4. Blow up in finite time

In this section, we assume that \( d \geq 0 \) and \( \tau \geq \frac{\delta}{\tau} \), and we consider the solutions obtained in the previous section. We will prove that we can choose the initial state in order that \( T^*(u) < \infty \). We then say that the corresponding solution "blows up in finite time."

Let us consider the (NLS) equation with \( d = 1 \):

\[
(NLS) \quad i \partial_t u - \Delta u = |u|^{p-1} u.
\]

We consider also the function space

\[
\Sigma = \{ u \in L^2(\mathbb{R}^n), \nabla u \in L^2(\mathbb{R}^n), \; xu \in L^2(\mathbb{R}^n) : H'(\mathbb{R}^n) \cap F' H'(\mathbb{R}^n) \}
\]

We first prove the following lemma:

**Lemma 1:**

Let \( u_0 \in \Sigma \) and \( u \in C(\mathbb{R}, H(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)) \), the maximal solution of (NLS) given by the theorem in the previous section, with \( u(0) = u_0 \). Then \( u \in C(\mathbb{R}, H(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)) \). Moreover, the function \( I \) defined by

\[
I(t) = \frac{1}{2} \int_{\mathbb{R}^n} |x|^2 |u(t,x)|^2 \, dx
\]

is \( C^1 \) on \( \mathbb{R} \rightarrow \mathbb{R} \), and

\[
\frac{d}{dt} I(t) = -2 \text{Im} \int_{\mathbb{R}^n} x \cdot \nabla u(t,x) \bar{u}(t,x) \, dx
\]

**Proof:**

We use a cut-off in space in order to justify the computations:

Let \( \psi \in C^0(\mathbb{R}) \), \( \psi \geq 0 \) and \( \psi \equiv 1 \) on \( B(0,1) \), \( \text{supp} \psi \subset B(0,2) \). We set \( \psi_n(x) = \psi\left(\frac{x}{n}\right) \). Then \( \psi_n u \in C\left([0,T^*), H^1\right) \) and for \( t \in [0,T] \) with \( T < T^* \).
\[
\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^n} \left| \nabla \phi_n \cdot u(t, x) \right|^2 \, dx = \Re \left( \partial_t \phi_n, \nabla^2 \phi_n \cdot u \right)_{H^{-1,1} \rightarrow H^{1,1}} \\
= \Re \left( \partial_t \phi_n, i \nabla \phi_n \cdot u \right)_{H^{-1,1} \rightarrow H^{1,1}} + \Re \left( i \nabla \phi_n, \nabla \phi_n \cdot u \right)_{H^{-1,1} \rightarrow H^{1,1}}
\]
when we have used the \((NIS)\) equation.

\[
= \text{Im} \left( \partial_t \phi_n, \nabla \phi_n \cdot u \right) = -\text{Im} \left( \nabla \phi_n, \nabla \phi_n \cdot u \right) \\
= -\text{Im} \left( \nabla \phi_n, \nabla \phi_n \cdot u \right) - 2 \text{Im} \left( \nabla \phi_n, \partial_t \phi_n \cdot u \right).
\]

On the other hand, there are constants \( C_1, C_2 \) independent of \( n \) such that \( \| \phi_n \|_{L^\infty} \leq C_1 \) and

\[
\| \nabla \phi_n \|_{L^\infty} = \| \frac{x}{n} \nabla \left( \frac{1}{n} \right) \|_{L^\infty} \leq C_2, \quad \forall n
\]

since \( \text{supp} \nabla \phi \left( \frac{x}{n} \right) \subset B(0, 2n) \).

Integrating the previous equality between \( 0 \) and \( t \), we deduce that

\[
\frac{1}{2} \int_{\mathbb{R}^n} \left| \nabla \phi_n \cdot u(t) \right|^2 \, dx \leq \frac{1}{2} \int_{\mathbb{R}^n} \left| \nabla \phi_n \cdot u(0) \right|^2 \, dx + 2 C_2 \int_0^t \| \nabla u(\tau) \|_{L^2} \left\| \nabla \phi_n \cdot u(\tau) \right\|_{L^2} \, d\tau \\
+ 2 C_1 \int_0^t \| \nabla u(\tau) \|_{L^2} \left\| \nabla \phi_n \cdot u(\tau) \right\|_{L^2} \, d\tau \\
\leq \frac{1}{2} \int_{\mathbb{R}^n} \left| \nabla \phi_n \cdot u(0) \right|^2 \, dx + (C_2 + C_1) \sup_{\tau \in [0,T]} \| \nabla u(\tau) \|_{L^2}^2 \\
+ (C_2 + C_1) \int_0^t \left\| \nabla \phi_n \cdot u(\tau) \right\|_{L^2}^2 \, d\tau.
\]

Gronwall's lemma then implies

\[
\int_{\mathbb{R}^n} \left| \nabla \phi_n \cdot u(t) \right|^2 \, dx \leq C(T, \int_{\mathbb{R}^n} \left| \nabla \phi_n \cdot u(0) \right|^2 \, dx, \left\| \nabla u \right\|_{L^2(0,T;H^1)})
\]

where the constant \( C \) does not depend on \( n \). Taking the limit as \( m \) goes to infinity, we deduce thanks to Fatou's lemma that

\[
\int_{\mathbb{R}^n} \left| \nabla \phi_n \cdot u(t) \right|^2 \, dx \leq C(T, \int_{\mathbb{R}^n} \left| \nabla \phi_n \cdot u(0) \right|^2 \, dx, \left\| \nabla u \right\|_{L^2(0,T;H^1)}).
\]

Coming back to the expression of \( \frac{1}{2} \| \nabla \phi_n \cdot u(t) \|_{L^2}^2 \), we may pass to the limit and get

\[
I(t) = I(0) - 2 \text{Im} \int_0^t \int_{\mathbb{R}^n} x \cdot \nabla u(s, x) \cdot \nabla (s, x) \, dx \, ds.
\]
This proves that $I$ is $t \mapsto I(t)$ is continuous, and using the equality again that $I$ is $C^2$. In order to show that
\[ u \in C([0,T], \Sigma) \], it is then sufficient to prove the continuity of $t \mapsto u(t)$ with values in $L^2$ weak. This follows easily from the fact that $x \in L^2(\Omega; L^2(\mathbb{R}^n))$, approximating the test function in $L^2(\mathbb{R}^n)$ by a sequence of functions in $C^0(\mathbb{R}^n)$. \[ \square \]

**Lemma E:**

Under the assumptions of Lemma 1, the function $t \mapsto I(t)$ is $C^2$ on $(0, T^*)$ and
\[
\frac{d^2}{dt^2} I(t) = 4\text{Re} E(u) - 2(\sigma n - 2) \| \partial u(t) \|_{L^2}^2
\]

**proof (indication):**

For a rigorous proof, one has to use a cut-off argument as in the proof of Lemma 1, together with a regularization procedure. We only give the formal arguments.

\[
\frac{d}{dt} I(t) = \int_{\mathbb{R}^n} x \cdot \partial u(t) \overline{\partial u(t)} \, dx
\]

\[= \int_{\mathbb{R}^n} x \cdot \partial u(t) \overline{\partial u(t)} \, dx + \int_{\mathbb{R}^n} x \cdot \partial^2 u(t) \overline{\partial u(t)} \, dx
\]

\[= \int_{\mathbb{R}^n} x \cdot \partial u(t) \overline{\partial u(t)} \, dx - n \int_{\mathbb{R}^n} \text{Re} \int_{\mathbb{R}^n} \overline{\partial u(t)} \, dx
\]

where we have integrated the second term by parts. Using the (NLS) equation, we get

\[
\frac{d}{dt} I(t) \overline{\partial u(t)} \, dx = \partial I(t) \overline{\partial u(t)} \, dx
\]

\[= \int_{\mathbb{R}^n} \text{Re} \int_{\mathbb{R}^n} \overline{\partial u(t)} \, dx
\]

The last term may be written as...
\[- n \int_\mathbb{R}^n |\nabla u|^2 \, dx + n \int_\mathbb{R}^n |\nabla u^{2+\varepsilon}| \, dx. \]

For the other term, we have
\[2 \Re \int_\mathbb{R}^n x \cdot \nabla u \Delta \overline{u} \, dx = -2 \Re \int_\mathbb{R}^n \nabla (x \cdot \nabla u) \cdot \nabla \overline{u} \, dx \]
\[= -2 \Re \sum_{k} \int_\mathbb{R}^n \partial_{x_k} \left( \sum_j x_j \partial_{x_j} u \right) \partial_{x_k} \overline{u} \, dx \]
\[= -2 \int_\mathbb{R}^n |\nabla u|^2 \, dx - 2 \sum_{j,k} \int_\mathbb{R}^n x_j \partial_{x_j} \left( \frac{4}{\varepsilon} |\nabla u|^{2+\varepsilon} \right) \, dx \]
\[= (n-2) \int_\mathbb{R}^n |\nabla u|^2 \, dx \]

and
\[2 \Re \int_\mathbb{R}^n x \cdot \nabla u \nabla u^{2+\varepsilon} \overline{u} \, dx \]
\[= \frac{1}{s+1} \int_\mathbb{R}^n x \cdot \nabla \left( |u|^{2+\varepsilon} \right) \, dx \]
\[= -\frac{n}{s+1} \int_\mathbb{R}^n |u|^{2+\varepsilon} \, dx. \]

So that finally,
\[
\frac{d}{dt} \Im \int_\mathbb{R}^n x \cdot \nabla u \overline{u} \, dx = -2 \int_\mathbb{R}^n |\nabla u|^2 \, dx + n \left( 1 - \frac{s}{s+1} \right) \int_\mathbb{R}^n |u|^{2+\varepsilon} \, dx
\]
\[= -2 \int_\mathbb{R}^n |\nabla u|^2 \, dx + \frac{n \varepsilon}{s+1} \int_\mathbb{R}^n |u|^{2+\varepsilon} \, dx. \]

Using Lemma 1, we get
\[
\frac{d^2}{dt^2} \Im (t) = 4 \int_\mathbb{R}^n |\nabla u|^2 \, dx - \frac{2n \varepsilon}{s+1} \int_\mathbb{R}^n |u|^{2+\varepsilon} \, dx
\]
\[= 4 n \varepsilon E(u(t)) + (4 - 2n\varepsilon) \|u(t)\|_{L^2}^2. \]

We can now give the result about blowing-up solutions.

**Theorem:**

Assume \( d = 1, \sigma \geq \frac{2}{n} \) and \( u_0 \in H^\sigma (\mathbb{R}^n) \) with \( E(u_0) < 0 \). then \( T^*(u_0) < \infty \).
Proof:

If $0 \geq \frac{\alpha}{n}$, then Lemma 2 implies that for $t < T^*$,
\[
\frac{d^2}{dt^2} I(t) = \frac{1}{2} \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 \leq 4Tn \ E(u_0) < 0.
\]
Hence,
\[
\frac{d}{dt} \|xu(t)\|_{L^2}^2 \leq 8Tn + E(u_0) + \frac{d}{dt} \|xu(t)\|_{L^2}^2 \bigg|_{t=0}
\]
\[
\leq 8Tn + E(u_0) - 4 \text{ Im} \int \int x \partial_x u_0 \overline{u_0} \ dx
\]
and
\[
\|xu(t)\|_{L^2}^2 \leq 4Tn E(u_0) t^2 - 4T \text{ Im} \int \int x \partial_x u_0 \overline{u_0} \ dx + \|xu_0\|_{H^1}^2.
\]
Since the r.h.s. goes to $-\infty$ as $t \to 0$ and the l.h.s. is positive, we necessarily have $T^* < 100$.  

Remark:

- In order to find an initial data which satisfies the condition of the theorem, one can fix $Q \in H^1(\mathbb{R}^n)$, then for any $x \in \mathbb{R}$, \[ E(x^2) = \frac{d^2}{2} \int |V|^2 dx - \frac{d^2}{2} \int |V|^2 \overline{x}^2 + \|xu_0\|_{H^1}^2 \]
so that if $d$ is sufficiently large, then \[ E(x^2) > 0. \]
- the condition $E(u_0) < 0$ is not a necessary condition for blow up. In fact, one can give weaker sufficient conditions, using the fact that $\|xu(t)\|_{L^2}$ is bounded above by a second order polynomial in $t$.
- In the critical case $\alpha = \frac{2}{n}$, $E(u_0) < 0$ implies that $\|u_0\|_{L^2}$ is sufficiently large, so the result in not in contradiction with the global existence result.