Consider a light wave propagating in a dielectric medium, described by Maxwell's equations (1860):

\[
\begin{align*}
\text{div } \mathbf{D} &= 0 \\
\text{div } \mathbf{B} &= 0 \\
\text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\text{curl } \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} \\
\text{div } \mathbf{E} &= \nabla \cdot \mathbf{E} \\
\text{curl } \mathbf{H} &= \nabla \times \mathbf{H} \\
D &= \varepsilon_0 E + P \\
B &= \mu_0 H
\end{align*}
\]

No charge and no current is present (dielectric medium). \( \mathbf{E} \) and \( \mathbf{H} \) are resp. electric and magnetic fields; \( \mathbf{D} \) and \( \mathbf{B} \) are resp. electric and magnetic inductions. We add the constitutive relations:

\[
\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0 \mathbf{H}
\]

\( \varepsilon_0 \) = vacuum permeability (\( \varepsilon_0 = \text{electric constant of vacuum} \)

\( \varepsilon_0 \) = electric permittivity, \( c \) = light velocity)

\( \mathbf{P}(\mathbf{r}, t) \): polarization of the medium; depends on \( \mathbf{E} \) in a way that depends on the medium.

Simplest case: isotropic, linear and instantaneous medium; then \( \mathbf{P} = \varepsilon_0 \mathbf{X} \mathbf{E} \), where \( \mathbf{X} = \text{electric susceptibility} = \text{const} \)

Here, we consider nonlinear media, that do not react instantly; then

\[
\mathbf{P}(\mathbf{r}, t) = \varepsilon_0 \int_{-\infty}^{+\infty} \mathbf{X}(t-z) \mathbf{E}(\mathbf{r}, \tau) \, d\tau + \mathbf{P}_{\text{NL}}(\mathbf{r}, t)
\]

with

\[
\mathbf{P}_{\text{NL}}(\mathbf{r}, t) = \varepsilon \int_{\mathbb{R}^3} \mathbf{X}(t-z) \mathbf{E}(\mathbf{r}, \tau) \mathbf{E}(\mathbf{r}, \tau) \, d\mathbf{r} \, d\tau \, d\mathbf{z}
\]

(obtained using a dot of \( \mathbf{P} \) in terms of "power" of \( \mathbf{E} \) cut-off).
Here, we have also assumed that the medium is center-symmetric, i.e., if \( E \) is changed to \( -E \), then \( P \) changes to \( -P \) (no quadratic term) \( X(1) \) and \( X(3) \) are (real-valued) scalar functions; moreover,

\[
\text{Supp} \ X(1) = \{ t \in \mathbb{R} \mid X(1)(t, \cdot) \not= 0 \} \subset \mathbb{R}^+, \qquad \text{which means that} \quad P(t) \ \text{depends only on the values} \ \bar{E}(z) \ \text{for} \ z \leq t.
\]

In the same way, \( \text{Supp} \ X(3) \subset (\mathbb{R}^+)^3 \).

Using the 3rd eqn in Maxwell's system, we obtain

\[
\begin{align*}
\nabla \times (\nabla \times \bar{E}) &= -\frac{\partial}{\partial t} \nabla \times \bar{B} = -\frac{1}{\varepsilon_0} \varepsilon \frac{\partial}{\partial t} \nabla \times \bar{H} \\
&= \frac{1}{\varepsilon_0 c^2} \frac{\partial \bar{E}}{\partial t^2} = -\frac{1}{c^2} \frac{\partial \bar{E}}{\partial t^2} - \frac{1}{\varepsilon_0 c^2} \frac{\partial^2 \bar{P}}{\partial t^2}
\end{align*}
\]

On the other hand,

\[
\nabla \times (\nabla \times \bar{E}) = \nabla (\text{div} \bar{E}) - \Box \bar{E}
\]

where \( \Box \bar{E} = (\Box E_x, \Box E_y, \Box E_z) \), \( \Box E_x = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \)


etc., and we obtain

\[
(\Box) \quad \Box \bar{E} - \nabla (\text{div} \bar{E}) = \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 \bar{P}}{\partial t^2} + \int_{\mathbb{R}} X(1)(z, \cdot) \bar{E}(z) \, dz
\]

Now, we define the Fourier transform in time of \( \bar{E} \) as

\[
\hat{E}(\omega, \vec{x}) = \int_{-\infty}^{\infty} e^{i \omega t} \bar{E}(\vec{x}, t) \, dt \quad \hat{E}(\omega, \vec{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \delta_{\omega} E(\omega, \vec{x}) \, d\omega
\]

(i.e., we use, only in this section, the physicist's notation). Then, \( \hat{E} \) satisfies:

\[
(3) \quad \Box \hat{E} - \nabla (\text{div} \hat{E}) + \frac{\omega^2}{\varepsilon_0 c^2} n_e^2(\omega, \vec{x}) \hat{E} = \frac{\omega^2}{\varepsilon_0 c^2} \hat{P}_{\text{NL}}
\]

with \( n_e^2(\omega, \vec{x}) = 1 + \chi_1(\omega) \). In an optical fiber, the function \( n(\omega, \vec{x}) \) (which is called the linear refractive index) depends only on \( r \) (=distance to the fiber axis) and has the following behaviour:
It means that the medium is considered to be infinite, but the refractive index \( n(\omega, r) \) is equal to 1 when \( r \geq r_c \), i.e., outside the fiber. We assume that the core of the fiber is equal to \( \Omega_c \), and we denote \( z^+ = (x, y) \) so that \( r = \sqrt{x^2 + y^2} \).

We also denote by \( \hat{L}(\partial_x^2, \partial_y^2, \partial_z^2, \omega) \) the real, symmetric operator

\[
\hat{L}(\partial_x^2, \partial_y^2, \partial_z^2, \omega) = \Delta - D(\partial r^2) + \frac{n^2 \omega^2}{c^2} \\
= \left( \begin{array}{ccc}
\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{n^2 \omega^2}{c^2} & -\frac{\partial^2}{\partial x \partial y} & -\frac{\partial^2}{\partial x \partial z} \\
-\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{n^2 \omega^2}{c^2} & -\frac{\partial^2}{\partial y \partial z} \\
-\frac{\partial^2}{\partial x \partial z} & -\frac{\partial^2}{\partial y \partial z} & \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} + \frac{n^2 \omega^2}{c^2}
\end{array} \right)
\]

Then equation (3) is rewritten as

\[
(4) \quad \hat{L}(\partial_x^2, \partial_y^2, \partial_z^2, \omega) \hat{\mathbf{E}}(\omega, x, y, z) = \frac{\omega^2}{\varepsilon_0 c^2} \hat{P}_{NL}(\omega, x, y, z).
\]

Now, we assume that

\[
\hat{\mathbf{E}}(t, x^+, z) = \varepsilon \hat{\mathbf{E}}_0(t, x^+, z) + \varepsilon^2 \hat{\mathbf{E}}_1(t, x^+, z) + \varepsilon^3 \hat{\mathbf{E}}_2(t, x^+, z) + o(\varepsilon^3)
\]

with

\[
\hat{\mathbf{E}}_0(t, x^+, z) = \text{Re} \left[ \hat{\mathbf{U}}_0(t, x^+) \ast \left( A(\varepsilon t, \varepsilon z, \varepsilon^2) e^{i(k_0 z - \omega t)} \right) \right] \\
= \hat{\mathbf{U}}_0(t, x^+) \ast \left( A(\varepsilon t, \varepsilon z, \varepsilon^2) e^{i(k_0 z - \omega t)} \right) + \text{c.c.,}
\]

where \( \ast \) is the convolution in time i.e.,

\[
(f \ast g)(t) = \int_{-\infty}^{\infty} f(t - \tau) g(\tau) \, d\tau,
\]

and \( \varepsilon \) is a small parameter, which measures the length of the support (in \( \omega \)) of \( \hat{\mathbf{E}}_0(\omega, x^+, z) \).
monochromatic wave \( e^{i(\mathbf{k} \cdot \mathbf{z} - \omega t)} \)

Here \( A \) is a complex-valued function which describes the wave "envelope".

The ansatz (assumption on the form) for \( \vec{E}_1 \) and \( \vec{E}_2 \) will be set more precisely later. The method now consists in plugging (5) and (6) into equ. (2) (or equivalently equ. (4)) and writing that the factor of \( \delta^2 \) in the left and right-hand side of the resulting expression must be equal, for \( \delta = 1, 2, 3 \).

In all what follows, we denote \( \vec{E}_1 = \mathcal{E}_1 \), \( \vec{E}_2 = \mathcal{E}_2 \), \( T = \mathcal{E} t \) and we denote by \( L(\partial_x, \partial_y, \partial_z, \partial_t) \) the operator which is such that \( L(\partial_x, \partial_y, \partial_z, \partial_t) \mathcal{E}(t, x, y, z) (w) = \mathcal{E}(\partial_x, \partial_y, \partial_z, \partial_t, w) \mathcal{E}(w) \),

which means that equ. (2) may be written as:

\[
L(\partial_x, \partial_y, \partial_z, \partial_t) \mathcal{E} = \frac{1}{\varepsilon_0 c^2} \frac{\partial^2}{\partial t^2} \mathcal{E}
\]

1. Expansion of \( L(\partial_x, \partial_y, \partial_z, \partial_t) \mathcal{E}_0 \) in terms of \( \varepsilon \)

First, we note that if \( T \) is an operator defined as a "Fourier multiplier" by \( \hat{T}f(w) = m(w) \hat{f}(w) \), and if moreover \( f(t) \) has the form \( \hat{f}(t) = e^{-i \omega_0 t} \hat{f}(\varepsilon t) \), with \( \varepsilon \) small, then, at least formally, assuming that \( \hat{f} \) has compact support:

\[
(Tf)(t) = \int_{\mathbb{R}} m(z) \hat{f}(t-z) \, dz = \int_{\mathbb{R}} m(z) e^{-i \omega_0(t-z)} \hat{f}(\varepsilon(t-z)) \, dz
\]
\[ e^{-i\omega t} \int m(\tau) \left[ \tilde{f}(\tau t) - \varepsilon z \partial_{\tau} \tilde{f}(\tau t) + \varepsilon^2 \varepsilon \partial_{\tau}^2 \tilde{f}(\tau t) + o(\varepsilon) \right] \ dx_{\tau} \partial_{x_{\tau}} \partial_{z} \]

We recall that
\[ \hat{E}_0 (t, x^+, z) = U_0 (t, x^+) * \tilde{A} (t, z) + cc \]

where we have set
\[ \hat{A} (t, z) = A (xt, z, z^2, z^3) e^{i(\kappa z - \omega t)} \]

hence
\[ \hat{E}_0 (t, x^+, z) = \hat{U}_0 (t, x^+) \hat{A} (t, z) + cc \]

and
\[ L (\delta x, \delta y, \delta z, \delta t) \hat{E}_0 (\omega) = L (\delta x, \delta y, \delta z, \omega) \hat{U}_0 (\omega, x^+) \hat{A} (\omega, z) + cc \]

We use (8), and the above computation with
\[ \hat{m} (\omega) = L (\delta x, \delta y, \delta z, \omega) \hat{U}_0 (\omega, x^+) \]

and
\[ \tilde{f} (\tau) = e^{-ik_0 z} A (\tau, z_2, z_3) \]

We deduce
\[ L (\delta x, \delta y, \delta z, \delta t) \hat{E}_0 (t, x^+, z) = e^{-i\omega t} \int L (\omega) \left[ \hat{U}_0 (\omega, x^+) e^{i \kappa z A} \right] \]

\[ + i \varepsilon \partial_{\omega} L (\omega) \left( \partial_{\omega} \hat{U}_0 (\omega, x^+) e^{i \kappa z A} \right) \]

\[ + i \varepsilon^2 \partial_{\omega}^2 L (\omega) \left( \partial_{\omega} \hat{U}_0 (\omega, x^+) e^{i \kappa z A} \right) \]

\[ - \varepsilon \partial_{\omega} L (\omega) \left( \partial_{\omega} \hat{U}_0 (\omega, x^+) e^{i \kappa z A} \right) \]

\[ + i \varepsilon^2 \partial_{\omega}^2 L (\omega) \left( \partial_{\omega} \hat{U}_0 (\omega, x^+) e^{i \kappa z A} \right) \]

\[ + \varepsilon \partial_{\omega} L (\omega) \left( \partial_{\omega} \hat{U}_0 (\omega, x^+) e^{i \kappa z A} \right) \]

\[ + \varepsilon^2 \partial_{\omega}^2 L (\omega) \left( \partial_{\omega} \hat{U}_0 (\omega, x^+) e^{i \kappa z A} \right) \]

\[ + cc \]

(9)
where we have used $L(w_0)$ for $L(x, y, z, w_0)$.

On the other hand, we have

$$
\partial^2_{z^2} \left[ e^{ik_0 z} A(T, x, y, z) \right] 
= e^{ik_0 z} \left[ \partial^2_{z^2} A(T, x, y, z) + \varepsilon \varepsilon^3 \partial_z A(T, x, y, z) + \varepsilon \varepsilon^3 \partial_{z^2} A(T, x, y, z) \right]
$$

and

$$
\partial^2_{z^2} \left[ e^{ik_0 z} A(T, x, y, z) \right] 
= e^{ik_0 z} \left[ (ik_0 + \varepsilon \partial z + \varepsilon \varepsilon^3 z^2)^2 A(T, x, y, z) \right].
$$

Since $\hat{\Phi}_0$ depends only on $x^1 = (x, y)$ and not on $z$, we may write (formally):

$$
L(\partial_x, \partial_y, \partial_z, w_0) \left( \hat{\Phi}_0 (w_0, x^1) e^{ik_0 z} A(T, x, y, z) \right) 
= e^{ik_0 z} \left[ L(\partial_x, \partial_y, ik_0) (\hat{\Phi}_0) A + \varepsilon^3 L(\partial_x, \partial_y, ik_0)(\hat{\Phi}_0) \partial_z A \right] 
+ e^{ik_0 z} \left[ \varepsilon^2 \partial^2 L(\partial_x, \partial_y, ik_0)(\hat{\Phi}_0) \partial_{z^2} A \right] 
+ o(e^2).
$$

Here we have denoted

$$
L(\partial_x, \partial_y, ik_0) = \left( \begin{array}{ccc} 
\partial^2_{y^2} & -k_0^2 + \frac{n^2w_0^2}{c^2} & -\frac{\partial}{\partial xy} \\
-\frac{\partial}{\partial xy} & \partial^2_{x^2} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial^2_{y^2} + \frac{n^2w_0^2}{c^2} \frac{\partial^2}{\partial y^2}}{c^2} 
\end{array} \right) 
$$

and

$$
\partial^3_{z^2} L(\partial_x, \partial_y, ik_0) = \left( \begin{array}{ccc} 
2ik_0 & 0 & -\frac{\partial}{\partial x} \\
0 & 2ik_0 & -\frac{\partial}{\partial y} \\
-\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 
\end{array} \right) 
$$
and
\[ \partial_x^2 L(\alpha_x, \alpha_y, i \kappa_0) = \left( \begin{array}{ccc} \kappa^2 & 0 & 0 \\ 0 & \kappa^2 & 0 \\ 0 & 0 & 0 \end{array} \right). \]

In the above computations, \( \partial_x \) is considered as a variable and \( L(\alpha_x, \alpha_y, \alpha_z, \omega) \) is then a matrix-valued polynomial in this variable; \( \partial_x L(\alpha_x, \alpha_y, i \kappa_0) \) is the derivative of the matrix operator with respect to this variable, in which we replace \( \partial_x \) by \( i \kappa_0 \).

In the same way as for (10), we may write
\[
\partial_0 L(\alpha_x, \alpha_y, \alpha_z, \omega) \left( \widehat{U}_0(\omega, x^+ \ k) e^{i k_0 z} \right) = e^{i k_0 z} \partial_0 L(\alpha_x, \alpha_y, i \kappa_0 + \varepsilon \alpha_z, \omega) \left( \widehat{U}_0(\omega, x^+ \ k) e^{i k_0 z} \right) = e^{i k_0 z} \left( \partial_0 L(\alpha_x, \alpha_y, i \kappa_0, \omega) \widehat{A}_0 \right) e^{i k_0 z} + \varepsilon \partial_3 \partial_0 L(\alpha_x, \alpha_y, i \kappa_0, \omega) \left( \widehat{U}_0(\omega, x^+ \ k) e^{i k_0 z} \right)
\]
\[ = e^{i k_0 z} \left( \partial_0 \partial_3 L(\alpha_x, \alpha_y, i \kappa_0, \omega) \widehat{A}_0 \right) e^{i k_0 z} + o(\varepsilon) \]

Since \( \partial_3 \partial_0 L = 0 \).

Finally,
\[
\partial_3^2 \partial_0 L(\alpha_x, \alpha_y, \alpha_z, \omega) \left( \widehat{U}_0(\omega, x^+ \ k) e^{i k_0 z} \right) = e^{i k_0 z} \left( \partial_3^2 \partial_0 L(\alpha_x, \alpha_y, i \kappa_0, \omega) \widehat{U}_0 \right) e^{i k_0 z} A + o(\varepsilon).
\]

Now, using (9), (10), (11) and (12), we may write the expansion of \( L(\alpha_x, \alpha_y, \alpha_z, \alpha_t) E_0(\omega, x^+ \ k) \) in terms of \( \varepsilon \) as follows:
\[
\begin{align*}
L((x, y, z, t)) \overset{\rightarrow}{E_0} (t \pm \frac{1}{2})
&= e^{i (k_0 z - \omega t)} \left[ L(\partial_x, \partial_y, i k_0, \omega_0) \hat{U}_0(\omega_0) \right] A(t, z, x, y) \\
&+ E e^{i (k_0 z - \omega t)} \left[ (\partial_x L(\partial_x, \partial_y, i k_0) \hat{U}_0(\omega_0)) \partial_x A + i (\partial_2 L(\partial_x, \partial_y, i k_0) \hat{U}_0(\omega_0) \partial_2 A \right. \\
&+ i (\partial_3 L(\partial_x, \partial_y, i k_0) \hat{U}_0(\omega_0) \partial_3 A - \frac{1}{2} (\partial^2_{x_0 x} L(\partial_x, \partial_y, i k_0) \hat{U}_0(\omega_0) \partial^2_{x_0 A} \\
&- (\partial_0 L(\partial_x, \partial_y, \omega_0) \hat{U}_0(\omega_0)) \partial^2_{x_0 A} - \frac{1}{2} (L(\partial_x, \partial_y, i k_0, \omega_0) \partial^2_{x_0 A} \partial^2_{x_0 A} \\
&+ \cdots
\right]
\end{align*}
\]

2. Solvability condition of (2)-(5) at order \( E \)

We remark that when we replace \( \hat{E} \) by \( \hat{E}_0 + E \hat{E}_1 + E^2 \hat{E}_2 \)
in the left hand side of equ. (2), then the only terms of order \( E \) are those coming from the above expansion. On the other hand, due to the expression of \( P_{nl} \) in terms of \( \hat{E} \) (which is cubic), the right hand side of equ. (2) will be, with the ansatz (5), at least of order \( E^2 \). Hence, we may write at order \( E \):

\[ e^{i (k_0 z - \omega t)} (L(\partial_x, \partial_y, i k_0, \omega_0) \hat{U}_0(\omega_0)) A = 0. \]

Since we want a nontrivial envelope \( A \), the equation becomes

\[ L(\partial_x, \partial_y, i k_0, \omega_0) \hat{U}_0(\omega_0, x, y) = 0, \]

that is actually a sufficient condition in order that equ. (2) is satisfied at order \( E \).
We will assume that the fiber is "monomode" and "polarization preserving". This means that for any \( w \in \mathbb{R} \), there is a unique \( \psi_e (w) \in \mathbb{R}^n \) such that equ. (13) with \( k_0 = k_e (w) \) has a localized solution \( \hat{U}_e (w, x, y) \) (which decay to 0 as \( x^2 + y^2 \to \infty \)).

Moreover, the solution \( \hat{U}_e \) is unique, up to notations with respect to the \( z \)-axis, and renormalization. Assume also that \( w \to k_e (w) \) is regular (it is the dispersion relation), and \( k_e (w) \neq 0 \) (\( w \neq 0 \)).

From now on, we take \( \psi_e = \psi_e (w) \).

We will come back to these assumptions later.

3 - Solvability condition of (3) - (5) at order \( E^2 \)

Since the right hand side of equ. (2) (with the ansatz (5)) has no \( \epsilon^2 \)-terms, we take \( E_1 (t, x^1, y) \) in (5) of the form

\[
\hat{E}_1 (t, x^1, y) = \text{Re} \left[ \hat{U}_1 (t, x^1) e^{i (k_0 y - \omega t)} \right] = \hat{U}_1 (t, x^1) e^{i (k_0 y - \omega t)} + \text{c.c.} = \hat{U}_1 (t, x^1) e^{i (k_0 y - \omega t)}
\]

in order to be able to compensate for the fast oscillation of the phase of the terms coming from the expansion of \( E (\partial \hat{y}, \partial \hat{y}, \partial \hat{z}, \partial \hat{z}) \).

The above computation then shows that

\[
L \left( \partial \hat{x}, \partial \hat{y}, \partial \hat{z}, \partial \hat{z} \right) E_1 (t, x^1, y) = e^{i (k_0 y - \omega t)} L \left( \partial \hat{x}, \partial \hat{y}, i k_0, \omega \right) \hat{U}_1 (\omega) + \text{c.c.}
\]

Hence, using the expansion of \( E \hat{E}_0 \) computed in section 4, we may write the equation coming from the terms factor of \( \epsilon^2 \) in the expansion of \( E \hat{E}_0 + \epsilon^2 E \hat{E}_1 \) as:

\[
L \left( \partial \hat{x}, \partial \hat{y}, i k_0, \omega \right) \hat{U}_1 (\omega) = - \partial_3 L \left( \partial \hat{x}, \partial \hat{y}, i k_0 \right) \hat{U}_0 (\omega) \partial_2 A
\]

\[
i \left[ L \left( \partial \hat{x}, \partial \hat{y}, i k_0, \omega \right) \partial_0 \hat{U}_0 (\omega) + \partial_3 L \left( \partial \hat{x}, \partial \hat{y}, i k_0, \omega \right) \hat{U}_1 (\omega) \right] \partial_1 A
\]
On the other hand, taking the derivative with respect to $\omega$ of
\[
L(\partial_x, \partial_y, i k(\omega), \omega) \hat{U}_0(\omega) = 0, \quad \forall \omega \in \mathbb{R}
\]
(which comes from the choice of the function $\omega \mapsto k(\omega)$),
we obtain
\[
\frac{d}{d\omega} \left[ L(\partial_x, \partial_y, i k(\omega), \omega) \hat{U}_0(\omega) \right] \bigg|_{\omega = \omega_0} = 0
\]
\[
= (\partial_x \partial_y) (\partial_x, \partial_y, i k(\omega), \omega) \hat{U}_0(\omega) + i k'(\omega_0) \partial_3 L(\partial_x, \partial_y, i k(\omega), \omega) \hat{U}_0
\]
\[
+ L(\partial_x, \partial_y, i k(\omega), \omega) \partial_3 \hat{U}_0(\omega)
\]
from which we deduce
\[
L(\partial_x, \partial_y, i k(\omega), \omega) \partial_3 \hat{U}_0(\omega)
\]
\[
= (\partial_x \partial_y) (\partial_x, \partial_y, i k(\omega), \omega) \hat{U}_0(\omega) + i k'(\omega_0) \partial_3 L(\partial_x, \partial_y, i k(\omega), \omega) \hat{U}_0(\omega)
\]
Inserting this expression in (14), and denoting $k' = k'(\omega_0)$,
we obtain (with $L = L(\partial_x, \partial_y, i k(\omega), \omega)$, etc.)
\[
L \hat{U}_1(\omega_0) = - \partial_3 L \hat{U}_0 \partial_3 A + i (\partial_3 L) \hat{U}_0 \partial_3 A - k' \partial_3 L \hat{U}_0 \partial_3 A
\]
\[
- i (\partial_3 L) \hat{U}_0 \partial_3 A
\]
(16)
\[
= - (\partial_3 A + k' \partial_3 A) (\partial_3 L) \hat{U}_0(\omega_0)
\]
We recall that $\hat{U}_0(\omega_0, x^1)$ is now known, and that
we want to solve the above equation in terms of $\hat{U}_1(\omega_0, x^1)$,
and $A(1, \mathbb{Z}, \mathbb{Z})$.

We define the hermitian product on $(L^2(\mathbb{R}^2, \mathbb{C}))^3$ by
\[
\langle U, V \rangle = \int_{\mathbb{R}^2} U(x^1) \overline{V}(x^1) \, dx^1 dy
\]
for $U, V$, square integrable functions on $\mathbb{R}^2$ with values in $\mathbb{C}^3$. 

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then, it is not difficult to see (using integration by parts), that the operator $\mathcal{L}$, which we recall is defined by

$$
\mathcal{L}(\phi, \psi, \lambda, \omega) = \begin{pmatrix}
\frac{\partial^2}{\partial x^2} - k_0^2 + \frac{n_0^2 \omega^2}{c^2} & -i k_0 \phi \\
\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} - k_0^2 + \frac{n_0^2 \omega^2}{c^2} \\
-\frac{\partial}{\partial x} & -i k_0 \phi \\
- \frac{\partial}{\partial y} & -i k_0 \phi \frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2} + \frac{n_0^2 \omega^2}{c^2}
\end{pmatrix}
$$

is symmetric with respect to the above product, i.e.

$$
\langle \mathcal{L} \hat{U}, \hat{V} \rangle = \langle \hat{U}, \mathcal{L} \hat{V} \rangle \quad \text{for any } \hat{U}, \hat{V} \in L^2(\mathbb{R}^3, \mathbb{C})^3.
$$

Taking then the products of the left and right hand sides of (16) with $\hat{U}_0(\omega)$ (assuming of course that every function that we consider is sufficiently regular and decays sufficiently fast at infinity) we obtain

$$
\langle \mathcal{L} \hat{U}_1(\omega), \hat{U}_0(\omega) \rangle = \langle \hat{U}_1(\omega), \mathcal{L} \hat{U}_0(\omega) \rangle = 0 \tag{17}
$$

On the other hand, taking the product of (15) with $\hat{U}_0(\omega)$, and taking account of the symmetry property of $\mathcal{L}$, we also have

$$
\langle \mathcal{L} \omega \hat{U}_0(\omega), \hat{U}_0(\omega) \rangle = \langle \omega \phi \mathcal{L} \hat{U}_0(\omega), \hat{U}_0(\omega) \rangle = 0
$$

$$
= - \langle (\partial_3 \omega \phi) \hat{U}_0(\omega), \hat{U}_0(\omega) \rangle + i k_0^2 \langle \partial_3 \hat{U}_0(\omega), \hat{U}_0(\omega) \rangle
$$

so that

$$
\langle (\partial_3 \omega \phi) \hat{U}_0(\omega), \hat{U}_0(\omega) \rangle = - \frac{i}{k_0^2} \langle \partial_3 \hat{U}_0(\omega), \hat{U}_0(\omega) \rangle \tag{18}
$$

Note that $\partial_3 \omega = \frac{d}{d\omega} \left[ \frac{n_0^2(\omega) \omega^2}{e^2} \right] \omega = \omega_0$. Thus the right hand side of equ. (18) is not

$$
\text{det} \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$
equal to 0 in general, and in order to solve (17), we impose

(19) \[ \partial_t A + \kappa'(\omega_0) \partial_\tau A = 0 \]

which is equivalent to (with \( \kappa'(\omega_0) = \kappa'(\omega_1) \))

(20) \[ A(T, z_1, z_2) = A_0(T - \kappa'_1 z_1, z_2) \]

with \( A_0(T, z_2) = A(T, 0, z_2) \).

Hence equation (20) does not allow to solve completely the
equation for \( A \), but it allows to eliminate one of the variables. This equation, which is a transport equ in the \((z_1, T)\) variable,
says that, at the main order, the envelope \( A \) travels with
the group velocity \( \kappa'(\omega_0) \) (we recall that here \( z_1 \) is consid-
ered as the evolution variable).

In all what follows, we set \( \tau = T - \kappa'_1 z_1 = \tau' + \kappa'_0 \tau \)
and the aim is now to find \( A_0(\tau, z_2) \). This will be done
by computing the factor of \( \tau' \) in equ. (2) with the ansaty (5).

Once equ. (19) is satisfied, we may take \( \hat{A}_1(\omega_0) = \tilde{A}_0 \),
for any \( \hat{d} \in \mathbb{R} \), in order to solve (16), hence (14) is
satisfied.

4 - Expansion of the right-hand side of (2)

We now need to go back to the expression of \( \hat{p}_{N_0} \), which we
recall is given by

\[ \hat{p}_{N_0} = \mathcal{E}_0 \int_{\mathbb{R}^3} X^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) \left[ \hat{E}(\tau_1) - \hat{E}(\tau_3) \right] \hat{E}(\tau_2) d\tau_1 d\tau_2 d\tau_3 \]

(everything here also depends on \((x, y, \tau)\); \( X^{(3)} \) actually
depends only on \( \tau^+ = (x, y) \)). We note that we may replace
\[ E \] by \( E_0 \) in the right hand side above (see (5)), since any contribution of a term involving \( E_1 \) or \( E_2 \) would be of order at least \( \varepsilon^4 \). Moreover, we recall that

\[ E_0(t, x^+, z) = U_0(t, x^+) \ast \left[ A(\varepsilon t, z_1, z_2) e^{i(k_0 z - \omega_0 t)} \right] + c.c., \]

where \( \ast \) denotes the convolution in time, i.e.

\[ E_0(t, x^+, z) = \int \limits_{\mathbb{R}} U_0(z, x^+) A(\varepsilon(\varepsilon t - z), z_1, z_2) e^{i(k_0 z - \omega_0 t)} d\varepsilon. \]

It follows from the computations of section 1 that

\[ \hat{E}_0(t, x^+, z) = \hat{U}_0(\omega_0) e^{i(k_0 z - \omega_0 t)} A(\tau, z_1, z_2) + c.c. + o(1). \]

Hence, since \( A(\tau - \varepsilon z) = A(\tau) + o(1) \):

\[ \begin{align*}
\hat{P}_{\varepsilon} &= \varepsilon \varepsilon^3 \int \limits_{\mathbb{R}^3} X^{(3)}(z_1, z_2, z_3) \left( \hat{E}_0(t - \tau, z) \cdot \hat{E}_0(t - \tau, z) \right) e^{i(k_0 z - \omega_0 (t - \tau))} A(T, z_1, z_2) + c.c. \\
&= \varepsilon \varepsilon^3 \int \limits_{\mathbb{R}^3} X^{(3)}(z_1, z_2, z_3) \left[ \hat{U}_0(\omega_0) e^{i(k_0 z - \omega_0 (t - \tau))} A(T, z_1, z_2) + c.c. \right] \\
&\quad \times \left[ \hat{U}_0(\omega_0) e^{i(k_0 z - \omega_0 (t - \tau))} A(T, z_1, z_2) + c.c. \right] \\
&= \varepsilon \varepsilon^3 \int \limits_{\mathbb{R}^3} X^{(3)}(z_1, z_2, z_3) \\
&\quad \left[ e^{i\omega_0(z_1 + z_2 + z_3)} \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) e^{i(k_0 z - \omega_0 t)} A^3 \right. \\
&+ e^{i\omega_0(z_1 - z_2 + z_3)} \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) e^{i(k_0 z - \omega_0 t)} A \hat{A} \hat{A} \hat{A} \\
&+ e^{i\omega_0(z_1 + z_2 - z_3)} \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) e^{i(k_0 z - \omega_0 t)} A \hat{A} \hat{A} \hat{A} \\
&+ e^{i\omega_0(-z_1 + z_2 + z_3)} \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) e^{i(k_0 z - \omega_0 t)} A \hat{A} \hat{A} \hat{A} \\
&+ e^{i\omega_0(-z_1 + z_2 - z_3)} \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) e^{i(k_0 z - \omega_0 t)} A \hat{A} \hat{A} \hat{A} \\
&+ e^{i\omega_0(z_1 - z_2 - z_3)} \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) \hat{U}_0(\omega_0) e^{i(k_0 z - \omega_0 t)} A \hat{A} \hat{A} \hat{A} + c.c. \right) d\varepsilon. \\
\end{align*} \]
We denote by \( \hat{X}(^{(3)})(\omega, \omega, \omega_3) \) the Fourier transform (in time) of \( X^{(3)}(t) \):

\[
\hat{X}^{(3)}(\omega, \omega, \omega_3) = \int_{\mathbb{R}^3} X^{(3)}(t_1, t_2, t_3) \, e^{-i(\omega t_1 + \omega t_2 + \omega_3 t_3)} \, dt_1 dt_2 dt_3.
\]

We assume that

\[
\hat{X}^{(3)}(\omega, \omega, -\omega_3) = \hat{X}^{(3)}(\omega, \omega, \omega_3) \quad \text{(this assumption amounts to assuming some symmetry in the material response with respect to the frequency)}.
\]

Then, we denote by \( \hat{X}^{(3)}(\omega, \omega, \omega_3) \) this quantity, while we set \( \hat{X}^{(3)}(\omega, \omega, \omega_3) = \hat{X}^{(3)}(\omega, \omega, \omega_3) \).

Then, we find

\[
P_{NL}(t, \mathbf{x}^3, \mathbf{r}) = e \cdot e^3 \, \hat{X}^{(3)}(\mathbf{A}_0, \mathbf{A}_0) \mathbf{A}_0(\omega, \mathbf{x}^3) e^{-i(k_{03} - \omega_0 t)} A(\mathbf{t}, \mathbf{z}, \mathbf{z}_0) \\

+ e \cdot e^3 \, \hat{X}^{(3)}(2(\mathbf{A}_0, \mathbf{A}_0) \mathbf{A}_0 + (\mathbf{A}_0, \mathbf{A}_0) \mathbf{A}_0) \mathbf{A}_0(\omega, \mathbf{x}^3) e^{-i(k_{03} - \omega_0 t)} A(\mathbf{t}, \mathbf{z}, \mathbf{z}_0) \\

\quad \times e^{i(k_{03} - \omega_0 t)} \, 1 A^2 A(\mathbf{t}, \mathbf{z}, \mathbf{z}_0) + \text{c.c.} + o(e^3).
\]

Moreover, when computing the derivative in the \( t \)-variable of the above expression, one may see that each time the derivative acts on \( A \) (or on \( \mathbf{A}_0 \)) then this gives rise to another \( e \) factor, so one only has to take into account the derivatives of the four phases terms \( e^{i(k_{03} - \omega_0 t)} \) and \( e^{i(k_{03} - \omega_0 t)} \) (\( + \text{c.c.} \)). It follows

\[
\frac{1}{c^2} \frac{\partial^2 P_{NL}}{\partial t^2} = - \frac{2\omega_0}{c^2} e \cdot e^3 \, \hat{X}^{(3)}(\mathbf{A}_0, \mathbf{A}_0) \mathbf{A}_0(\omega, \mathbf{x}^3) e^{-i(k_{03} - \omega_0 t)} A(\mathbf{t}, \mathbf{z}, \mathbf{z}_0) \\

\quad \times e^{i(k_{03} - \omega_0 t)} \, 1 A^2 A(\mathbf{t}, \mathbf{z}, \mathbf{z}_0) + \text{c.c.} + o(e^3)
\]

\[\text{(21)}\]
S - Solvability condition of (3) - (5) at order $E^3$

The only terms factors of $E^3$ when plugging the ansatz (5) into the left hand side of equation (2) come from

- the terms factor of $E^2$ in the development of $L(\delta_x, \delta_y, \delta_z, \delta_t)\tilde{E}_0$
  (see p. 14),

- the terms of order $E^0$ in the development of $L(\delta_x, \delta_y, \delta_z, \delta_t)\tilde{E}_0$.

Indeed, there is no term factor of $E$ in the development of $L\tilde{E}_0$, since we recall that

$$\tilde{E}_0(t, x^1, y) = \hat{U}_0(t, x^1) \ast e^{i(k_0z - \omega_0t)} + c.c.$$

$$= \hat{U}_0(\omega_0, x^1) \ast e^{i(k_0z - \omega_0t)} + c.c.$$

For the solvability condition, the sum of the two terms above must be equal to the right hand side of equ. (21). In view of the fast oscillating terms in the rhs of (21) and in the expansion of $L\tilde{E}_0$, we take $\tilde{E}_2(t, x^1, y)$ in the form

$$\tilde{E}_2(t, x^1, y) = \hat{U}_2(3)(t, x^1) \ast e^{3i(k_0z - \omega_0t)} + c.c.$$

$$+ U_2^{(1)}(t, x^1) \ast e^{i(k_0z - \omega_0t)} + c.c.$$

$$= \hat{U}_2^{(3)}(3\omega_0, x^1) \ast e^{3i(k_0z - \omega_0t)} + \hat{U}_2^{(1)}(\omega_0, x^1) \ast e^{i(k_0z - \omega_0t)} + c.c.$$

In order to satisfy the solvability condition, we need to find $\hat{U}_2^{(3)}$ and $\hat{U}_2^{(1)}$ such that:

$$\begin{align*}
\mathcal{L}(\delta_x, \delta_y, 3i\kappa_0, 3\omega_0) \hat{U}_2^{(3)} &= -\frac{9\omega_0^2}{c^2} \hat{X}_0^{(2)}(\tilde{\omega}_0, \tilde{\omega}_0) A^3 \hat{U}_0 \\
\mathcal{L}(\delta_x, \delta_y, i\kappa_0, \omega_0) \hat{U}_2^{(1)} &= -\frac{\omega_0^2}{c^2} \hat{X}_1^{(1)}[2(\hat{U}_0, \hat{\tilde{U}}_0) \hat{U}_0 + (\hat{U}_0, \hat{\tilde{U}}_0) \hat{U}_0] 1A^1 \hat{A} \ldots \\
\end{align*}$$
\[
\frac{1}{\varepsilon^2} \frac{\partial^2}{\partial z^2} \hat{A} - \frac{1}{3} \frac{\partial^2}{\partial y^2} \hat{A} - i k \frac{\partial}{\partial y} \hat{A} + \frac{1}{h^2} \frac{\partial^2}{\partial y^2} \hat{A} = 0 \]

Equation (22) may be solved under the "non-resonance condition":
\[
(24) \quad k(3\omega) \neq 3 k(\omega)
\]

where we recall that \(k(\omega) = k\) is the unique value such that the operator \(\mathcal{L}(\partial_x, \partial_y, ik, \omega)\) has a nontrivial nullspace, i.e., the equation \(\mathcal{L}(\partial_x, \partial_y, ik, \omega) \hat{U}(\omega, x, y) = 0\) has a localized solution \(\hat{U}(\omega, \cdot)\) (i.e., a solution that decay to zero as \(x^2 + y^2 \to \infty\)).

Indeed, if (24) is satisfied, then the nullspace of the operator \(\mathcal{L}(\partial_x, \partial_y, ik, \omega)\) (considered, e.g., as an unbounded operator on \(L^2(\mathbb{R}^2)\), with a dense domain that would have to be specified) is reduced to \(\{0\}\). On the other hand, \(\mathcal{L}(\partial_x, \partial_y, ik, \omega)\) is a symmetric operator, and we assume it is selfadjoint (i.e., we replace \(\mathcal{L}\) by its selfadjoint extension).

But then,
\[
\text{Im} \mathcal{L} = \text{N}(\mathcal{L})^\perp = L^2(\mathbb{R}^2)
\]

(see e.g., H. Brezis, Functional analysis) and it follows that, provided \(\hat{U}\) is sufficiently localized in \(x^2\), the right hand side of equ. (22) is, up to a \(O(\varepsilon)\) term, in \(\text{Im} \mathcal{L}\), i.e., equ. (22) has a solution \(\hat{U}_\varepsilon(\omega)\) up to a \(O(\varepsilon)\) term. Now, the fact that the right hand side of equ. (22) is localized follows from the fact that \(\hat{U}_\varepsilon(\omega)\) is localized, and we may assume that \(\hat{X}_0(\omega)\) is also localized (note that \(\hat{X}_0^2\) will give a correction term to the refractive index \(n(x, x^2)\), hence
we may assume that it has compact support in \( x^1 \). Note also that here \( A^3 \) is considered as a constant since it does not depend on \( x^1 \).

So we concentrate now on equation (23). We recall (see eq. (20)) that \( A_0 (T, \vec{z}, \vec{z}_0) = A_0 (T, \vec{z}) \) with \( \vec{z} = T - \frac{b}{c} \vec{z}_1 \).

It follows that \( \partial_A (T, \vec{z}, \vec{z}_0) = \partial_A A_0 (T, \vec{z}) \), \( \partial_A \vec{z} A (T, \vec{z}, \vec{z}_0) = \partial_A \vec{z} A_0 (T, \vec{z}) \) and \( \partial_{\vec{z}_0} A (T, \vec{z}, \vec{z}_0) = -b_0 \partial_{\vec{z}_0} A_0 \).

\( \partial_{\vec{z}} \vec{z} A (T, \vec{z}, \vec{z}_0) = (b_0 \partial_{\vec{z}} A_0 (T, \vec{z}) - b_0 \partial_{\vec{z}} A_0 (T, \vec{z}) \).

Inserting these expressions in eq. (23), we find

\[
L \widehat{A}^{(2)}_0 = -\frac{\omega_0^2}{c_2^3} \widehat{X}_1 (z) \left[ \hat{E} (\widehat{\omega}_0, \vec{0}) \widehat{\omega}_0 + (\hat{E} (\widehat{\omega}_0, \vec{0}) \vec{0} \right] A_0^2 \nabla A_0
+ \left[ \frac{1}{2} \left( \partial_{\omega_0}^2 \right) A_0 + ik_0 \left( \partial_{\omega_0} \right) \left( \partial_{\omega_0} \right) \vec{\hat{A}}_0 + k_0 \left( \partial_{\omega_0} \right) \left( \partial_{\omega_0} \right) \vec{\hat{A}}_0 \right]
- \frac{1}{2} \left( \partial_{\omega_0} \right)^2 \left( \partial_{\omega_0} \right) \vec{\hat{A}}_0 + \frac{1}{2} \left( \partial_{\omega_0} \right)^2 \left( \partial_{\omega_0} \right) A_0
- \left( \partial_{\omega_0} \right)^2 A_0 \vec{\hat{A}}_0 .
\]  

We may simplify the right hand side of eq. (25) as follows: we recall that for all \( \omega \), the following relation holds:

\[
L \left( \partial_{\omega}, \partial_{\omega}, \partial_{\omega} \right) \vec{\hat{A}}_0 (\omega) = 0 .
\]

We then take the second order derivative with respect to \( \omega \) of the above expression, which gives at \( \omega = \omega_0 \):

\[
\frac{d^2}{d\omega_0^2} \left( \hat{L} \left( \partial_{\omega}, \partial_{\omega}, \partial_{\omega} \right) \vec{\hat{A}}_0 (\omega) \right) \bigg|_{\omega = \omega_0} = 0
= \left( \partial_{\omega_0}^2 \right) \vec{\hat{A}}_0 - \left( \partial_{\omega_0} \right)^2 \left( \partial_{\omega_0} \right) \vec{\hat{A}}_0 + \left( \partial_{\omega_0}^{(2)} \right) \vec{\hat{A}}_0
- 23.
\]

---
$$+ 2i \omega \left( \dot{e}_3 \right) \omega \dot{\omega}_0 + 2 \left( \ddot{\omega} \right) \left( \omega \dot{\omega}_0 \right) + i \alpha \epsilon_0 \dot{\omega}_0 = 0$$

Hence, equ.(25) reduces to:

$$n \ddot{U}_0 = - \frac{\omega}{c^2} \left( \ddot{U}_0 \right) \dot{U}_0 + \left( \dot{\omega}_0 \dot{U}_0 \right) \omega \left( \dot{U}_0 \right)$$

$$+ \left( \omega \ddot{\omega}_0 \right) \hat{e}_0 \epsilon_0 A_0 \left[ 1A_0 \right]$$

$$+ \frac{i}{2} \left( \omega \dot{\omega}_0 \right) \left( \hat{e}_0 \right) \dot{A}_0 \epsilon_0 A_0 \left[ 1A_0 \right]$$

(26)

Note that the operator $L = L \left( \omega, \dot{U}_0, \ddot{U}_0, \omega \dot{U}_0, \omega \ddot{U}_0 \right)$ in the left hand side of (26) has a null space which is not reduced to $\{0\}$ (contrary to the operator $L \left( \omega, \dot{U}_0, \ddot{U}_0, \omega \dot{U}_0, \omega \ddot{U}_0 \right)$ in the left hand side of (22)) since we have precisely chosen $k_0 = k_0 \left( \omega \right)$. However, the operator $L$ above is still symmetric and (assuming that it is self-adjoint) we still have:

$$\text{Im} \ L = \text{N} \left( L \right)^{-1}$$

As a consequence, in order to solve equ. (26) up to a $O(\epsilon)$ term, as we did for equ. (22), we need to check that the right hand side of equ. (26) belongs to $\text{N} \left( L \right)^{-1}$. This will actually give us the solvability condition.

Since $\text{N} \left( L \right)$ is spanned by $\dot{U}_0$, we take the inner product of the r.h.s. of (26) with $\dot{U}_0$ (see p. 16 for a definition of the inner product) and we find:

$$\langle \ddot{U}_0, \dot{U}_0 \rangle \left( \dot{\epsilon}_0 \epsilon_0 A_0 + \frac{i}{2} \left( \omega \dot{\omega}_0 \right) \epsilon_0 A_0 \right)$$

$$+ \frac{\omega}{c^2} \left[ \langle \hat{e}_0 \dot{U}_0, \dot{U}_0 \rangle \dot{U}_0, \dot{U}_0 \rangle + \langle \hat{e}_0 \ddot{U}_0, \dot{U}_0 \rangle \dot{U}_0, \dot{U}_0 \rangle \right]$$

$$\text{Im} A_0 \epsilon_0 A_0 = 0$$
We finally use equ. (18), i.e.,
\[
\left< \frac{1}{2} \hat{\mathbf{p}}_2 \hat{\mathbf{u}}_0 (\omega_0), \hat{\mathbf{u}}_0 (\omega_0) \right> = \frac{i}{\hbar} \left< \hat{\mathbf{p}}_2 \hat{\mathbf{u}}_0 (\omega_0), \hat{\mathbf{u}}_0 (\omega_0) \right>,
\]
and we moreover define
\[
n_\omega = \frac{n_\omega}{c} \left< \hat{\mathbf{u}}_0 (\omega_0), \hat{\mathbf{u}}_0 (\omega_0) \right> + \frac{1}{2} \left< \hat{\mathbf{u}}_0 (\omega_0), \hat{\mathbf{u}}_0 (\omega_0) \right> \frac{\mathbf{X}_0 (\omega_0) \hat{\mathbf{u}}_0 (\omega_0)}{\left< \hat{\mathbf{u}}_0 (\omega_0), \hat{\mathbf{u}}_0 (\omega_0) \right>}
\]

Then, the previous equation (which we recall is the solvability condition at order \( \mathcal{E}^1 \)) reduces to the (NLS) equation:
\[
(27) \quad \left\{ \begin{array}{l}
\hbar^2 \partial_{\omega} A_0 - \frac{i}{\hbar} \frac{1}{2} \hbar \frac{\mathbf{X}_0 (\omega_0)}{\left< \hat{\mathbf{u}}_0 (\omega_0), \hat{\mathbf{u}}_0 (\omega_0) \right>} A_0 + \frac{\omega}{c} n_\omega |A_0|^2 A_0 = 0.
\end{array} \right.
\]

As for equ. (22), once this solvability condition is satisfied, it is possible to find \( \hat{\mathbf{u}}_0 (\omega) \) satisfying equ. (23) up to a \( O(\mathcal{E}) \) term, since the right hand side of (23) is in \( \text{Im} \).  

6. Some comments

- In assuming that \( \bcancel{X}^1 (\omega) \) is a real valued function, we have neglected attenuation effects; more precisely, the above computations are valid provided \( \text{Im} (\bcancel{X}^1 (\omega)) \) is of order at least \( O(\mathcal{E}^3) \). If one had assumed that \( \text{Im} (\bcancel{X}^1 (\omega)) = O(\mathcal{E}) \), then the computations above would have led to an additional "damping term" in the NLS equation, namely:

\[
i \frac{1}{2} \hbar \frac{\mathbf{X}_0 (\omega_0)}{\left< \hat{\mathbf{u}}_0 (\omega_0), \hat{\mathbf{u}}_0 (\omega_0) \right>} A_0 + \frac{\omega}{c} n_\omega |A_0|^2 A_0 + \text{Im} \frac{\omega}{c} A_0 = 0.
\]

- It is possible (see Newell, Section 6e.) to compute explicitly (in terms of "special functions" i.e. eigenfunctions of the Laplace operator on the disc, written in \( (r, \theta) \) coordinate) the function

\[
-25 -
\]
$u_0 (\omega, x^+)$ (which we recall is the solution of $\nabla \cdot B = 0$) in the simplified case of a cylindrical waveguide with constant refractive indices $n_0$ and $n_1$ inside and outside:

![Diagram of a cylindrical waveguide with refractive indices $n_0$ and $n_1$.]

It is possible to prove in particular that, provided $n_0^2 - n_1^2$ is sufficiently small, and for some precise value of $k_0 \omega$, there is a unique localized solution, which satisfies continuity conditions at the boundary $x^2 + y^2 = L^2$.

- A lot of higher order effects have been neglected and could be added in the model, e.g., higher order dispersion (linear or nonlinear); delay in $z(t)$ (which gives rise to a derivative in the nonlinear term); birefringence, which occurs when
the two components of \( \mathbf{E} \) do not travel with the same group velocity; in this case the medium is not polarization preserving and one cannot assume that the slowly varying envelope \( A \) is a scalar function; in this case, the equation for the envelope is a vector-valued (or system of) nonlinear Schrödinger equation linearly and nonlinearly coupled.

The sign of \( \omega_0 e^{(w_0)} \) in eqn. (27) is important: it in particular determines the long-time behaviour of the envelope \( A \), as we will see in the next chapter.