9. Properties of the free equation

The aim of this section is to give estimates on the solution (denoted $U(t)u_0$) of the equation

\[ i \frac{d}{dt} u = \Delta u, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R} \]

and also on the solution of the inhomogeneous equation

\[ i \frac{d}{dt} u = \Delta u + f, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R} \]

\[ u(0, x) = u_0(x) \]

where $f = f(t, x)$ is given.

We first note that the operator $A$ defined by $D(A) = H^2(\mathbb{R}^n)$ and $Au = -\Delta u$ for $u \in D(A)$ is self-adjoint, and so by Stone's theorem, $iA$ generates a $C^0$-group of unitary operators denoted by $(U(t))_{t \in \mathbb{R}}$. We may extend $U(t)$ to $D(A)^* = H^{-2}(\mathbb{R}^n)$ and, for $u_0 \in L^2(\mathbb{R}^n)$, $U(t)u_0$ is the unique solution of

\[ i \frac{du}{dt} = \Delta u \text{ in } H^{-2}, \quad t \in \mathbb{R} \]

which satisfies $u \in C(\mathbb{R}; L^2(\mathbb{R}^n)) \cap C^1(\mathbb{R}; H^{-2}(\mathbb{R}^n))$. In fact, we have an explicit expression of the kernel of the operator $U(t)$ (or equivalently, of the fundamental solution).

Prop 1

Let $u_0 \in L^2(\mathbb{R}^n)$; then for all $t \neq 0$,

\[ (U(t)u_0)(x) = \frac{1}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} \exp(-i \frac{|x-y|^2}{4t})u_0(y)dy \]

i.e. $U(t)$ is the convolution operator by the kernel $k(x, t) = \frac{1}{(4\pi it)^{n/2}} e^{-i|x|^2/4t}$.

Proof: Let $u_0 \in L^2(\mathbb{R}^n)$ and $u(t) = U(t)u_0$. Then $u$ satisfies

\[ i \frac{du}{dt} = \Delta u \text{ in } L^2(\mathbb{R}^n) \]

hence we can take
The space Fourier transform to obtain:
\[
\frac{\partial \hat{u}}{\partial t} = -i\xi \hat{u} + (t, \xi) \quad \text{i.e.} \quad \hat{u}(t, \xi) = e^{-i\xi^2 t} \hat{u}(0, \xi), \quad \text{i.e.} \quad u(t) = u(0) \ast \mathcal{F}^{-1}(e^{-i\xi^2 t}).
\]

Let us prove that \( \mathcal{F}^{-1}(e^{-i\xi^2 t}) = K(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-i\frac{\xi x}{2\sqrt{t}}} \).

First, \( \mathcal{F}^{-1}(e^{i\tau \xi^2 t}) = \mathcal{F}^{-1}(e^{i\tau \xi^2 t} \otimes \mathcal{F}^{-1}(e^{i\tau \xi^2 t}) \otimes \ldots \otimes \mathcal{F}^{-1}(e^{i\tau \xi^2 t})) \).

Hence it suffices to consider the case where \( \frac{\xi}{\sqrt{t}}, x \in \mathbb{R}. \)

Let \( T = T^{-1}(e^{i\tau \xi^2 t}) \in \mathcal{B} \) where \( \frac{\xi}{\sqrt{t}} \in \mathbb{R}. \) Then
\[
T' = T^{-1}(i\xi \xi^2 t) = T^{-1}(\frac{1}{2t} \frac{\partial}{\partial t} (e^{i\tau \xi^2 t})).
\]

Hence \( T(t) = \alpha(t) \exp(-\frac{i\tau \xi^2 t}{2t}). \)

Let us now compute \( \alpha(t) \) assuming \( t > 0, \)
\[
\alpha(t) = \left. \frac{\partial}{\partial t} \left. T(0) \right|_{t} \right|_{t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi^2 t} \, d\xi.
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} e^{i\xi^2 \eta} \, d\xi \, d\eta = \frac{1}{\pi \sqrt{2\pi t}} \left. \int_{0}^{\infty} e^{i\xi^2 \eta} \, d\eta \right|_{0}^{\infty}.
\]

Now, \( e^{i\xi^2 \eta} \) is a holomorphic function on \( \mathbb{C}, \) hence
\[
\int_{0}^{\infty} e^{i\xi^2 \eta} \, d\eta = \int_{0}^{\pi/4} e^{i(\Re(\eta e^{i\pi}))^2} e^{i\pi} \, d\eta.
\]

Note that the modulus of the 2nd term on the left hand side above is bounded by
\[
\int_{0}^{\pi/4} e^{(\Re(\eta e^{i\pi}))^2} \, d\eta \leq \int_{0}^{\pi/4} e^{-\frac{\eta^2}{2}} \, d\eta \leq \int_{0}^{\pi/4} e^{-\frac{\eta^2}{2}} \, d\eta.
\]

\[
\int_{0}^{\pi/4} e^{-\frac{\eta^2}{2}} \, d\eta 
\leq \int_{0}^{\pi/4} e^{-\frac{\eta^2}{2}} \, d\eta = \frac{1}{\sqrt{\pi}} e^{-\frac{\pi}{4}} \leq \frac{1}{\sqrt{\pi}} e^{-\frac{\pi}{4}}.
\]

\[
\int_{0}^{\pi/4} e^{-\frac{\eta^2}{2}} \, d\eta 
\leq \int_{0}^{\pi/4} e^{-\frac{\eta^2}{2}} \, d\eta = \frac{1}{\sqrt{\pi}} e^{-\frac{\pi}{4}} \leq \frac{1}{\sqrt{\pi}} e^{-\frac{\pi}{4}}.
\]

\[
\int_{0}^{\pi/4} e^{-\frac{\eta^2}{2}} \, d\eta 
\leq \int_{0}^{\pi/4} e^{-\frac{\eta^2}{2}} \, d\eta = \frac{1}{\sqrt{\pi}} e^{-\frac{\pi}{4}} \leq \frac{1}{\sqrt{\pi}} e^{-\frac{\pi}{4}}.
\]
hence this term goes to zero as $R \to \infty$ and we deduce that
\[ \int_0^\infty e^{in^2} \, dn = e^{i\pi/4} \int_0^\infty e^{-\eta^2} \, d\eta = e^{i\pi/4} \sqrt{\frac{\pi}{2}}. \]
So that $a(t) = \frac{1}{2 \sqrt{\pi t}} e^{i\pi/4} = \frac{1}{\sqrt{-4\pi t}}$. \qed

**Corollary 1:** Let $q \in \mathbb{R}$ with $2 \leq q \leq \infty$ and let $t \neq 0$.
Then $U(t) \in L^p(L^q; L^q(R^n))$, where $\frac{1}{p} + \frac{1}{q} = 1$, and $\forall u_0 \in L^p(R^n)$,
\[ \|U(t)u_0\|_{L^q} \leq (4\pi |t|)^{-n(\frac{1}{2} - \frac{1}{q})} \|u_0\|_{L^p}. \]

**proof:** It follows from Prop 1 that $U(t)$ extends to a linear continuous application from $L^1$ into $L^\infty$, for $t \neq 0$, satisfying
\[ \|U(t)u_0\|_{L^\infty} \leq (4\pi |t|)^{-n} \|u_0\|_{L^2}. \]
On the other hand, $U(t)$ is unitary in $L^2$, hence is an isometry, i.e.
\[ \|U(t)u_0\|_{L^2} = \|u_0\|_{L^2}. \]
We then apply the Riesz-Thorin interpolation theorem
with $p_0 = 1$, $p_1 = 2$, $q_0 = \infty$, $q_1 = 2$ and $\Pi_0 = (4\pi |t|)^{-n/2}$, $\Pi_1 = 1$. We obtain that $U(t)$ is linear continuous from $L^p$ into $L^q$, as soon as
\[ \frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}, \]
i.e.
\[ q = \frac{2}{\frac{1}{q_0} + \frac{1}{q_1}} \quad \text{and} \quad \frac{1}{p} = \frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{q} + \frac{1}{q_1}, \]
and that its norm is bounded by
\[ (4\pi |t|)^{-n(1/q_0/2)} = (4\pi |t|)^{-n(1/2 - 1/q_1)}. \]
\[ \square \]

**Remarks:**

- The expression $\widehat{U(t)u_0}(\xi) = e^{it\xi + \xi^2 \xi_0}/(\xi_0)$ shows that $U(t)$ extends to a $C^0$-group on $\mathcal{S}'(R^n)$ (by a duality argument). Hence we may in particular take $u_0 = \delta_0 \in \mathcal{S}'(R^n)$ and we get
\[ U(t)\delta_0 = \mathcal{F}^{-1}(e^{it|\xi|^2}) = \frac{1}{(4\pi |t|)^n/2} \exp(-\frac{\xi_0^2}{4t}) \quad \text{for} \quad t > 0. \]
Hence, $U(t)\varphi_0$ is a $C^0(\mathbb{R}_+)$ function for any $t \geq 0$ ($\Rightarrow$ smoothing effect).

The same expression implies that $U(t)$ is an isometry in all Sobolev space $H^s(\mathbb{R}^n)$, with $s \in \mathbb{R}$ (use the characterisation of $H^s$ using the Fourier transform).

Let $\Omega \subset \mathbb{R}^n$ be a bounded regular open set. Let $A$ be defined by $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ and $Au = -\Delta u$, $u \in D(A)$. Then, one can prove that $A$ is self-adjoint (see Brezis—Functional analysis); hence $iA$ generates a unitary group on $L^2(\mathbb{R})$, by Stone's theorem. However, the estimate of corollary 1 cannot be true in this case. Indeed, if $\Omega$ is bounded, then by Hölder's inequality, $L^p(\Omega) \subset L^q(\Omega)$ is continuous, if $1 \leq p \leq 2$. Then, if the estimate of corollary 1 was true, one would have for $u \in L^2(\Omega)$ and $\frac{1}{p} + \frac{1}{q} = 1$:

$$\|U(t)u\|_{L^p(\Omega)} \leq C \|U(t)u\|_{L^q(\Omega)} \leq C_1t^{-n(\frac{1}{2} - \frac{1}{q})} \|U(t)u\|_{L^2(\Omega)}$$

$$\leq C_2 t^{-n(\frac{1}{2} - \frac{1}{q})} \|U(t)u\|_{L^2(\Omega)} = C_2 t^{-n(\frac{1}{2} - \frac{1}{q})} \|u\|_{L^2(\Omega)}$$

which is absurd, since $L^2(\Omega) \neq L^q(\Omega)$ for $q > 2$.

The estimate of Corollary 1 is called dispersive estimate. We now prove that we also have space-time estimates, called Strichartz inequality, allowing to control the action of $U(t)$.

Def:

A pair of real numbers $(\varepsilon, \eta)$ is called admissible if

$2 \leq \eta < \frac{2n}{n-2}$ ($2 \leq \eta \leq \infty$ if $n = 1$ and $2 \leq \eta < \infty$ if $n = 2$) and

$$\frac{\varepsilon}{\eta} = n \left(\frac{\eta}{2} - \frac{1}{q}\right)$$

($n$ is here the space dimension).

Note that if $(\varepsilon, \eta)$ is admissible, then $2 \leq \eta \leq \infty$, and $(\varepsilon, 2)$ is always an admissible pair.
**Theorem (Stochastic Inequalities)**

For $f \in L^p(0,T; H^{-2})$, we set $\Lambda f(t) = \int_0^t u(t,s) f(s) \, ds$, t.e.R.

Then, the following properties hold:

(i) For any $u_0 \in L^p(\mathbb{R}^n)$, and for any admissible pair $(r,q)$, $t \to u(t) u_0 \in C(\mathbb{R}; L^r(\mathbb{R}^n)) \cap L^q(\mathbb{R}; L^q(\mathbb{R}^n))$.

Moreover, there is a constant $C > 0$ depending only on $q$ such that

$$\| u(\cdot) u_0 \|_{L^r(\mathbb{R}; L^q(\mathbb{R}^n))} \leq C \| u_0 \|_{L^p(\mathbb{R}^n)}$$

(ii) Let $(q', p)$ be an admissible pair and $f \in L^{q'}(0,T; L^p(\mathbb{R}^n))$ (with $\frac{1}{q'} + \frac{1}{p} = 1$, $\frac{1}{p} + \frac{1}{p} = 1$) then

$$\Lambda f \in C(\mathbb{R}^n; L^p(\mathbb{R}^n)) \cap L^{q'}(0,T; L^q(\mathbb{R}^n))$$

for any admissible pair $(r, q)$. In addition, there is a constant $C > 0$, depending only on $p$ and $q$ (or $r$ and $r$) such that

$$\| \Lambda f \|_{L^p(0,T; L^q(\mathbb{R}^n))} \leq C \| f \|_{L^{q'}(0,T; L^p)}$$

for any $f \in L^{q'}(0,T; L^p(\mathbb{R}^n))$.

**Remarks:**

- Let $(q, p)$ be an admissible pair, then $2 < p < \frac{2n}{n-2}$ hence $H^{-1}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ with continuous embedding. Hence $L^p \subset H^{-1} \subset H^{-2}$ and if $f \in L^{q'}(0,T; L^p)$, then $\Lambda f$ is well defined.

- Note that in (ii), the pairs $(r, q)$ and $(q', p)$ are completely decoupled. For instance, if $n=1$, one can...
take $r = 4$ and $q = 10$, even if one only has $f \in L^1([0,T];L^3)$ - this will allow us to get more integrability properties on the solution than the initial state.

Proof of the theorem

Step 1: We first show that if $(r, q)$ is an admissible pair, then $\Lambda$ is continuous from $L^r([0, T], L^q)$ into $L^r([0, T], L^q)$.

By a density argument, we may assume that $f \in C\left([0, T]; L^q\right)$. Then, by Nikolskii inequality,

$$\| \Lambda f(t) \|_{L^q(\mathbb{R}^n)} = \| \int_0^t U(t-s) f(s) \, ds \|_{L^q(\mathbb{R}^n)} \leq \int_0^t \| U(t-s) f(s) \|_{L^q(\mathbb{R}^n)} \, ds$$

and by Corollary 1:

$$\leq C \int_0^t \| f(s) \|_{L^q(\mathbb{R}^n)} \, ds$$

We then notice that $\frac{n}{2} - \frac{1}{r} - \frac{1}{q} = \frac{n}{2} - \frac{1}{r} - \frac{1}{q}$, and we use the Corollary of the Hardy-Littlewood-Sobolev inequality with $n = 1$, $d = \frac{2}{r'}$ so that $\lambda = \frac{n}{r'}$, and hence

$$\| \Lambda f(t) \|_{L^q(\mathbb{R}^n)} \leq C(r) \| f \|_{L^q(\mathbb{R}^n)}$$

which gives the result.

Note that we did not use the fact that the integral in time was between 0 and $t$: the same is true for

$$\int_0^t U(t-s) f(s) \, ds \text{ or } \int_0^t U(s-t) f(s) \, ds$$

useful in the sequel.

Step 2: Let now $(r, q)$ be an admissible pair, and let us prove that $\Lambda$ is continuous from $L^r([0, T]; L^q)$ into $C([0, T]; L^q(\mathbb{R}^n))$. -44-
Again by density, we may assume that $f \in C_c \left( \mathbb{R}^n ; \mathbb{R}^n \right)$
and $\nabla f \in \mathcal{L}^2 \left( \mathbb{R}^n \right)$. In this case, it is clear that $\Lambda f \in C \left( \mathbb{R}^n ; \mathbb{L}^2 \right)$
and it suffices to prove the estimate
\[
\sup_{t \in [0,T]} \| \Lambda f(t) \|_{L^2} \leq C \| f \|_{L^2 \left( \mathbb{R}^n \right)} \quad \text{with} \quad C \text{ independent of } t.
\]

We have
\[
\| \Lambda f(t) \|_{L^2}^2 = \left( \int_0^t \int_0^t \left( U(t-s) f(s), U(t-s) f(s) \right) \, ds \, dr \right) \left( \int_0^t \int_0^t \left( U(t-s) f(s), U(t-s) f(s) \right) \, ds \, dr \right)
\]
\[
= \int_0^t \int_0^t \left( \nabla f(s), U(s-t) \nabla f(s) \right) \, ds \, ds = \int_0^t \left( \nabla f(s), \nabla \tilde{U}_t f(s) \right) \, ds
\]
where $\tilde{U}_t f(s) = \int_0^t U(s-t) \nabla f(s) \, ds$. We have used the fact that $U(t-s)^* = U(s-t)$. We then apply H"older's inequality, first
in space and then in time:
\[
\| \Lambda f(t) \|_{L^2} \leq \int_0^t \| f(s) \|_{L^{2r/3}} \| \nabla \tilde{U}_t f(s) \|_{L^{2r/3}} \, ds \quad \text{with} \quad \frac{1}{3} + \frac{1}{3} = 1
\]
\[
\leq \| f \|_{L^r \left( 0,T ; L^{2r/3} \right)} \| \nabla \tilde{U}_t f \|_{L^2 \left( 0,T ; L^{2r/3} \right)}
\]
\[
\leq C \| f \|_{L^r \left( 0,T ; L^{2r/3} \right)} \| \tilde{U}_t \|_{L^2 \left( 0,T ; L^{r} \right)} \leq C \| f \|_{L^r \left( 0,T ; L^{2r/3} \right)}
\]

Where we have used the remark following Step 1. Hence, the estimate follows by taking the supremum in time of the left-hand side.

**Step 3:** We now prove that if $(r,q)$ is an admissible pair, $\Lambda$ is continuous from $L^r \left( 0,T ; L^q \left( \mathbb{R}^n \right) \right)$ into $L^2 \left( 0,T ; L^q \left( \mathbb{R}^n \right) \right)$.

Let $f \in L^r \left( 0,T ; L^q \left( \mathbb{R}^n \right) \right)$ and let $\varphi \in C_c \left( \mathbb{R}^n ; \mathbb{R} \right)$.

Then
\[
\int_0^T \int_0^t \left( \Lambda f(s), \varphi(t) \right) \, ds \, dt = \int_0^T \int_0^t \left( U(t-s) f(s), \varphi(t) \right) \, ds \, dt
\]
\[
= \int_0^T \int_0^t \left( f(s), U(t-s) \varphi(t) \right) \, dt \, ds
\]

Where $U(t)$ is the heat semigroup.
\[
\int_{0}^{T} (f(\varphi), \int_{0}^{T} u(s-t) \varphi(t) \, dt) \, ds'
\]
\[\leq \|f\|_{L^1(\Omega T; L^p)} \| \int_{0}^{T} u(s-t) \varphi(t) \, dt \|_{L^0(\Omega T; L^q)}
\]
\[\leq C \|f\|_{L^1(\Omega T; L^p)} \| \varphi \|_{L^\infty(\Omega T; L^q)}
\]

where we have used Step 2 and the same remark as in Step 1.

Now, since
\[\|\Lambda f\|_{L^p(\Omega T; L^q)} = \sup_{\varphi \in C^0(\Omega T \times \mathbb{R}^2)} \| \int_{0}^{T} (\Lambda f(t), \varphi(t)) \, dt \|_{L^q(\Omega T; L^q)}
\]

it follows that
\[\|\Lambda f\|_{L^p(\Omega T; L^q)} \leq C \|f\|_{L^1(\Omega T; L^p)}
\]

**Step 4:** Let us prove part (ii) of the theorem. Let \((\xi, \varphi)\) be an admissible pair. From Steps 1 and 2, \(\Lambda\) is continuous from \(L^0(\Omega T; L^p)\) into \(L^0(\Omega T; L^q) \cap L^\infty(\Omega T; L^q)\). Consider first an admissible pair \((\xi, \varphi)\) with \(0 \leq q \leq p\); then for some \(\theta\) with \(0 < \theta < 1\), \(1 - \theta = \frac{\theta}{q} + \frac{1 - \theta}{p}\). Since \((\xi, \varphi)\) is admissible,
\[
\frac{1}{q} = \frac{n}{p} - \frac{1 - \theta}{p} = \frac{\theta}{q} + \frac{1 - \theta}{p}.
\]

Hölder's inequality implies then, for \(f \in L^0(\Omega T; L^p)\):
\[
\|\Lambda f(\xi, \varphi)\|_{L^q} \leq \|\Lambda f\|_{L^1(\Omega T; L^p)} \|f\|_{L^1(\Omega T; L^0)}
\]

and
\[
\|\Lambda f\|_{L^q(\Omega T; L^q)} \leq \|\Lambda f\|_{L^1(\Omega T; L^p)} \|f\|_{L^1(\Omega T; L^0)}
\]

\[
\leq C \|f\|_{L^0(\Omega T; L^p)} \|f\|_{L^1(\Omega T; L^0)}
\]

from Steps 1 and 2.
This proves (ii) in the case where \( 2 \leq q \leq p \).

Let now \((r, q)\) be an admissible pair, with \( p < q \); then since
\[
2 \leq p < q, \quad 3 \mu \in (0, 1], \quad \text{such that} \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{q} \quad \text{(hence also} \quad \frac{1}{r} = \frac{\mu}{q} + \frac{1-\mu}{r}) \quad \text{and from the preceding computation, we also have} \quad \frac{1}{r} = \frac{\mu}{r} + \frac{1-\mu}{r},
\]

it then follows from steps 1 and 3 that \( \Lambda \) is continuous from \( L^r(\Omega; L^q) \) into \( L^r(\Omega; L^q) \) and from \( L^r(\Omega; L^p) \) into \( L^r(\Omega; L^q) \). We apply another interpolation theorem (see Begehr–Lofstrom, Interpolation Spaces, th. 5.1.2) to conclude that in this case, \( \Lambda \) is continuous from \( L^r(\Omega; L^q) \) into \( L^r(\Omega; L^q) \) for any \((r, p)\) satisfying
\[
\frac{1}{r} = \frac{\theta}{r} + \frac{1-\theta}{q}, \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{q}, \quad \text{for some} \ \theta \in (0, 1).
\]

It suffices then to choose \( \theta = 1 \) to find that \( \Lambda \) is continuous from \( L^r(\Omega; L^q) \) into \( L^r(\Omega; L^q) \).

Note that the case \( q > p \) will not be used for the proof of existence of a solution to the nonlinear equation; however, we state it since it allows to get additional information on the solution.

Step 5: It remains to prove (i).

Let \((r, q)\) be an admissible pair, and let \( u_0 \in L^r(\Omega) \). As in step 3, we use the fact that
\[
\|u(t) - u_0\|_{L^r(\Omega; L^q)} = \sup_{c \in C^1(\Omega; \mathbb{R}^m)} \left| \int_\Omega (u(t)u_0, cp) \, dx \right|
\]
\[
\|u_0\|_{L^r(\Omega; L^q)} = 1
\]

let \( c \in C^1(\Omega; \mathbb{R}^m) \), then
\[
\left| \int_\Omega (u(t)u_0, cp) \, dx \right| = \left| \int_\Omega (u_0, u(t)c) \, dx \right|
\]

-47-
\[
\| \sum_{t=0}^{t=T} \int_{R} U(-t) \varphi(t) \, dt \|_{L^2} \leq \| U \|_{L^2} \| \sum_{t=0}^{t=T} \int_{R} U(-t) \varphi(t) \, dt \|_{L^2}
\]

when we recall that \( (U, \varphi) = \sum_{t=0}^{T} \int_{R} U(x) \varphi(x) \, dx \). Then, by the same computation as in step 1, we have
\[
\| \sum_{t=0}^{t=T} \int_{R} U(-t) \varphi(t) \, dt \|_{L^2}^2 = \left( \sum_{t=0}^{T} \int_{R} U(-t) \varphi(t) \, dt, \sum_{t=0}^{T} \int_{R} U(-t) \varphi(t) \, dt \right)
\]
\[
= \sum_{t=0}^{T} \int_{R} \varphi(t) \int_{R} U(-t) \varphi(t) \, dt \, dt
\]
\[
\leq \| \varphi \|_{L^2(0, T; L^2)} \int_{R} \int_{R} U(-t) \varphi(t) \, dt \, dt
\]
\[
\leq C \| \varphi \|_{L^2(0, T; L^2)}^2 \text{ by Step 1 and the remark at the end.}
\]

It follows that
\[
\| U(\cdot) \|_{L^2(0, T; L^2)} \leq C \| w(t) \|_{L^2}.
\]

**Remark:** The original proof of Strichartz was different and consisted in using a Fourier restriction lemma. The present proof by duality is due to Ginibre and Velo.

**Corollary:** Estimate on the solution of the inhomogeneous equation: let \( u_0 \in L^2(\mathbb{R}^n) \) and \( f \in L^{q'}(0, T; L^q) \) where \( (q, p) \) is an admissible pair; let \( u \in C([0,T]; L^2(\mathbb{R}^n)) \) be a solution of
\[
\Delta u + u = f \quad \text{for all } t, \text{ in } H^{-2}(\mathbb{R}^n),
\]
\[
u(0) = u_0
\]
then \( u \in L^q(0, T; L^q) \) for any admissible pair \((r, q)\); in particular if \( n = 1 \), then \( u \in L^4(0, T; L^4(\mathbb{R})) \).

**Proof:**

Since \( f \in L^{q'}(0, T; L^q) \) \( \subset L^{q'}(0, T; H^{-2}) \) and \( u \in C([0,T]; L^2) \) we know that \( u \) is solution of the above equation if and only if it satisfies the integral formulation (or modified...
equation) (see Section 1.6):
\[ u(t) = u(t) u_0 + \int_0^t u(t-s) (-i f(s)) \, ds; \]
the result then follows from (i) and (ii) of Strichartz inequalities. \[ \square \]

**Corollary 3:** Dispersion of the free solutions

Let \( u_0 \in \mathcal{H}^s(\mathbb{R}^n) \) and \( q \) with \( 2 < q < \frac{2n}{n-2} \) \((2 < q < \infty \) if \( n = 1 \)
or \( 2 \leq q \leq \infty \) if \( n = 2 \)); then \( \lim_{t \to \pm \infty} \| u(t) u_0 \|_{L^q} = 0. \)

**Proof:**

Let \( r \) be such that \( (r,q) \) is an admissible pair, then from (i) of Strichartz inequalities, \( u(\cdot) u_0 \in L^r(\mathbb{R}, L^q(\mathbb{R}^n)) \) hence the result will follow if we prove that \( t \mapsto u(t) u_0 \)
is uniformly continuous with values in \( L^q(\mathbb{R}^n) \). Let \( u(t) = U(t) u_0 \); by the Gagliardo-Nirenberg inequalities
\[ \| u(t) - u(s) \|_{L^q} \leq C \| u(t) - u(s) \|_{H^s} \]
with \( a = n (\frac{1}{2} - \frac{4}{q}) = \frac{n}{r} \); since \( U(t) \) is an isometry in \( H^s(\mathbb{R}^n) \) for all \( s \), we deduce
\[ \| u(t) - u(s) \|_{L^q} \leq C (\| u(t) \|_{H^s} + \| u(s) \|_{H^s}) \frac{\| u(t) - u(s) \|_{L^r}}{\| u(t) \|_{H^s}} \leq 2 C \| u_0 \|_{H^s} \frac{1}{r} \| u(t) - u(s) \|_{L^r} \]
On the other hand, \( \frac{du}{dt} = -i A u(t) \), hence
\[ \frac{du}{dt} \|_{H^s} = \| A u(t) \|_{H^s} \leq \| u(t) \|_{H^s} = \| u_0 \|_{H^s}, \quad \forall t \in \mathbb{R}. \]
We deduce that
\[ \| u(t) - u(s) \|_{H^s} \leq \| u_0 \|_{H^s} |t-s|, \quad \forall t, s \in \mathbb{R} \]
and
\[ \| u(t) - u(s) \|_{L^q} = \langle u(t) - u(s), u(t) - u(s) \rangle_{H^s, H^s} \]
\[ \leq \| u(t) - u(s) \|_{H^s} (\| u(t) \|_{H^s} + \| u(s) \|_{H^s}) \]
\[ \leq 2 \| u_0 \|_{H^s} |t-s|, \]
Finally, \( \| u(t) - u(s) \|_{L^q} \leq 2 C \| u_0 \|_{H^s} |t-s| \frac{1}{r} \)
and \( t \mapsto u(t) u_0 \) is uniformly continuous with values in \( L^q \).
in $L^q$, as soon as $r > 2$, which is the case if $q < \frac{2n}{n+2}$. 

Remarks:
- Strichartz inequalities are still valid in the case $q = \frac{2n}{n+2}$, $n \geq 3$ (i.e. $r = 2$), but the proof is much more complicated.
- The only necessary ingredients in the proof of Strichartz inequalities are
  - the unitarity of $U(t)$ in $L^2$
  - the $L^1 - L^\infty$ estimate (i.e. dispersive estimate) on $U(t)$

Those inequalities generalize to the case where $U(t) = e^{itA}$ for a self-adjoint operator $A$ on $L^2$, such that $U(t)$ is continuous from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$, at least for $|t|$ small, with a norm bounded by $C|t|^{-n/2}$; in this latter case, the norm in Strichartz estimates may depend on $T$ and Corollary 3 may not be true.

Examples:

\[ A = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \quad a_{ij} \in \mathbb{R}, \quad a_{ij} = a_{ji}; \quad \text{assume that} \]

the matrix $A = (a_{ij})$ is $n \times n$ is invertible, but not necessarily positive definite, so that $A$ is not necessarily elliptic.

Using the Fourier transform, it is easily seen that $iA$ generates a unitary group in all $H^s(\mathbb{R}^n)$.

Exercise: Compute the convolution kernel of the group generated by $iA$ and show that the $L^1 - L^\infty$ estimate holds.

Note that the domain on which $A$ is self-adjoint is not necessarily equal to $H^s$ if $A$ is not elliptic.

Applications:
- $A = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$, or $A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$
Note that the latter operator corresponds to the case $k'(w_0)k''(w_0) > 0$ in Chapter I.

$$ A = -\frac{1}{2} \Delta + V(x) \text{ where } V(x) = \frac{\omega^2}{2} |x|^2 + \frac{\omega^2}{2} \sum_{i=1}^{n} \xi_i x_i^2, \omega \in \mathbb{R} $$

Then one can check (see Feynman, Hibbs, Quantum Mechanics) that $A$ generates a $C^0$ group of unitary operators $T(t)$ on $L^2(\mathbb{R}^n)$ which satisfies: $\forall \psi \in L^1(\mathbb{R}^n)$, $\forall t$ with $|t| < \frac{\pi}{\omega}$,

$$ T(t) \psi(x) = \int_{\mathbb{R}^n} k(t, x, y) \psi(y) dy $$

with

$$ k(t, x, y) = \left( \frac{i}{\pi \sin \omega t} \right)^{n/2} \exp \left[ -i \left( \cos \omega t \right) (|x|^2 + |y|^2) - \frac{2 \omega x \cdot y}{2 \sin \omega t} \right] $$

This operator is called the quantum harmonic oscillator; the above expression for the kernel $k$ has been generalized to more general potentials $V$.

It follows that the group $T(t)$ satisfies the $L^1-L^\infty$ estimate, and hence also the Strichartz estimates (with $\mathbb{R}$ replaced by $[0, T]$). Note that $-\Delta + \omega^2 |x|^2$ is an operator with compact inverse, hence its spectrum is purely discrete; it follows that Strichartz inequalities cannot be global in time; indeed if $A \psi = d \psi$ then $T(t) \psi = e^{i t \omega} \psi$ and $\| T(t) \psi \|_{L^4} = \| \psi \|_{L^4}$ for all $\psi$, i.e. $T(\cdot) \psi \in L^4(\mathbb{R}; L^4)$.

3. The local Cauchy problem

We now consider the nonlinear equation

$$(NLS) \quad \begin{cases} i \partial_t u - \Delta u = d |u|^2 u, \\ u(0, x) = u_0(x) \end{cases}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, $u(t, x) \in C$, $d \in \mathbb{R}$, $\sigma > 0$. We will consider the integral formulation.