap" permettent souvent d'itérer cette régularité et atteindre C^∞ ou analytique si les données et le problème le permettent.

En fait, nous sommes surtout intéressés *par la toute première étape dans l'étude de la régularité* qui consiste, *à partir de "rien"* (= la seule régularité -ou plutôt irrégularité- donnée par le théorème d'existence), à atteindre un minimum de régularité sur la solution. On peut, par exemple, viser à prouver que la forme optimale est un ouvert à bord lipschitzien (ou C^1). Alors, on peut entrer dans la philosophie précédente et espérer atteindre la régularité optimale, compte-tenu du problème.

Notons que cette toute première étape d'obtention d'un minimum d'information sur la régularité est en général très difficile. Entre autre difficulté, il est souvent impossible de dériver efficacement par rapport à la forme, ou tout au moins d'écrire de façon exploitable cette dérivée par manque de régularité a priori sur la solution optimale.

Nous suivrons le plan suivant pour la discussion de ces diverses questions:

A) Isoperimetric problems and quasi-minimizers (from the theory of minimal surfaces).

B) A model problem: optimal shapes for the Dirichlet energy with prescribed volume.

C) Adding surface tension.

D) More examples. Optimal eigenvalues of the Laplacian, convexity constraints, polygons as optimal shapes.

Faisons quelques commentaires sur chacun des points avant de les aborder successivement.

A) Nous rappelons les résultats de régularité classiques pour les *problèmes isopérimétriques*, ici la minimisation du périmètre à volume constant et contrainte d'inclusion. Ils sont obtenus comme conséquences de résultats profonds de la théorie des surfaces minimales qui ont été obtenus progressivement au cours des cinquante dernières années. Ils utilisent de façon essentielle les outils de la théorie géométrique de la mesure et le concept de quasi-minimiseur.

B) Nous utilisons comme problème modèle la forme optimale pour l'énergie de Dirichlet avec contrainte de volume. Ainsi, la fonctionnelle de forme à minimiser se calcule via la solution d'une équation aux dérivées partielles (ici l'une des plus simples qui soit). La première étape consiste à montrer que cette solution de l'équation, dite d'état, est lipschitzienne. C'est beaucoup plus facile lorsque la fonction d'état est de signe constant. D'ailleurs, des points de rebroussement peuvent apparaître au bord là où cette fonction change de signe et, bien sûr, la régularité lipschitzienne y est plus difficile à établir: elle nécessite l'utilisation d'extensions adéquates du fameux lemme de monotonie de Alt-Caffarelli-Friedman ([3, 11]). L'étape suivante est d'étudier la frontière elle-même, après avoir prouvé que le gradient ne dégénère pas.

Bien qu'encore simple, ce modèle d'optimisation de forme apparaît déjà dans beaucoup d'applications où l'énergie de Dirichlet est naturellement en jeu (voir par exemple [33, 34, 35, 32, 29] et pour les simulations numériques [15, 16, 57, 55, 13] et pour des bulles en 3-d [56, 50, 52, 39]).

C) Ce cas est la somme de A) et B): nous ajoutons à l'énergie de Dirichlet un terme d'énergie de tension superficielle, traditionnellement proportionnel au périmètre du domaine occupé. Ce terme supplémentaire est bien sûr régularisant. Mais, il ne rend pas pour autant plus facile l'étude de la régularité, car les techniques pour A) et B) sont assez différentes, et il n'est pas si facile de les combiner.

D) Dans ce dernier paragraphe, nous évoquons le cas des formes minimisant les valeurs propres de l'opérateur Laplacien avec conditions de Dirichlet. Nous ajoutons quelques commentaires sur l'optimisation de forme avec contrainte de convexité: d'une part, nous mentionnons un résultat surprenant de régularité $C^{1,\frac{1}{2}}$ à la jonction entre les parties saturée-non saturée; d'autre part, nous indiquons une famille de fonctionnelles pour lesquelles les formes optimales sont des polygones.

2 Isoperimetric problems and quasi-minimizers (from the theory of minimal surfaces)

Let us start with the following geometrical model problem where D is a given open subset of $\mathbb{R}^d$ and $m \in (0, |D|)$ (we denote by $|\cdot|$ the Lebesgue measure in $\mathbb{R}^d$). Let us consider a measurable subset $\hat{\Omega}$ of D solution of the following problem:

$$|\hat{\Omega}| = m,\ P(\hat{\Omega}) = \min\{P(\Omega); \Omega \subset D \text{ measurable},\ |\Omega| = m\}. \tag{1}$$

Here $P(\Omega)$ denotes the perimeter of the measurable set Ω, that is, if the boundary $\partial\Omega$ of Ω is regular enough, $P(\Omega) = \int_{\partial\Omega} d\sigma$ (a convenient extended definition of $P(\Omega)$ may be given for any measurable set Ω).

It is easy to see (at least for $D = \mathbb{R}^d$) that (1) is equivalent to the following "dual" problem -which is a true isoperimetric problem-,

$$P(\hat{\Omega}) = p,\ |\hat{\Omega}| = \max\{|\Omega|; \Omega \subset \mathbb{R}^d \text{ measurable},\ P(\Omega) = p\},$$

where $p > 0$ is given.

It is classical that, if $D = \mathbb{R}^d$, or if D is large enough to contain a ball of measure m, then the solution of (1) is precisely this ball. This is a consequence of the famous isoperimetric inequality which states that

$$\forall\, \Omega \subset \mathbb{R}^d \text{ measurable},\ P(\Omega) \geq c_d |\Omega|^{(d-1)/d},\ \text{where } c_d = d\,|B(0,1)|^{1/d}, \tag{2}$$

and *the equality holds if and only if* Ω *is a ball* (see a.e. [24],[49] for proofs and references).

Now, if D is too small to contain a ball, there exists nevertheless an optimal measurable shape $\hat{\Omega}$ for the problem (1) and the existence proof is easy.

A natural question to ask is: how does $\hat{\Omega}$ look like? Is it at least an open set? Does it have a regular boundary? In dimension $d = 2$, is $\partial\hat{\Omega}$ built out of pieces of circles?

The same questions are of interest if one works with the *relative perimeter* to D which is defined for regular sets by $P_D(\Omega) = \int_{\partial\Omega\cap D} d\sigma$, that is the part of the perimeter of Ω which is inside D. Then we consider the corresponding isoperimetric problem

$$|\hat{\Omega}| = m,\ P_D(\hat{\Omega}) = \min\{P_D(\Omega);\ \Omega \subset D \text{ measurable},\ |\Omega| = m\}. \tag{3}$$

Here, the optimal shape has a tendency to "stick" to the boundary of D in order to reduce its relative perimeter.

With respect to the regularity of the optimal set $\hat{\Omega}$, we have the following main result.

Theorem 1 *Let $\hat{\Omega}$ be a solution of* (1) *or of* (3)*. Then,*
- if $d \leq 7$, $\partial\hat{\Omega} \cap D$ is an analytic hypersurface.
- if $d \geq 8$, then there exists a "small" set $\Sigma \subset \partial\hat{\Omega}$ of $s-$Hausdorff measure zero for all $s > d-8$ such that $\partial\hat{\Omega} \setminus \Sigma$ is an analytic hypersurface.

Thus, in dimension $d = 2$, once regularity is proved for $\hat{\Omega}$, it is indeed easy to prove that $\partial\hat{\Omega}$ is a union of pieces of circles.

Note that the value $d \leq 7$ is optimal since actual singularities may occur if $d \geq 8$. An example is given in [6].

We find the above result in [27] or also in [26]. It follows and uses a long series of previous results by E. De Giorgi [19], M. Miranda [48], H. Federer [23], E. Bombieri, E. De Giorgi & E. Giusti [6], E. Giusti [25]. One of the main intermediate tool is the *notion of quasiminimizers* and their regularity (for this notion and its use, see for instance [5]). They apply to a great variety of geometric problems like finding sets of prescribed mean curvature, in capillarity theory, etc...(see the examples and references in [59]).

A more precise statement of Theorem 1 gives more information on the singular set Σ. It says for instance that it is included in the complement of the so-called *reduced boundary* $\partial^*\hat{\Omega}$ of $\hat{\Omega}$: in other words, the reduced boundary itself is regular.

This kind of regularity carries over to quite more general situations involving for instance area-minimizing rectifiable currents: we refer for instance to the comments and references in the books [24] and [49].

3 A model problem: optimal shape for the Dirichlet energy with prescribed volume

Let $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be given. For each bounded open subset Ω of $\mathbb{R}^d$, we denote as usual by $H_0^1(\Omega)$ the closure for the norm

$$\|v\|_{H^1} = \Big\{ \int_\Omega v^2 + |\nabla v|^2 \Big\}^{1/2},$$

of the space of infinitely differentiable functions with compact support in Ω (denoted by $\mathcal{C}_0^\infty(\Omega)$). We denote by u_Ω the solution of the Dirichlet problem

$$-\Delta u_\Omega = f \text{ in } \Omega, \;\; u_\Omega = 0 \text{ on } \partial\Omega,$$

which is more precisely defined as the unique solution of the variational problem

$$u_\Omega \in H_0^1(\Omega), \; \forall\, v \in H_0^1(\Omega), \; \int_D \nabla u_\Omega \nabla v = \int_D f\, v. \tag{4}$$

We know that u_Ω minimizes the Dirichlet energy in $H_0^1(\Omega)$, that is

$$G(u_\Omega) = \{\min G(v);\; v \in H_0^1(\Omega)\}, \text{ where } G(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - f\, v.$$

Given D a bounded open subset in $\mathbb{R}^d$, and $m \in (0, |D|)$, we consider the following shape optimization problem

$$\hat{\Omega} \subset D,\ |\hat{\Omega}| = m,\ G(u_{\hat{\Omega}}) = \min\{G(u_\Omega);\ \Omega \subset D,\ |\Omega| = m\}. \tag{5}$$

This problem appears in many applications, besides the one described in Section 1 (see for instance [33], the books [21], [34], [58] and the references in them). It is not hard to show that the problem (5) has a solution (see e.g. [17], or [29], [10], [58]). However, the general existence theory does not provide an open set, but only *a quasi-open* set. Thus, it is already a regularity result to prove that we may obtain an open set as an optimal set (this is actually not true if for instance f is only in H^{-1}). A first result is the following. Recall that $f \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

Theorem 2 *Let $\hat{\Omega}$ be a solution of (5). Then, $u_{\hat{\Omega}}$ is Lipschitz continuous on $D_\delta = \{\xi \in D; d(\xi, \partial D) \geq \delta\}$ for all $\delta > 0$.*

As explained above, this is the first step in studying the regularity of $\hat{\Omega}$ itself. The case where $f \geq 0$ (or more generally when $u_{\hat{\Omega}} \geq 0$) is a little easier. It may be found in [29] and [7] or in [28] for a penalized version, and the proof relies on tools from [2]. See also [1] for a similar problem. The case where u changes sign is more involved (see [8]) and requires the monotonicity lemma of [3] in its nonhomogeneous form given in [11].

Now, we come to the regularity of $\partial\hat{\Omega}$ itself. We cannot expect full regularity in all cases: for instance, it is easy to prove that, if the solution $u_{\hat{\Omega}}$ changes sign near the boundary of $\hat{\Omega}$, then necessarily the boundary has a singularity, this even in dimension 2. On the other hand, we do expect regularity in the positive case. We can indeed prove the following:

Theorem 3 *Let $\hat{\Omega}$ be a solution of (5) and assume $f \geq 0$. Then, there exists a set $\Sigma \subset \partial\hat{\Omega}$ such that*
- the $(d-1)-$Hausdorff measure of Σ is zero,
- $\partial\hat{\Omega} \setminus \Sigma$ is regular.
If moreover $d = 2$, then the full boundary $\partial\hat{\Omega}$ is regular.

Here, by regular, we mean that the boundary is at least $C^{1,\alpha}$. If f is more regular, like $\mathcal{C}^\infty$, then so is the boundary.

We refer to [7] for a proof (based on tools from [2]) and for more general assumptions. See also [28], [1] for results of this kind.

It is very likely that the estimate of the size of the singular set Σ in Theorem 3 is not optimal. The functional involved here is very much like the one appearing in similar problems in [2], [3], [61], [12], [20], at least in the nonnegative case (the case without sign is different since cusps appear in 2-d here and not in [3]). In [20], it is shown for these problems that singularities appear in dimension $d = 7$ (note the similarity with the minimal surfaces in Section 2, but where the critical dimension is $d = 8$). It follows from the results and discussions in [61], [12] that full regularity is obtained for $d = 3$ and very likely for $d \leq 6$.

Finally, let us mention that, for these problems, one may also look directly for regular solutions by different approaches. See an example in [31].

4 Adding surface tension

We now consider the energy by adding a surface tension energy to the Dirichlet energy of the previous section namely

$$E(\Omega) = \tau P(\Omega) + G(u_\Omega), \tag{6}$$

where $P(\Omega)$ is defined in Section 2 and $G(u_\Omega)$ is defined in Section 3. And the shape optimization problem is the following

$$\widehat{\Omega} \subset D,\ |\widehat{\Omega}l = m,\ E(\widehat{\Omega}) = \min\{E(\Omega);\ \Omega \subset D,\ |\Omega| = m\}. \tag{7}$$

We may expect that the optimal shape is more regular here since the surface tension has a regularizing effect. However, it does not imply that the analysis is simpler and a generalization of the two approaches is not obvious at all.

Much progress has been done on this question in [45, 46, 47]. Let us summarize the results:

- For $f \in L^\infty(D)$, the state function u_Ω is $\frac{1}{2}$-Hölder-continuous. As a consequence, there is at least one optimal shape which is open.
- If moreover $f \geq 0$, then u_Ω is locally Lipschitz-continuous in D. It turns out that, as soon as one knows that the state function is Lipschitz, then it is possible to prove that *the optimal shape is a quasi-minimizer for the perimeter*, in the sense of Section 2. In some sense, the Dirichlet energy becomes "negligible" with respect to the perimeter energy once the Lipschitz continuity holds. It follows that the optimal shape has then the same regularity as for the isoperimetric problem of Section 2: we could state here the same result as Theorem 1.
- When f changes sign, the question of Lipschitz-continuity of u_Ω is open. It is interesting to notice that, it would be sufficient to prove that u_Ω is $\left[\frac{1}{2} + \epsilon\right]$-Hölder-continuous (with $\epsilon > 0$) to deduce that the optimal shape is a quasi-minimizer and to be essentially in the same situation as the one just described.

5 More examples: optimal eigenvalues, convexity constraint, polygons as optimal shapes.

We may also consider the regularity question of the optimal shapes for the eigenvalues of the Laplacian operator with Dirichlet boundary conditions. Let us look at it for the first eigenvalue. Recall that, for a bounded open set Ω, the first eigenvalue $\lambda_1(\Omega)$ may be defined by

$$\lambda_1(\Omega) = \inf\{\int_\Omega |\nabla v|^2;\ v \in H_0^1(\Omega), \int_\Omega v^2 = 1\}.$$

Then, we consider the shape optimization problem

$$\hat{\Omega} \subset D,\ |\hat{\Omega}| = m,\ \lambda_1(\hat{\Omega}) = \min\{\lambda_1(\Omega);\ \Omega \subset D,\ |\Omega| = m\}.$$

Here again, if D contains a ball of measure m, then the ball is the minimum (see [22, 40, 41]). If not, there exists at least a quasi-open minimal set (see [10]). Next, regularity results similar to Theorems 2 and Theorem 3 may be proved (see [30], [9]). Other results may be found in [60]. A few results may also be obtained for the other eigenvalues (see [54]). For instance, it is proved that the second eigenfunction is Lipschitz-continuous if the optimal shape is (quasi-)connected.

Let us mention that regularity questions turn out to be an important issue in many mathematical models. We may refer to [4] for a design problem, to [14] for a compliance problem and to the book [18] for the famous conjecture on the Mumford-Shah problem.

We end this survey by a few remarks on shape optimization with *convexity constraint*. When we minimize a functional among convex sets, at least we start with shapes whose boundary is the graph of a Lipschitz function (see the remarks of Section 1). However, it might still be difficult to go a little farther for the regularity. Let us take the following example:

$$\lambda_2(\widehat{\Omega}) = \min\{\lambda_2(\Omega); \Omega \ convex,\ |\Omega| = m\}$$

where $\lambda_2(\Omega)$ denotes the second eigenvalue of the Laplacian operator with homogeneous Dirichlet boundary conditions. We expect the solution to look like the convex envelop of two identical tangent balls (although it cannot be strictly so, see [37], [38], [53]). But, even in dimension two, the regularity is not clear. Many qualitative results have been shown in the three just mentioned papers, assuming enough a priori regularity. But, Lipschitz regularity is not enough to start. A nice result has been proved in [42], [43]: it says that, in dimension 2, at the junction point between a flat part and a strictly convex part of the boundary, the regularity is $C^{1,\frac{1}{2}}$!. In particular, the curvature blows up.

Another family of results is also interesting in this framework of convexity constraint. In dimension 2, the domain may be represented through a periodic function $u = [0, 2\pi] \to \mathbb{R}$ so that the boundary of $\Omega = \Omega_u$ is the curve $r = 1/u(\theta)$ in polar coordinates. Convexity is then equivalent to $u+u'' \geq 0$. Let us consider the minimization problem: $\min\{G(u(\theta), u'(\theta))\, d\theta\}$, in the set

$$u : [0, 2\pi] \to \mathbb{R},\ 2\pi - periodic,\ u + u'' \geq 0, \int_0^{2\pi} u^{-2} d\theta = m\},$$

where the last constraint means that the Lebesgue measure of Ω_u is prescribed to $2m$. Then the result says essentially (see [44], [42] for details): if G is concave with respect to u', then the optimal shapes are polygons. This may be extended to other natural constraints like $0 < a \leq u \leq b$ appearing in other interesting problems.

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