2 - Properties of the free equation

The aim of this section is to give estimates on the solution (denoted $U(t)u_0$) of the equation

$$\begin{cases} i\partial_t u = \Delta u, & x \in \mathbb{R}^n, t \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases}$$

and also on the solution of the inhomogeneous equation

$$\begin{cases} i\partial_t u - \Delta u = f, & x \in \mathbb{R}^n, t \in \mathbb{R} \\ u(0,x) = 0 \end{cases}$$

where $f = f(t,x)$ is given.

We first note that the operator A defined by $D(A) = H^2(\mathbb{R}^n)$ and $Au = -\Delta u$ for $u \in D(A)$ is selfadjoint, and so by Stone's theorem, iA generates a C^0-group of unitary operators denoted by $(U(t))_{t\in\mathbb{R}}$. We may extend $U(t)$ to $D(A)^* = H^{-2}(\mathbb{R}^n)$ and, for $u_0 \in L^2(\mathbb{R}^n)$, $u(t) = U(t)u_0$ is the unique solution of

$$i\frac{du}{dt} = \Delta u \text{ in } H^{-2}, \quad \forall t \in \mathbb{R}$$

which satisfies $u \in C(\mathbb{R}; L^2(\mathbb{R}^n)) \cap C^1(\mathbb{R}; H^{-2}(\mathbb{R}^n))$. In fact, we have an explicit expression of the kernel of the operator $U(t)$ (or equivalently, of the fundamental solution).

<u>Prop 1</u>

Let $u_0 \in \mathcal{S}(\mathbb{R}^n)$; then for all $t \neq 0$,

$$(U(t)u_0)(x) = \frac{1}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-i\frac{|x-y|^2}{4t}\right) u_0(y)\,dy$$

i.e. $U(t)$ is the convolution operator by the kernel

$$K(x,t) = \frac{1}{(4\pi i t)^{n/2}} e^{-i|x|^2/4t}$$

<u>Proof</u>: Let $u_0 \in \mathcal{S}(\mathbb{R}^n)$ and $u(t) = U(t)u_0$. Then u satisfies

$$i\frac{du}{dt} = \Delta u \text{ in } L^2(\mathbb{R}^n)$$

, hence we can take

the space Fourier transform to obtain:

$$i\frac{d\hat{u}}{dt} = -|\xi|^2 \hat{u}(t,\xi) \quad \text{i.e} \quad \hat{u}(t,\xi) = e^{i|\xi|^2 t}\hat{u}_0(\xi), \quad \text{i.e}$$

$$u(t) = u_0 * \mathcal{F}^{-1}\left(e^{i|\xi|^2 t}\right).$$

Let us prove that $\mathcal{F}^{-1}\left(e^{i|\xi|^2 t}\right) = K(x,t) = \frac{1}{(-4\pi i t)^{n/2}} e^{-i|x|^2/4t}$.

First, $\mathcal{F}^{-1}\left(e^{i|\xi|^2 t}\right) = \mathcal{F}^{-1}\left(e^{i\xi_1^2 t}\right) \otimes \mathcal{F}^{-1}\left(e^{i\xi_2^2 t}\right) \otimes \dots \otimes \mathcal{F}^{-1}\left(e^{i\xi_n^2 t}\right)$

Hence it suffices to consider the case where $\xi, x \in \mathbb{R}$.

Let $T = \mathcal{F}^{-1}\left(e^{i\xi^2 t}\right) \in \mathcal{S}'$ where $\xi \in \mathbb{R}$. Then

$$T' = \mathcal{F}^{-1}\left(i\xi\, e^{i\xi^2 t}\right) = \mathcal{F}^{-1}\left(\frac{1}{2t}\partial_\xi\left(e^{i\xi^2 t}\right)\right)$$

$$= -\frac{ix}{2t}\mathcal{F}^{-1}\left(e^{i\xi^2 t}\right) = -\frac{ix}{2t}T$$

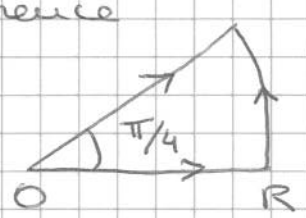
Hence $T(t) = \alpha(t)\exp\left(-\frac{ix^2}{4t}\right)$.

Let us now compute, assuming $t>0$,

$$\alpha(t) = \text{"}T(0)\text{"} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{it\xi^2}\,d\xi$$

$$= \frac{1}{\pi}\int_0^{+\infty} e^{it\xi^2}\,d\xi = \frac{1}{\pi\sqrt{t}}\int_0^{+\infty} e^{i\eta^2}\,d\eta$$

Now, $e^{i\xi^2}$ is a holomorphic function on $\mathbb{C}$, hence



$$\int_0^R e^{i\eta^2}\,d\eta + \int_0^{\pi/4} e^{i(Re^{i\theta})^2} iRe^{i\theta}\,d\theta$$

$$= \int_0^R e^{i(\eta e^{i\pi/4})^2} e^{i\frac{\pi}{4}}\,d\eta$$

Note that the modulus of the 2nd term on the left hand side above is bounded by

$$\left|\int_0^{\pi/4} e^{i(Re^{i\theta})^2} iRe^{i\theta}\,d\theta\right| \le \int_0^{\pi/4} e^{-R^2\sin 2\theta} R\,d\theta$$

$$\le \underbrace{\int_0^{1/R^{3/2}} e^{-R^2\sin 2\theta} R\,d\theta}_{\le \frac{R}{R^{3/2}} = \frac{1}{\sqrt{R}}} + \underbrace{\int_{1/R^{3/2}}^{\pi/4} e^{-R^2\sin\frac{2}{R^{3/2}}} R\,d\theta}_{\le \frac{\pi}{4} R e^{-R^2\sin\frac{2}{R^{3/2}}}}$$

hence this term goes to zero as $R \to +\infty$ and we deduce that

$$\int_0^{+\infty} e^{i\eta^2} d\eta = e^{i\pi/4} \int_0^{+\infty} e^{-\eta^2} d\eta = e^{i\pi/4} \frac{\sqrt{\pi}}{2}$$

So that $\alpha(t) = \frac{1}{2\sqrt{\pi t}} e^{i\pi/4} = \frac{1}{\sqrt{-4\pi i t}}$. $\square$

<u>Corollary 1:</u> Let $q \in \mathbb{R}$ with $2 \leq q \leq +\infty$ and let $t \neq 0$; then $U(t) \in \mathcal{L}(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$, where $\frac{1}{p} + \frac{1}{q} = 1$, and $\forall u_0 \in L^p(\mathbb{R}^n)$,

$$\| U(t) u_0 \|_{L^q} \leq (4\pi |t|)^{-n\left(\frac{1}{2} - \frac{1}{q}\right)} \|u_0\|_{L^p}.$$

<u>proof:</u>

It follows from Prop 1 that $U(t)$ extends to a linear continuous application from L^1 into L^∞, for $t \neq 0$, satisfying

$$\|U(t) u_0\|_{L^\infty} \leq (4\pi |t|)^{-n/2} \|u_0\|_{L^1}.$$

On the other hand, $U(t)$ is unitary in L^2, hence is an isometry, i.e $\|U(t) u_0\|_{L^2} = \|u_0\|_{L^2}$.

We then apply the Riesz-Thorin interpolation theorem with $p_0 = 1$, $p_1 = 2$, $q_0 = \infty$, $q_1 = 2$ and $M_0 = (4\pi|t|)^{-n/2}$, $M_1 = 1$. We obtain that $U(t)$ is linear continuous from L^p into L^q, as soon as $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}$ and $\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$, $0 \leq \theta \leq 1$ i.e $q = \frac{2}{\theta} \geq 2$ and $\frac{1}{p} = 1 - \frac{\theta}{2} = 1 - \frac{1}{q}$, and that its norm is bounded by $(4\pi|t|)^{-n(1-\theta)/2} = (4\pi|t|)^{-n(1/2 - 1/q)}$. $\square$

<u>Remarks:</u>

- the expression $\widehat{U(t)u_0}(\xi) = e^{it|\xi|^2} \hat{u}_0(\xi)$ shows that $U(t)$ extends to a C^0-group on $\mathcal{S}'(\mathbb{R}^n)$ (by a duality argument). Hence we may in particular take $u_0 = \delta_0 \in \mathcal{S}'(\mathbb{R}^n)$ and we get

$$U(t)\delta_0 = \mathcal{F}^{-1}\left(e^{it|\xi|^2}\right) = \frac{1}{(-4\pi i t)^{n/2}} \exp\left(-\frac{i|x|^2}{4t}\right) \text{ for } t > 0;$$

Hence, $U(t)\delta_0$ is a $C^\infty(\mathbb{R}^n)$ function for any $t>0$ ($\Rightarrow$ smoothing effect).

- the same expression implies that $U(t)$ is an isometry in all Sobolev spaces $H^s(\mathbb{R}^n)$, with $s\in\mathbb{R}$ (use the caracterisation of H^s using the Fourier transform).
- let $\Omega\subset\mathbb{R}^n$ be a bounded regular open set. let A be defined by $D(A)=H^2(\Omega)\cap H^1_0(\Omega)$ and $Au=-\Delta u$, $u\in D(A)$. Then, one can prove that A is selfadjoint (see Brezis - Functional analysis); hence iA generates a unitary group on $L^2(\Omega)$, by Stone's theorem. However, the estimate of corollary 1 cannot be true in this case. Indeed, if Ω is bounded, then by Hölder's inequality, $L^2(\Omega)\hookrightarrow L^p(\Omega)$ is continuous, if $1\le p\le 2$. Then, if the estimate of corollary 1 was true, one would have for $u\in L^2(\Omega)$ and $\frac{1}{p}+\frac{1}{q}=1$:

$$\|u\|_{L^q} = \|U(t)U(-t)u\|_{L^q} \le C|t|^{-n\left(\frac{1}{2}-\frac{1}{q}\right)}\|U(-t)u\|_{L^p}$$

$$\le C_\Omega |t|^{-n\left(\frac{1}{2}-\frac{1}{q}\right)}\|U(-t)u\|_{L^2} = C_\Omega |t|^{-n\left(\frac{1}{2}-\frac{1}{q}\right)}\|u\|_{L^2}$$

which is absurd, since $L^2(\Omega)\neq L^q(\Omega)$ for $q>2$.

The estimate of Corollary 1 is called dispersive estimate. We now prove that we also have space-time estimates, called Strichartz inequalities, allowing to control the action of $U(t)$.

Def:

A pair of real numbers (r,q) is called admissible if $2\le q<\frac{2n}{n-2}$ ($2\le q\le+\infty$ if $n=1$ and $2\le q<+\infty$ if $n=2$) and $\frac{2}{r}=n\left(\frac{1}{2}-\frac{1}{q}\right)$ (n is here the space dimension).

Note that if (r,q) is admissible, then $2\le r\le+\infty$, and $(+\infty,2)$ is always an admissible pair.

Theorem (Strichartz inequalities)

For $f \in L^1(0,T; H^{-2})$, we set $\Lambda f(t) = \int_0^t U(t-s) f(s)\, ds$, $t \in \mathbb{R}$.
Then, the following properties hold:

(i) For any $u_0 \in L^2(\mathbb{R}^n)$, and for any admissible pair (r,q),
$t \mapsto U(t) u_0 \in C(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^r(\mathbb{R}, L^q(\mathbb{R}^n))$.
Moreover, there is a constant $C > 0$ depending only on q such that

$$\| U(\cdot) u_0 \|_{L^r(\mathbb{R}; L^q(\mathbb{R}^n))} \leq C \| u_0 \|_{L^2(\mathbb{R}^n)}$$

(ii) let (γ, ρ) be an admissible pair and $f \in L^{\gamma'}(0,T; L^{\rho'}(\mathbb{R}^n))$ (with $\frac{1}{\gamma'} + \frac{1}{\gamma} = 1$, $\frac{1}{\rho'} + \frac{1}{\rho} = 1$); then

$$\Lambda f \in C([0,T]; L^2(\mathbb{R}^n)) \cap L^r(0,T; L^q(\mathbb{R}^n))$$

for any admissible pair (r,q). In addition, there is a constant $C > 0$, depending only on ρ and q (or γ and r) such that

$$\| \Lambda f \|_{L^\infty(0,T; L^2) \cap L^r(0,T; L^q)} \leq C \| f \|_{L^{\gamma'}(0,T; L^{\rho'})}$$

for any $f \in L^{\gamma'}(0,T; L^{\rho'}(\mathbb{R}^n))$.

Remarks:

- Let (γ, ρ) be an admissible pair, then $2 \leq \rho < \frac{2n}{n-2}$ hence $H^1(\mathbb{R}^n) \subset L^\rho(\mathbb{R}^n)$ with continuous embedding. Hence $L^{\rho'} \subset H^{-1} \subset H^{-2}$ and if $f \in L^{\gamma'}(0,T; L^{\rho'})$, then Λf is well defined
- Note that in (ii), the pairs (r,q) and (γ, ρ) are completely decoupled. For instance, if $n = 1$, one can

take $r=4$ and $q=\infty$, even if one only has $f \in L^1(0,T;L^2)$. This will allow us to get more integrability properties on the solution than the initial state.

Proof of the theorem

Step 1 : We first show that if (r,q) is an admissible pair, then Λ is continuous from $L^{r'}(0,T,L^{q'})$ into $L^r(0,T;L^q)$. By a density argument, we may assume that $f \in C_c(]0,T[;L^{q'})$. Then, by Minkowski inequality,

$$\|\Lambda f(t)\|_{L^q(\mathbb{R}^n)} = \left\| \int_0^t U(t-s) f(s)\,ds \right\|_{L^q(\mathbb{R}^n)}$$

$$\leq \int_0^t \|U(t-s) f(s)\|_{L^q(\mathbb{R}^n)}\,ds$$

and by Corollary 1:

$$\leq C \int_0^t |t-s|^{-n\left(\frac{1}{2}-\frac{1}{q}\right)} \|f(s)\|_{L^{q'}(\mathbb{R}^n)}\,ds$$

We then notice that $n\left(\frac{1}{2}-\frac{1}{q}\right) = \frac{2}{r}$, and we use the Corollary of the Hardy-Littlewood-Sobolev inequality with $n=1$, $\lambda = \frac{2}{r}$ so that $\lambda + \frac{1}{r'} = 1 + \frac{1}{r}$; hence

$$\left\| \int_{\mathbb{R}} |\cdot - s|^{-2/r} \|f(s)\|_{L^{q'}}\,ds \right\|_{L^r_t(\mathbb{R})} \leq C(r) \|f\|_{L^{r'}_t(\mathbb{R};L^{q'}(\mathbb{R}^n))}$$

which gives the result.

Note that we did not use the fact that the integral in time was between 0 and t: the same is true for $\int_s^T U(t-s) f(t)\,dt$ or $\int_0^t U(s-\sigma) f(\sigma)\,d\sigma$; this will be usefull in the sequel.

Step 2 : Let now (γ,ρ) be an admissible pair, and let us prove that Λ is continuous from $L^{\gamma'}(0,T;L^{\rho'})$ into $C([0,T];L^2(\mathbb{R}^n))$.

Again by density, we may assume that $f \in C_c([0,T[; L^{\rho'}(\mathbb{R}^n)) \cap C_c([0,T[; L^2(\mathbb{R}^n))$; in this case, it is clear that $\Lambda f \in C([0,T]; L^2)$ and it suffices to prove the estimate

$$\sup_{t\in[0,T]} \|\Lambda f(t)\|_{L^2} \leq C \|f\|_{L^{\gamma'}(0,T;L^{\rho'})} \quad \text{with } C \text{ indep of } T.$$

We have

$$\|\Lambda f(t)\|^2_{L^2} = \left(\int_0^t U(t-s)f(s)\,ds, \int_0^t U(t-s)f(\sigma)\,d\sigma\right)$$

$$= \int_0^t\int_0^t (U(t-s)f(s), U(t-\sigma)f(\sigma))\,d\sigma\,ds$$

$$= \int_0^t\int_0^t (f(s), U(s-\sigma)f(\sigma))\,d\sigma\,ds = \int_0^t (f(s), \tilde{\Lambda}_t f(s))\,ds$$

where $\tilde{\Lambda}_t f(s) = \int_0^t U(s-\sigma)f(\sigma)\,d\sigma$; we have used the fact that $U(t-s)^* = U(s-t)$. We then apply Hölder's inequality, first in space and then in time:

$$\|\Lambda f(t)\|^2_{L^2} \leq \int_0^t \|f(s)\|_{L^{\rho'}_x} \|\tilde{\Lambda}_t f(s)\|_{L^{\rho}_x}\,ds \quad \text{with } \frac{1}{\rho}+\frac{1}{\rho'}=1;$$

$$\leq \|f\|_{L^{\gamma'}(0,t;L^{\rho'}_x)} \|\tilde{\Lambda}_t f\|_{L^{\gamma}_s(0,t;L^{\rho}_x)}$$

$$\leq C\|f\|^2_{L^{\gamma'}(0,t;L^{\rho'}_x)} \leq C\|f\|^2_{L^{\gamma'}(0,T;L^{\rho'}_x)}$$

where we have used the remark following Step 1. Hence, the estimate follows by taking the supremum in time of the left hand side.

<u>Step 3</u>: We now prove that if (r,q) is an admissible pair, Λ is continuous from $L^1(0,T;L^2(\mathbb{R}^n))$ into $L^r(0,T;L^q(\mathbb{R}^n))$.

Let $f \in L^1(0,T;L^2(\mathbb{R}^n))$ and let $\varphi \in C_c([0,T[; C_0^\infty(\mathbb{R}^n))$.

Then $\left|\int_0^T (\Lambda f(t), \varphi(t))\,dt\right| = \left|\int_0^T\int_0^t (U(t-s)f(s), \varphi(t))\,ds\,dt\right|$

$$= \left|\int_0^T\int_s^T (f(s), U(t-s)\varphi(t))\,dt\,ds\right|$$

$$= \left| \int_0^T \left(f(s), \int_s^T U(s-t)\varphi(t)\,dt \right) ds \right|$$

$$\leq \|f\|_{L^1(0,T;L^2)} \left\| \int_\cdot^T U(\cdot - t)\varphi(t)\,dt \right\|_{L^\infty_s(0,T;L^2_x)}$$

$$\leq C \|f\|_{L^1(0,T;L^2)} \|\varphi\|_{L^{r'}(0,T,L^{q'})}$$

where we have used Step 2 and the same remark as in Step 1.

Now, since

$$\|\Lambda f\|_{L^r(0,T;L^q)} = \sup_{\substack{\varphi \in C_0^\infty(]0,T[\times\mathbb{R}^n) \\ \|\varphi\|_{L^{r'}(0,T,L^{q'})} = 1}} \left| \int_0^T (\Lambda f(t), \varphi(t))\,dt \right|$$

it follows that

$$\|\Lambda f\|_{L^r(0,T;L^q)} \leq C \|f\|_{L^1(0,T;L^2)}$$

<u>Step 4</u>: Let us prove part (ii) of the theorem. Let (γ, ρ) be an admissible pair. From Steps 1 and 2, Λ is continuous from $L^{\gamma'}(0,T;L^{\rho'})$ into $L^\gamma(0,T;L^\rho) \cap L^\infty(0,T;L^2)$. Consider first an admissible pair (r,q) with $2 \leq q \leq \rho$; then for some θ with $0 \leq \theta \leq 1$, $\frac{1}{q} = \frac{\theta}{\rho} + \frac{1-\theta}{2}$. Since (r,q) is admissible,

$$\frac{2}{r} = n\left(\frac{1}{2} - \frac{1}{q}\right) = n\left(\frac{1}{2} - \frac{\theta}{\rho} - \frac{1-\theta}{2}\right) = \theta n\left(\frac{1}{2} - \frac{1}{\rho}\right) = \frac{2\theta}{\gamma}$$

since (γ, ρ) is also admissible. It follows that

$$\frac{1}{r} = \frac{\theta}{\gamma} = \frac{\theta}{\gamma} + \frac{1-\theta}{\infty}.$$

Hölder's inequality implies then, for $f \in L^{\gamma'}(0,T;L^{\rho'})$:

$$\|\Lambda f(t)\|_{L^q} \leq \|\Lambda f(t)\|_{L^\rho}^{\theta} \|\Lambda f(t)\|_{L^2}^{1-\theta}$$

and

$$\|\Lambda f\|_{L^r(0,T,L^q)} \leq \|\Lambda f\|^{\theta}_{L^{\theta r}(0,T;L^\rho)} \|\Lambda f\|^{1-\theta}_{L^\infty(0,T;L^2)}$$

$$\leq \|\Lambda f\|^{\theta}_{L^\gamma(0,T;L^\rho)} \|\Lambda f\|^{1-\theta}_{L^\infty(0,T,L^2)}$$

$$\leq C \|f\|_{L^{\gamma'}(0,T;L^{\rho'})} \quad \text{from Steps 1 and 2.}$$

This proves (ii) in the case where $2 \le q \le p$.

Let now (r,q) be an admissible pair, with $p<q$; then since $2 \le p < q$, $\exists\, \mu \in]0,1]$ such that $\frac{1}{p} = \frac{\mu}{2} + \frac{1-\mu}{q}$ (hence also $\frac{1}{p'} = \frac{\mu}{2} + \frac{1-\mu}{q'}$) and from the preceding computations, we also have $\frac{1}{\gamma} = \frac{\mu}{\infty} + \frac{1-\mu}{r}$ and $\frac{1}{\gamma'} = \frac{\mu}{1} + \frac{1-\mu}{r'}$.

It then follows from Steps 1 and 3 that Λ is continuous from $L^{r'}(0,T;L^{q'})$ into $L^r(0,T;L^q)$ and from $L^1(0,T;L^2)$ into $L^r(0,T;L^q)$. We apply another interpolation theorem (see Bergh-Löfstrom, Interpolation Spaces, th. 5.1.2) to conclude that in this case, Λ is continuous from $L^\sigma(0,T;L^\beta)$ into $L^r(0,T;L^q)$ for any (σ,β) satisfying

$$\frac{1}{\sigma} = \frac{\theta}{1} + \frac{1-\theta}{r'} \quad \text{and} \quad \frac{1}{\beta} = \frac{\theta}{2} + \frac{1-\theta}{q'} \quad \text{for some } \theta \in [0,1].$$

It suffices then to choose $\theta = \mu$ to find that Λ is continuous from $L^{\gamma'}(0,T;L^{p'})$ into $L^r(0,T;L^q)$.

Note that the case $q>p$ will not be used for the proof of existence of a solution to the nonlinear equation; however we state it since it allows to get additional information on the solution.

<u>Step 5</u> : It remains to prove (i).

Let (r,q) be an admissible pair, and let $u_0 \in L^2(\mathbb{R}^n)$. As in Step 3, we use the fact that

$$\| U(\cdot)u_0 \|_{L^r(\mathbb{R};L^q)} = \sup_{\substack{\varphi \in C_0^\infty(\mathbb{R}^{n+1}) \\ \|\varphi\|_{L^{r'}(\mathbb{R},L^{q'})}=1}} \left| \int_{\mathbb{R}} (U(t)u_0, \varphi(t))\, dt \right|$$

Let $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$, then

$$\left| \int_{\mathbb{R}} (U(t)u_0, \varphi(t))\, dt \right| = \left| \int_{\mathbb{R}} (u_0, U(-t)\varphi(t))\, dt \right|$$

$$= \left|\left(u_0, \int_{\mathbb{R}} U(-t)\varphi(t)\,dt\right)\right| \leq \|u_0\|_{L^2} \left\|\int_{\mathbb{R}} U(-t)\varphi(t)\,dt\right\|_{L^2}$$

where we recall that $(u,v) = \int_{\mathbb{R}^n} u(x)\,\bar{v}(x)\,dx$. Then, by the same computation as in step 2, we have

$$\left\|\int_{\mathbb{R}} U(-t)\varphi(t)\,dt\right\|_{L^2}^2 = \left(\int_{\mathbb{R}} U(-t)\varphi(t)\,dt, \int_{\mathbb{R}} U(-s)\varphi(s)\,ds\right)$$

$$= \int_{\mathbb{R}} \left(\varphi(t), \int_{\mathbb{R}} U(t-s)\varphi(s)\,ds\right) dt$$

$$\leq \|\varphi\|_{L^{r'}(0,T;L^{q'})} \left\|\int_{\mathbb{R}} U(\cdot - s)\varphi(s)\,ds\right\|_{L^r(0,T;L^q)}$$

$$\leq C\|\varphi\|^2_{L^{r'}(0,T,L^{q'})} \quad \text{by step 1 and the remark at the end.}$$

It follows that

$$\|U(\cdot)u_0\|_{L^r(\mathbb{R},L^q)} \leq C\|u_0\|_{L^2}. \qquad \square$$

<u>Remark</u>: The original proof of Strichartz was different and consisted in using a Fourier restriction lemma. The present proof by duality is due to Ginibre and Velo.

<u>Corollary 2</u>: estimate on the solution of the inhomogeneous equation: Let $u_0 \in L^2(\mathbb{R}^n)$ and $f \in L^{\gamma'}(0,T;L^{\rho'})$ where (γ,ρ) is an admissible pair; let $u \in C([0,T];L^2(\mathbb{R}^n))$ be a solution of

$$\begin{cases} i\partial_t u - \Delta u = f & \text{for a.all } t, \text{ in } H^{-2}(\mathbb{R}^n) \\ u(0) = u_0 \end{cases}$$

then $u \in L^r(0,T;L^q)$ for any admissible pair (r,q); in particular if $n=1$, then $u \in L^4(0,T;L^\infty(\mathbb{R}))$.

<u>Proof</u>:

Since $f \in L^{\gamma'}(0,T;L^{\rho'}) \subset L^1(0,T;H^{-2})$ and $u \in C([0,T];L^2)$ we know that u is solution of the above equation if and only if it satisfies the integral formulation (or mild

equation) (see Section 1-b):

$$u(t) = U(t)u_0 + \int_0^t U(t-s)\,(-if(s))\,ds\,;$$

the result then follows from (i) and (ii) of Strichartz inequalities. □

<u>Corollary 3: Dispersion of the free solutions</u>

Let $u_0 \in H^1(\mathbb{R}^n)$ and q with $2 < q < \frac{2n}{n-2}$ ($2 < q \leq +\infty$ if $n=1$ or $2 \leq q < +\infty$ if $n=2$); then $\lim_{t \to \pm\infty} \|U(t)u_0\|_{L^q} = 0$.

<u>Proof:</u>

Let r be such that (r,q) is an admissible pair, then from (i) of Strichartz inequalities, $U(\cdot)u_0 \in L^r(\mathbb{R}, L^q(\mathbb{R}^n))$ hence the result will follow if we prove that $t \mapsto U(t)u_0$ is uniformly continuous with values in $L^q(\mathbb{R}^n)$. Let $u(t) = U(t)u_0$; by the Gagliardo-Nirenberg inequalities

$$\|u(t)-u(s)\|_{L^q} \leq C\,\|u(t)-u(s)\|_{H^1}^{a}\,\|u(t)-u(s)\|_{L^2}^{1-a}$$

with $a = n\left(\frac{1}{2}-\frac{1}{q}\right) = \frac{2}{r}$; since $U(t)$ is an isometry in $H^s(\mathbb{R}^n)$ for all s, we deduce

$$\begin{aligned}\|u(t)-u(s)\|_{L^q} &\leq C\left(\|u(t)\|_{H^1} + \|u(s)\|_{H^1}\right)^{2/r}\,\|u(t)-u(s)\|_{L^2}^{1-2/r}\\ &\leq 2C\,\|u_0\|_{H^1}^{2/r}\,\|u(t)-u(s)\|_{L^2}^{1-2/r}\end{aligned}$$

On the other hand, $\frac{du}{dt} = -i\Delta u(t)$, hence

$$\left\|\frac{du}{dt}\right\|_{H^{-1}} = \|\Delta u(t)\|_{H^{-1}} \leq \|u(t)\|_{H^1} = \|u_0\|_{H^1}, \quad \forall t \in \mathbb{R}.$$

We deduce that

$$\|u(t)-u(s)\|_{H^{-1}} \leq \|u_0\|_{H^1}\,|t-s|, \quad \forall t,s \in \mathbb{R} \text{ and}$$

$$\begin{aligned}\|u(t)-u(s)\|_{L^2}^2 &= \langle u(t)-u(s), u(t)-u(s)\rangle_{H^{-1},H^1}\\ &\leq \|u(t)-u(s)\|_{H^{-1}}\left(\|u(t)\|_{H^1} + \|u(s)\|_{H^1}\right)\\ &\leq 2\,\|u_0\|_{H^1}^2\,|t-s|\end{aligned}$$

Finally, $\|u(t)-u(s)\|_{L^q} \leq \sqrt{2}C\,\|u_0\|_{H^1}\,|t-s|^{\frac{1}{2}-\frac{1}{r}}$

and $t \mapsto U(t)u_0$ is uniformly continuous with values

in L^q, as soon as $r > 2$, which is the case if $q < \frac{2n}{n-2}$. $\square$

Remarks:

- Strichartz inequalities are still valid in the case $q = \frac{2n}{n-2}$, $n \geq 3$ (i.e. $r = 2$), but the proof is much more complicated.
- the only necessary ingredients in the proof of Strichartz inequalities are
  - the unitarity of $U(t)$ in L^2
  - the L^1-L^∞ estimate (i.e. dispersive estimate) on $U(t)$

Those inequalities generalize to the case where $U(t) = e^{itA}$ for a selfadjoint operator A on L^2, such that $U(t)$ is continuous from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$, at least for $|t|$ small, with a norm bounded by $C|t|^{-n/2}$; in this latter case, the norm in Strichartz estimates may depend on T and Corollary 3 may not be true.

Examples:

1) $A = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$, $a_{ij} \in \mathbb{R}$, $a_{ji} = a_{ij}$; assume that the matrix $\mathcal{A} = (a_{ij})_{1 \leq i,j \leq n}$ is invertible, but not necessarily positive-definite, so that $-A$ is not necessarily elliptic. Using the Fourier transform, it is easily seen that iA generates a unitary group in all $H^s(\mathbb{R}^n)$

exercise: compute the convolution kernel of the group generated by iA and show that the L^1-L^∞ estimate holds.

Note that the domain on which A is selfadjoint is not necessarily equal to H^2 if $-A$ is not elliptic.

Applications: $A = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$, or $A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

Note that the latter operator corresponds to the case $k(\omega_0)k''(\omega_0)>0$ in chapter I.

2) $A = -\frac{1}{2}\Delta + V(x)$ where $V(x) = \frac{\omega^2}{2}|x|^2 = \frac{\omega^2}{2}\sum_{i=1}^{n} x_i^2$, $\omega \in \mathbb{R}$; then one can check (see Feynman, Hibbs, Quantum Mechanics) that A generates a C^0 group of unitary operators $T(t)$ on $L^2(\mathbb{R}^n)$ which satisfies: $\forall \varphi \in L^1(\mathbb{R}^n)$, $\forall t$ with $|t| < \frac{\pi}{2\omega}$,

$$T(t)\varphi(x) = \int_{\mathbb{R}^n} k(t,x,y)\,\varphi(y)\,dy$$

with

$$k(t,x,y) = \left(\frac{i}{2\pi \sin\omega t}\right)^{n/2} \exp\left[-i\,\frac{(\cos\omega t)(|x|^2+|y|^2) - 2x\cdot y}{2\sin\omega t}\right]$$

this operator A is called the quantum harmonic oscillator; the above expression for the kernel k has been generalized to more general potentials V.

It follows that the group $T(t)$ satisfies the L^1-L^∞ estimate, and hence also the Strichartz estimates (with $\mathbb{R}$ replaced by $[0,T]$). Note that $-\Delta + \omega^2|x|^2$ is an operator with compact inverse, hence its spectrum is purely discrete; it follows that Strichartz inequalities cannot be global in time: indeed if $A\varphi = \lambda\varphi$ then $T(t)\varphi = e^{i\lambda t}\varphi$ and $\|T(t)\varphi\|_{L^q} = \|\varphi\|_{L^q}$ for all q, ie $T(\cdot)\varphi \notin L^r(\mathbb{R}; L^q)$.

3 - The ~~local~~ Cauchy problem

We now consider the nonlinear equation

$$\text{(NLS)} \quad \begin{cases} i\partial_t u - \Delta u = \lambda |u|^{2\sigma} u, \\ u(0,x) = u_0(x) \end{cases}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, $u(t,x) \in \mathbb{C}$, $\lambda \in \mathbb{R}$, $\sigma > 0$. We will consider the integral formulation