4 - Blow up in finite time

In this section, we assume that $d>0$ and $\sigma \geq \frac{2}{n}$, and we consider the solutions obtained in the previous section. We will prove that we can choose the initial state in order that $T^*(u_0)<+\infty$. We then say that the corresponding solutions "blow up in finite time".

Let us consider the (NLS) equation with $d=1$:

(NLS) $\quad i\partial_t u - \Delta u = |u|^{2\sigma} u$.

We consider also the function space

$$\Sigma = \{u \in L^2(\mathbb{R}^n),\ \nabla u \in L^2(\mathbb{R}^n),\ xu \in L^2(\mathbb{R}^n)\} = H^1(\mathbb{R}^n) \cap \mathcal{F} H^1(\mathbb{R}^n)$$

We first prove the following lemma:

Lemma 1 :

Let $u_0 \in \Sigma$ and $u \in C([0,T^*(u_0)), H^1)$ the maximal solution of (NLS) given by the theorem in the previous section, with $u(0)=u_0$. Then $u \in C([0,T^*(u_0)), \Sigma)$. Moreover, the function I defined by

$$I(t) = \frac{1}{2}\int_{\mathbb{R}^n} |x|^2 |u(t,x)|^2 dx$$

is C^1 on $[0,T^*[$, and

$$\frac{d}{dt} I(t) = -2\,\mathrm{Im}\int_{\mathbb{R}^n} x.\nabla u(t,x)\, \bar{u}(t,x)\, dx$$

proof :

We use a cut-off in space in order to justify the computations. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $1\geq\varphi\geq 0$ and $\varphi \equiv 1$ on $B(0,1)$, $\mathrm{supp}\,\varphi \subset B(0,2)$. We set $\varphi_n(x) = \varphi(\frac{x}{n})$. Then $x\varphi_n u \in C([0,T^*), H^1)$ and for $t\in[0,T]$ with $T<T^*$,

$$\frac{d}{dt} \frac{1}{2}\int_{\mathbb{R}^n} |x\varphi_n u(t,x)|^2 dx = \mathrm{Re}\langle \partial_t u, |x|^2\varphi_n^2 u\rangle_{H^{-1};H^1}$$

$$= \mathrm{Re}\langle -i\Delta u, |x|^2\varphi_n^2 u\rangle_{H^{-1};H^1} + \underbrace{\mathrm{Re}\langle -i|u|^{2\sigma}u, |x|^2\varphi_n^2 u\rangle_{H^{-1};H}}_{0}$$

where we have used the (NLS) equation.

$$= \mathrm{Im}\langle \Delta u, |x|^2\varphi_n^2 u\rangle = -\mathrm{Im}\langle \nabla u, \nabla(|x|^2\varphi_n^2 u)\rangle$$

$$= -\mathrm{Im}\langle \nabla u, |x|^2\nabla(\varphi_n^2)u\rangle - 2\mathrm{Im}\langle \nabla u, x\varphi_n^2 u\rangle$$

On the other hand, there are constants C_1, C_2, independent of n such that $\|\varphi_n\|_{L^\infty} \le C_1$ and

$$\|x\nabla\varphi_n\|_{L^\infty} = \left\|\frac{x}{n}\nabla\varphi\left(\frac{\cdot}{n}\right)\right\|_{L^\infty} \le C_2, \ \forall n$$

since $\mathrm{supp}\,\nabla\varphi\left(\frac{\cdot}{n}\right) \subset B(0,2n)$.

Integrating the previous equality between 0 and t, we deduce that

$$\frac{1}{2}\|x\varphi_n u(t)\|_{L^2}^2 \le \frac{1}{2}\|x\varphi_n u_0\|_{L^2}^2 + 2C_2\int_0^t \|\nabla u(s)\|_{L^2}\|x\varphi_n u(s)\|_{L^2} ds + 2C_1\int_0^t \|\nabla u(s)\|_{L^2}\|x\varphi_n u(s)\|_{L^2} ds$$

$$\le \frac{1}{2}\|xu_0\|_{L^2}^2 + (C_2+C_1)T\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2}^2 + (C_2+C_1)\int_0^t \|x\varphi_n u(s)\|_{L^2}^2 ds.$$

Gronwall's lemma then implies

$$\|x\varphi_n u(t)\|_{L^2}^2 \le C(T, \|xu_0\|_{L^2}, \|u\|_{L^\infty(0,T,H^1)})$$

where the constant C does not depend on n. Taking the limit as n goes to infinity, we deduce thanks to Fatou's lemma that $\|xu(t)\|_{L^2}^2 \le C(T, \|xu_0\|_{L^2}, \|u\|_{L^\infty(0,T,H^1)})$

Coming back to the expression of $\frac{1}{2}\|x\varphi_n u(t)\|_{L^2}^2$, we may pass to the limit and get

$$I(t) = I(0) - 2\mathrm{Im}\int_0^t\int_{\mathbb{R}^n} x\cdot\nabla u(s,x)\,\bar{u}(s,x)\,dx\,ds.$$

This proves that I is $t \mapsto I(t)$ is continuous, and using the equality again that I is C^1. In order to show that $u \in C([0,T], \Sigma)$, it is then sufficient to prove the continuity of $t \mapsto x\,u(t)$ with values in L^2-weak. This follows easily from the fact that $xu \in L^\infty(0,T; L^2(\mathbb{R}^n))$, approximating the test function in $L^2(\mathbb{R}^n)$ by a sequence of functions in $C_0^\infty(\mathbb{R}^n)$. $\square$

<u>Lemma 2:</u>

Under the assumptions of lemma 1, the function $t \mapsto I(t)$ is C^2 on $[0,T^*)$ and

$$\frac{d^2}{dt^2} I(t) = 4\sigma n E(u_0) - 2(\sigma n - 2)\, \|\nabla u(t)\|_{L^2}^2$$

<u>proof</u> (indications)

For a rigorous proof, one has to use a cut-off argument as in the proof of lemma 1, together with a regularization procedure. We only give the formal arguments.

$$\frac{d}{dt} \operatorname{Im} \int_{\mathbb{R}^n} x \cdot \nabla u(t,x)\, \bar{u}(t,x)\, dx$$

$$= \operatorname{Im} \int_{\mathbb{R}^n} x \nabla u\, \partial_t \bar{u}\, dx + \operatorname{Im} \int_{\mathbb{R}^n} x \nabla \partial_t u \cdot \bar{u}\, dx$$

$$= 2 \operatorname{Im} \int_{\mathbb{R}^n} x \nabla u\, \partial_t \bar{u}\, dx - n \operatorname{Im} \int_{\mathbb{R}^n} \bar{u}\, \partial_t u\, dx$$

where we have integrated the second term by parts. Using the (NLS) equation, we get

$$\frac{d}{dt} \operatorname{Im} \int_{\mathbb{R}^n} x \cdot \nabla u\, \bar{u}\, dx = 2 \operatorname{Im} \int_{\mathbb{R}^n} x \cdot \nabla u \left(i \Delta \bar{u} + i |u|^{2\sigma} \bar{u} \right) dx$$

$$- n \int_{\mathbb{R}^n} \operatorname{Im} \bar{u} \left(-i \Delta u - i |u|^{2\sigma} u \right) dx$$

The last term may be written as

$$-n\int_{\mathbb{R}^n}|\nabla u|^2dx + n\int_{\mathbb{R}^n}|u|^{2\sigma+2}dx.$$

For the other term, we have

$$2\,\mathrm{Re}\int_{\mathbb{R}^n} x.\nabla u\,\Delta\bar{u}\,dx = -2\,\mathrm{Re}\int_{\mathbb{R}^n}\nabla(x.\nabla u).\nabla\bar{u}\,dx$$

$$= -2\,\mathrm{Re}\sum_k\int_{\mathbb{R}^n}\partial_{x_k}\Big(\sum_j x_j\partial_{x_j}u\Big)\partial_{x_k}\bar{u}\,dx$$

$$= -2\int_{\mathbb{R}^n}|\nabla u|^2dx - 2\sum_{j,k}\int_{\mathbb{R}^n}x_j\,\partial_{x_j}\Big(\frac{1}{2}|\partial_{x_k}u|^2\Big)dx$$

$$= (n-2)\int_{\mathbb{R}^n}|\nabla u|^2dx$$

and

$$2\,\mathrm{Re}\int_{\mathbb{R}^n} x.\nabla u\,|u|^{2\sigma}\bar{u}\,dx = \frac{1}{\sigma+1}\int_{\mathbb{R}^n}x.\nabla\big(|u|^{2\sigma+2}\big)dx$$

$$= -\frac{n}{\sigma+1}\int_{\mathbb{R}^n}|u|^{2\sigma+2}dx$$

so that finally,

$$\frac{d}{dt}\,\mathrm{Im}\int x.\nabla u\,\bar{u}\,dx = -2\int_{\mathbb{R}^n}|\nabla u|^2dx + n\Big(1-\frac{1}{\sigma+1}\Big)\int|u|^{2\sigma+2}dx$$

$$= -2\int_{\mathbb{R}^n}|\nabla u|^2dx + \frac{n\sigma}{\sigma+1}\int_{\mathbb{R}^n}|u|^{2\sigma+2}dx.$$

Using lemma 1, we get

$$\frac{d^2}{dt^2}I(t) = 4\int_{\mathbb{R}^n}|\nabla u|^2dx - \frac{2n\sigma}{\sigma+1}\int_{\mathbb{R}^n}|u|^{2\sigma+2}dx$$

$$= 4n\sigma\,E(u(t)) + (4-2n\sigma)\,\|\nabla u(t)\|_{L^2}^2. \quad \square$$

We can now give the result about blowing-up solutions.

Theorem :

Assume $d=+1$, $\sigma\geq\frac{2}{n}$ and $u_0\in H^1(\mathbb{R}^n)$ with $E(u_0)<0$. Then $T^*(u_0)<+\infty$.

Proof:

If $\sigma \geq \frac{2}{n}$, then Lemma 2 implies that for $t < T^*$,

$$\frac{d^2}{dt^2} I(t) = \frac{1}{2} \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 \leq 4\sigma n E(u_0) < 0.$$

Hence, $$\frac{d}{dt} \|xu(t)\|_{L^2}^2 \leq 8\sigma n t E(u_0) + \frac{d}{dt} \|xu(t)\|_{L^2}^2 \Big|_{t=0}$$

$$\leq 8\sigma n t E(u_0) - 4 \operatorname{Im} \int_{\mathbb{R}^n} x.\nabla u_0 \bar{u}_0 \, dx$$

and

$$\|xu(t)\|_{L^2}^2 \leq 4\sigma n E(u_0) t^2 - 4t \operatorname{Im} \int_{\mathbb{R}^n} x.\nabla u_0 \bar{u}_0 \, dx + \|xu_0\|_{L^2}$$

Since the r.h.s. goes to $-\infty$ as $t \to \infty$ and the l.h.s. is positive, we necessarily have $T^* < +\infty$. $\square$

Remark:

- In order to find an initial data which satisfies the condition of the theorem, one can fix $\varphi \in H^1(\mathbb{R}^n)$, then for any $\alpha \in \mathbb{R}$, $E(\alpha\varphi) = \frac{\alpha^2}{2} \int |\nabla\varphi|^2 dx - \frac{\alpha^{2\sigma+2}}{2\sigma+2} \int |\varphi|^{2\sigma+2}$ so that if α is sufficiently large, then $E(\alpha\varphi) < 0$.
- the condition $E(u_0) < 0$ is not a necessary condition for blow up. In fact, one can give weaker sufficient conditions, using the fact that $\|xu(t)\|_{L^2}^2$ is bounded above by a second order polynomial in t.
- In the critical case $\sigma = \frac{2}{n}$, $E(u_0) < 0$ implies that $\|u_0\|_{L^2}$ is suff. large, so the result in not in contradiction with the global existence result.