Remark. Once we have proved the inequality of Step 1, the argument used to prove the other inequalities, which is due to Glimm and Velo and usually called the "\( T^{*} T \)" argument, is the following: for a given linear operator \( T \) from \( X \) into \( Y \), where \( X \) is a Hilbert space and \( Y \) is a reflexive Banach space, these following properties are equivalent.

- \( T: X \rightarrow Y \) is continuous
- \( T^{*}: Y' \rightarrow X' \) is continuous (with \( X'=X \))
- \( TT^{*}: Y' \rightarrow Y \) is continuous

The proof of Step 1 corresponds to the proof of (iii), with \( T = S(t) \) and \( T^{*} f = \int_{0}^{t} S(t-s) f(s) \, ds \) so that \( TT^{*} f = \int_{0}^{t} S(t-s) f \, ds \).

\[ E = \text{the Cauchy problem} \]

We now consider the nonlinear equation

\[ (NLS) \quad i \partial_t u + \partial_x^2 u = \lambda |u|^2 u, \]
\[ u(0, x) = u_0(x) \]

with \( x \in \mathbb{R}, \, t \in \mathbb{R}^+, \, u(t, x) \in C, \, \lambda = \pm 1 \) and \( \delta > 0 \). We will consider the integral formulation

\[ u(t) = S(t) u_0 - i \int_{0}^{t} S(t-s) F(u(s)) \, ds \]

with \( F(u) = \lambda |u|^2 u \).

Remark. One may easily check the conservation of the energy and the \( L^2 \)-norm for a regular solution of the equation \((NLS)\). Indeed, let \( u \in C([0,T]; H^2(\mathbb{R})) \cap C^2([0,T], L^2(\mathbb{R})) \) be a solution of \((NLS)\). Then

- \( t \in [0,T], \quad \frac{d}{dt} |u(t)|_{L^2} = \frac{d}{dt} \int_{\mathbb{R}} |u(t,x)|^2 \, dx \)

---

1.1
\[ = \text{Re} \int_{\mathbb{R}} \bar{\omega}(t, x) \overline{\partial_t u(t, x)} \, dx \]

Moreover, using equ. (NLS), \( \partial_t u(t, x) = i \partial_x u(t, x) - i d L^1 u \)

hence,

\[ \frac{d}{dt} \| u(t) \|^2_{L^2} = -2 \text{Re} \int_{\mathbb{R}} \bar{\omega}(t, x) \partial_x u(t, x) \, dx + 2d \text{Im} \int_{\mathbb{R}} \bar{u} u^{20+2} \, dx \]

Note that all the above terms are well defined, since \( \partial_x u \in C([0, T]; L^2) \) and \( u \in C([0, T]; H^1) \subset C([0, T]; H^1) \subset C([0, T]; L^2(\mathbb{R})) \); hence,

\[ (4) \quad \| u(t) \|^2_{L^2} \leq \| u_0 \|^2_{L^2} \quad \| u(t) \|^2_{L^2} \leq C \| u \|^2_{H^1} \]

Now, integrating the first term of the right hand side of the above equality by parts, it is easy to see that

\[ \text{Im} \int_{\mathbb{R}} \bar{\omega} \partial_x u \, dx = -\text{Im} \int_{\mathbb{R}} \partial_x |u|^2 \, dx = 0, \]

and it follows that \( \| u(t) \|^2_{L^2} = \| u_0 \|^2_{L^2}, \quad t \in [0, T] \).

(1) Let \( E(0) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 \, dx + \frac{a}{20+2} \int_{\mathbb{R}} |u|^{20+2} \, dx \).

Note that \( E(0) \) is well defined for \( u \in H^1(\mathbb{R}) \), thanks to the inequality (4). Then

\[ \frac{d}{dt} E(u(t, x)) = \text{Re} \left\langle \partial_t \bar{\omega}(t, x), \partial_x u(t, x) \right\rangle_{H^{-1}, H^1} \]

\[ + d \int_{\mathbb{R}} |u(t, x)|^{20} \text{Re} (\bar{u} u^{20+2}) \, dx \]

\[ = \text{Re} \left[ \int_{\mathbb{R}} \partial_x \bar{\omega} \left( \partial_x^2 u - d L^1 u \right) \, dx \right] \]

where we have integrated the first term by parts; then, by (NLS) we have \( \partial_x^2 u - d L^1 u = -i \partial_x u \), and the right hand side above vanishes, from which it follows
that \( E(u(t)) = E(u_0) \), for all \( t \in [0, T] \). Again, note that all the above terms are well defined if \( u \in C([0, T]; H^2) \).

These two conserved quantities motivate the study of the Cauchy problem in \( L^2(\mathbb{R}) \) when possible, or in \( H^1(\mathbb{R}) \) in order to globalize the solution.

**Local and global existence in \( L^2(\mathbb{R}) \).**

**Theorem 1.** Assume \( 0 < c < 2 \); then, for any \( u_0 \in L^2(\mathbb{R}) \), there is a unique solution \( u \in C(\mathbb{R}^+; L^2(\mathbb{R})) \cap L^2(0, T; L^2(\mathbb{R})) \) for any \( T > 0 \), of the equation \( (\text{NLS}) \); here, \( r \) is such that \( \frac{r}{2} = \frac{1}{2} - \frac{1}{2c+2} \), i.e. \((r, 2c+2)\) is an admissible pair.

Moreover, for any \( T > 0 \) and \( u \in L^2(\mathbb{R}) \), \( \|u(t)\|_{L^2} = \|u_0\|_{L^2} \), and \( u \) depends continuously on \( u_0 \) in the following sense:

If \( u_1 \equiv u_0 \in L^2(\mathbb{R}) \), then for any \( T > 0 \), the solution \( u_1 \) of \( (\text{NLS}) \) with \( u_1(0) = u_0 \) converges to \( u \) in \( C([0, T]; L^2) \cap L^2(0, T; L^2) \).

**Remarks.**

- Uniqueness holds only for solutions \( u \in C(\mathbb{R}^+; L^2) \cap L^2(0, T; L^2) \).
- It is not known whether uniqueness is true for \( u \in C([0, T]; L^2) \).
- One can change \( t \rightarrow -t \) in the result of Theorem 1.
- The result is completely independent of the sign of \( \sigma \).

**Proof of Theorem 1.** We first note that since \( H^1(\mathbb{R}) \subset C(\mathbb{R}) \), \( H^1(\mathbb{R}) \subset L^{2c+2} \), as already seen above, we have \( L^{2c+2} \cap C(\mathbb{R}) \subset H^1(\mathbb{R}) \).
indeed, \( \frac{2r+1}{2r+2} + \frac{1}{2r+2} = 1 \), and \( L^{2r+1}/2r+2 = (L^{2r+2}(R)^*)' \),
the dual space of \( L^{2r+2}(R) \). It follows that if
\( u \in L^r(0,T; L^{2r+2}) \), then \( F(u) \in L^r(0,T; L^{2r+2}) \)
and \( \| F(u) \| \) is well defined. We will
actually solve the equation in the form (I).

The proof of the theorem will be done in two steps:

1. Construction of a solution of (I) on \([0,T]\) for \( T \) sufficiently
small
2. Globalization thanks to the conservation of the \( L^2 \) norm.

Step 1. We use the Banach fixed point theorem: let
\( X(T) = C([0,T]; L^2) \cap L^r(0,T; L^{2r+2}) \), where
we recall that \( \frac{2r}{r} = \frac{2r+1}{2r+2} \). For \( u \in X(T) \) and \( t \in (0,T) \),
we define
\[
(\mathbf{u}(t))(t) = S(t)u_0 - i \int_0^t S(t-s)F(u(s))ds = S(t)u_0 + i F(u(t))
\]
with the same notations as in the Strichartz theorem.

Assume now that \( F(u) \in L^r(0,T; (L^{2r+2})') \), then by
Strichartz theorem, \( \mathbf{u}(t) \in X(T) \) and

\[
(\mathbf{u}(t))(t) \leq C_0 \| u_0 \|_{L^2} + C_1 \| F(u) \|_{L^r(0,T; (L^{2r+2})')}.
\]

Since \( (2r+2)' = \frac{2r+1}{2r+2} \), and \( F(u) = d \| u \|_{L^{2r+2}} \)

\[
\| F(u) \|_{(L^{2r+2})'} = \| u \|_{L^{2r+2}}
\]

and

\[
\| F(u) \|_{L^r(0,T; (L^{2r+2})')} = \| u \|_{L^{2r+2}(0,T; L^{2r+2})}.
\]

Thanks to Hölder's inequality, the right-hand side above is bounded by \( \| u \|_{L^r(0,T; L^{2r+2})} \)
provided that \( \frac{1}{r} (2r+1) \leq 1 \), or \( \frac{1}{r} (2r+1) \leq 1 - \frac{1}{r} \), i.e.

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\[
\frac{2}{5} (\varepsilon + 1) \leq 1; \text{ since } \varepsilon = \frac{1}{2} - \frac{1}{2 \varepsilon} = \frac{1}{2} \frac{\varepsilon}{5}, \text{ the condition is equivalent to } \varepsilon \leq 2. \text{ If } \varepsilon < 2, \text{ then by Hölder's inequality, since } R(2\varepsilon + 1) < R \text{ in this case, we have}
\]
\[
\| u \|_{L^1(\varepsilon \omega)} \leq \| u \|_{L^1(\varepsilon \omega)} \leq \frac{1}{2} \| u \|_{L^1(\varepsilon \omega)}
\]
\[
\text{with } \varepsilon = \frac{1}{2} - \frac{1}{2 \varepsilon} > 0. \text{ We then deduce from the above inequality and (5) that}
\]
\[
\| u \|_{L^1(\varepsilon \omega)} \leq C_1 \| u \|_{L^1(\varepsilon \omega)} + C_2 T^{\varepsilon \omega \lambda} \| u \|_{L^1(\varepsilon \omega)}
\]
\[
\text{let } R = 2 C_1 \| u \|_{L^1(\varepsilon \omega)} + C_2 T^{\varepsilon \omega \lambda}, \text{ and let } B_R \text{ be the ball of radius } R \text{ (and center } O \text{) in } X(t). \text{ Then, if we choose } T \text{ small enough so that } C_2 T^{\varepsilon \omega \lambda} R^2 < \frac{1}{2}, \text{ it follows from (6) that for any } u \in B_R,
\]
\[
\| u \|_{X(t)} \leq \frac{R}{2} + C_2 T^{\varepsilon \omega \lambda} R^2 \leq R
\]
\[
i.e. \quad u \in B_R. \text{ Let us prove that, choosing } T \text{ smaller if necessary, } X \text{ is a strict contraction in } B_R.
\]
\[
\text{Let } u, v \in B_R. \text{ Then}
\]
\[
\| F(u(s)) - F(v(s)) \| = \int_t^t S(t-s) \left( F(u(s)) - F(v(s)) \right) ds
\]
\[
\text{Note that}
\]
\[
\| F(u(s)) - F(v(s)) \| = \int_0^1 \frac{d}{ds} F \left( \theta u(s) + (1-\theta) v(s) \right) d\theta \frac{1}{2}.
\]
\[
\text{Moreover, for } \varepsilon > 0, \text{ we may compute}
\]
\[
\frac{d}{d\theta} \left[ u + \theta (u-v) \right]^{\frac{5}{2}} (u-v)
\]
\[
= \left| \frac{d}{d\theta} (u-v) \right|^{\frac{5}{2}} (u-v) + \left| \frac{d}{d\theta} (u-v) \right|^{\frac{3}{2}} \cdot 2 \varepsilon \text{ Re} \left[ \left( \frac{1}{5} \right) (u-v) (u+\theta (u-v)) \right] (u+\theta (u-v))
\]
\[
-45-
\]
It follows that
\[
\sup_{0 \leq \varepsilon \leq 1} \int \left[ |v + \varepsilon (u - v)|^{2p} (u + \varepsilon (u - v)) \right]^{\frac{1}{p}} d\theta 
\leq C(\varepsilon) (|u|^{2p} + |v|^{2p}) |u - v|
\]
Using again Strichartz theorem, we deduce
\[
\|e^j_{-\varepsilon \phi} x(t) - \|F(u) - F(v)\|_{L^\infty(0,T; L^{2p+2})} \leq C_2 \|F(u) - F(v)\|_{L^r(0,T; L^{2p+2})}
\leq C_2 C(\varepsilon) (|u|^{2p} + |v|^{2p}) |u - v| \|x(t)\|_{L^p(0,T; L^{2p+2})}.
\]
By Hölder's inequality, with
\[
\frac{1}{(2p+2)'} = \frac{2p}{2p+2} = \frac{2p}{2p+2} + \frac{1}{2p+2},
\]
we obtain the bound
\[
\|e^j_{-\varepsilon \phi} x(t)\|_{L^{2p+2}} \leq (\|u\|_{L^{2p+2}} + \|v\|_{L^{2p+2}}) |u - v|
\]
Using again Hölder's inequality in the time variable, with
\[
\frac{1}{r} = \frac{2p}{8p+4} + \frac{1}{2p+2},
\]
\[
\|e^j_{-\varepsilon \phi} x(t)\|_{L^{2p+2}} \leq (\|u\|_{L^{2p+2}} \|r(2p+2)\|_{L^{2p+2}} + \|v\|_{L^{2p+2}} \|r(2p+2)\|_{L^{2p+2}}) \times \|e^j_{-\varepsilon \phi} x(t)\|_{L^{2p+2}}.
\]
Finally, since we recall that \(r(2p+2) \leq r\) and
\[
\|u\|_{L^{r(2p+2)}(0,T; L^{2p+2})} \leq T^{\frac{1}{r}} \|u\|_{L^{r}[0,T; L^{2p+2}]},
\]
we obtain
\[
\|e^j_{-\varepsilon \phi} x(t)\|_{L^{2p+2}} \leq C(\varepsilon) T^{\frac{1}{r(2p+2)}} 2R^{\frac{2p}{2p+2}} \|u - v\|_{L^{2p+2}}
\]
for \(u, v \in \mathbb{R}^2\). Since \(\varepsilon > 0\), we may choose \(T > 0\) sufficiently small so that
\[
2C_2 C(\varepsilon) T^{\frac{1}{r(2p+2)}} R^{\frac{2p}{2p+2}} \leq \frac{1}{2},
\]
and the
above inequality then shows that $C$ is a contraction mapping in $B_r$. Then, since $B_r$ is closed in the complete metric space $X(T)$, it follows from the Banach fixed point theorem that $C$ has a unique fixed point $u \in X(T)$, and it is clear that $u$ is then a solution of (I). Moreover, $u$ is a solution of (Nls) by Prop.4.

Note that, up to now, we only have proved the uniqueness in $B_r$. However, if $u, v \in X(T)$ are two solutions of (Nls), hence also of (I) by prop.4, then it follows from the previous computations that

$$\|u - v\|_{X(T)} = \|Ru - Cv\|_{X(T)} \leq 2C_2 C(\theta) T^{\theta(2\theta+1)} R^{2\theta} \|u - v\|_{X(T)}$$

with here $R = \max(\|u\|_{X(T)}, \|v\|_{X(T)})$; we deduce that $u = v$ on $[0, T]$, where $T$ is such that $2C_2 C(\theta) T^{\theta(2\theta+1)} R^{2\theta} = \frac{1}{2}$, and by iterating the argument on $[T, 2T], \ldots$ we finally obtain $u = v$ on $[0, T]$.

The continuity of the solution with respect to the initial data follows also from the Banach fixed point theorem; indeed, if $u^0 \sim u_0$ in $L^2(\mathbb{R})$, then setting $R = \sup \|u_0\|_{L^2(\mathbb{R})}$, it is not difficult to prove that for $T$ sufficiently small, depending only on $R$, the solution $u^n$ of $u^n = C^n u$ satisfies

$$\|u^n - u^0\|_{X(T)} \leq 2C_2 \|u^0 - u^0\|_{X(T)}$$

by the preceding estimates (the details are left as an exercise).

Iterating again the argument, one easily proves the continuity on $[0, T]$, where $T$ is such that $\sup \|u^n\|_{L^2(\mathbb{R})} T^{\theta(2\theta+1)} C_2^{2\theta} \leq \frac{1}{2}$.

**Step 2: Globalization.** From the previous step, one can
construct a maximal solution \( \mu \in L^2(\mathbb{R}^n) \), then there exists \( T^* > 0 \), and a unique solution \( u \) of (NLS) with

\[
\mu \in C([0,T^*]; L^2(\mathbb{R}^n)) \cap L^\infty_{loc}([0,T^*]; L^{2+\epsilon}(\mathbb{R}^n)),
\]

and with \( u(0) = u_0 \). Moreover, either \( T^* = +\infty \) or

\[
\lim_{t \to T^*(u_0)} \| u(t) \|_{L^2} = +\infty.
\]

The proof of this fact is the same as for the Cauchy-Lipschitz theorem. We deduce from the characterization of the existence time \( T^* \) that if \( \| \mu \|_{L^2} \) is finite, the solution is global. The fact that the \( L^2 \) norm of \( u \) is bounded will in turn follow from the conservation of the \( L^2 \) norm. Note, however, that we cannot apply directly the formal argument given on p. 42 for \( \mu \in X(T) \), since e.g. \( \int u \). \( \delta_u \) does not make sense for \( \mu \in X(T) \).

In order to justify the computations, we need to regularize the solution. Let \( p \in C^0(\mathbb{R}) \) with \( p \geq 0 \) and \( \int p(x) \, dx = 1 \); then if \( p^\varepsilon(x) = \frac{1}{\varepsilon} p(\frac{x}{\varepsilon}) \), we have \( \int p^\varepsilon(x) \, dx = 1 \) and \( p_\varepsilon \ast u \to u \) in \( L^2(\mathbb{R}) \), for any \( u \in L^2(\mathbb{R}) \).

Let \( u \in X(T) \) solution of (NLS), then, for all \( \varepsilon > 0 \) and all \( t \in [0,T] \), \( p^\varepsilon \ast u(t) \in C^0(\mathbb{R}^n) \) and \( p^\varepsilon \ast u \) satisfies

\[
i \partial_t (p^\varepsilon \ast u) + \partial_x (p^\varepsilon \ast u) = \Delta p^\varepsilon \ast (1|u|^2 \ast u).
\]

We then compute, noticing that all terms in the equation above are in \( L^2(\mathbb{R}^n) \):

\[
\frac{d}{dt} \| p^\varepsilon \ast u(t) \|_{L^2}^2 = 2 \text{Re} \int p^\varepsilon \ast u(t) p^\varepsilon \ast u(t) \, dx
\]

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\[
-2 \text{Im} \int_{\mathbb{R}} (e^{2x} \bar{u}) \cdot \partial_x (\rho^2 u) \, dx + 2 \text{Re} \int_{\mathbb{R}} (e^{2x} \bar{u}) \cdot (\rho^2 \partial_x u) \, dx
\]

We can integrate the first term on the right-hand side above by parts, and this term is equal to 0. It follows that

\[
\| \rho^2 u(t) \|_{L^6}^6 = \| \rho^2 u \|_{L^6}^6 + 2 \text{Re} \int_{\mathbb{R}} (e^{2x} \bar{u}) \cdot (\rho^2 \partial_x u) \, dx
\]

Now, \( u \in X(t) \Rightarrow u \in L^{\infty}([0, T) \times \mathbb{R}) \) with \( T > 0 \).\( \frac{\gamma}{2} = 2 - \frac{1}{2+2} = \frac{3}{2} \), hence \( u \in L^6([0, T) \times \mathbb{R}) \). Moreover, by Lebesgue theorem, since \( \rho^2 u \in L^{2+2} \) and \( \| \rho^2 u \|_{L^{2+2}} \leq \| u \|_{L^{2+2}} \), \( 2+2 \) and \( \rho^2 u \in L^{2+2} \), \( \| \rho^2 u \|_{L^{2+2}} \leq \| u \|_{L^{2+2}} \)

we deduce that

\[
\rho^2 u \in L^{6}([0, T) \times \mathbb{R})
\]

\[
\rho^2 u \in L^{6}([0, T) \times \mathbb{R})
\]

Taking the limits of all the terms in the above equation, we finally find

\[
\| u(t) \|_{L^6}^6 = \| u(0) \|_{L^6}^6
\]

That shows the conservation of the \( L^6 \)-norm of a solution \( u \) in \( X(t) \) and ends the proof of step 2. \( \square \)

Theorem 1 shows the existence and uniqueness of a global solution if \( \sigma < 2 \). When \( \sigma = 2 \), the situation changes (\( \Gamma = 2 \) is called the critical \( L^6 \)-case). Actually, the equation still possesses local solutions in this case, but they are not global in space. This is stated in the next theorem. Note that, if \( \sigma = 2 \), then \( 2+2 = 6 \), and that (4.6) is an admissible pair.
Theorem 2. Let $s = 2$; then for any $u_0 \in L^6(\mathbb{R})$, $\exists T^*(u_0)$ such that equation (NLS) has $u_t \in C([0, T^*(u_0)), L^2(\mathbb{R})) \cap L^6_c(0, T^*(u_0); L^6(\mathbb{R}))$ with $u(0) = u_0$. Moreover, either $T^*(u_0) = \infty$ or $\|u\|_{L^6(0, T^*(u_0); L^6(\mathbb{R}))} = \infty$.

Proof (indication). For $T > 0$, we set $X(T) = C([0, T]; L^6) \cap L^6(0, T; L^6(\mathbb{R}))$. Then the proof consists in showing that if we set

$$S = \{ u \in X(T), \| u \|_{L^6(0, T; L^6)} \leq R, \| u \|_{L^6(0, T; L^6)} \leq R \}$$

then $X$ sends $S$ into itself and is a contraction mapping for the $X(T)$ norm provided that

(i) $C R^{\alpha+1} \leq \| u \|_{L^6(0, T; L^6)}$

(ii) $\| S(\cdot) u_0 \|_{L^6(0, T; L^6)} \leq R$

(iii) $C R^{\alpha} \leq R$

(iv) $2 C \| S(\cdot) R^{\alpha} \|_{L^6(0, T; L^6)} \leq R$

where we have used the notation of the proof of theorem 1. This is easily obtained with the use of Strichartz estimates. Then for $u_0 \in L^6(\mathbb{R})$ fixed, we first choose $R$ sufficiently small so that (i), (iii) and (iv) are satisfied; then we choose $T$ sufficiently small ($T$ being now fixed) such that (ii) holds. This latter property is made possible by the fact that $S(\cdot) u_0 \in L^6(\mathbb{R}, L^6)$ by Strichartz Theorem, implying that $\| S(\cdot) u_0 \|_{L^6(0, T; L^6)} \to 0$. \(\square\)

Remark. Note that in the critical case $s = 2$, the conservation of the $L^6$-norm still holds (with the same arguments as in Theorem 1) but is not sufficient to globalize the solution.
b. Local and global existence in $H^1(\mathbb{R})$

When $\sigma > 2$, no local existence is available in $L^2(\mathbb{R})$ and one has to consider more regular solutions, typically $H^1$. One can then use the energy to get globalization in some cases. This is stated in the next theorem.

**Theorem 3.** Let $\sigma > 0$, $u_0 \in H^1(\mathbb{R})$ and $\sigma = \pm 1$. Then, there exists $T^* = T^*(u_0) > 0$ and a unique maximal solution $u \in C([0, T^*) ; H^1(\mathbb{R}))$ of (NLSE) such that $u(t, \cdot) = u_0(\cdot)$.

Moreover, either $T^* = \infty$ or $T^* < \infty$ and $\lim_{t \to T^*} \| \partial_x u(t) \|_{L^2} = \infty$.

In each of the following cases:

1. $\sigma < 2$
2. $\sigma = +1$ (defocusing case)
3. $\|u_0\|_{L^2}$ sufficiently small (if $\sigma = -1$)

the solution is globally defined i.e. $T^* = \infty$.

**Proof.** The local existence result is easily proved using a fixed point argument on the integral equation, as in Theorem 1. The use of Strichartz estimates is not even necessary, it suffices to use the fact that $e^{it\Delta}$ is an isometry in $H^1(\mathbb{R})$, together with the fact that if $u \in H^1(\mathbb{R})$, then for $\sigma > 0$,

$$\partial_x (1 + i u^{2\sigma} u) = 1 + i u^{2\sigma} \partial_x u + \sigma \partial_x u^{2\sigma} \Re (\bar{u} \partial_x u)$$

so that

$$\| \partial_x (1 + i u^{2\sigma} u) \|_{L^2} \leq C(\sigma) \| u \|^2_{H^1} \| \partial_x u \|_{L^2} \leq C(\sigma) \| u \|^2_{H^1}$$

and in the same way,

$$\| \partial_x (1 + i u^{2\sigma} u) - \partial_x (1 + i u^{2\sigma} u) \|_{L^2} \leq C(\sigma) \| u_1 - u_2 \|^2_{H^1}$$
The details are left as an exercise.

In order to get global existence, one first proves that the energy \( E(u(t)) \) is conserved i.e. \( E(u(t)) = E(u_0) \), \( t \in \mathbb{R} \). 

Again, this needs a regularization argument, as for the conservation of the \( L^2 \)-norm in the \( L^2 \) case, by considering \( \mu^p = \frac{1}{p} \rho(\frac{x}{\varepsilon}) \), \( \rho \in C_0^\infty(\mathbb{R}) \), \( p \geq 0 \) and \( \int_\mathbb{R} \rho(x) \, dx = 1 \).

The argument is similar and we do not give the details.

Now, if \( d = 1 \), then since

\[
E(u(t)) = \frac{1}{2} \int_\mathbb{R} |\partial_x u(t)|^2 \, dx + \frac{1}{2s+2} \int_\mathbb{R} |u(t)|^{2s+2} \, dx = E(u_0),
\]

we immediately deduce that

\[
\|\partial_x u(t)\|_{L^2}^2 \leq 2E(u_0)
\]

and since \( \|u(t)\|_{L^2} = \|u_0\|_{L^2} \), we have \( \|u(t)\|_{H^s} \leq C(\|u_0\|_{H^s}) \).

The characterization of \( T^*(u_0) \) given by local existence implies that \( T^*(u_0) = \infty \).

Assume now that \( 0 < \varepsilon \). We use here the Gagliardo-Nirenberg inequalities: from the Sobolev embedding \( H^s(\mathbb{R}) \subset C^{0,a} \), we know that \( \exists a, 0 < a < 1 \) such that

\[
\|u\|_{L^\infty} \leq C \|u\|_{L^2}^{1-a} \|u\|_{L^1}^a \text{ for any } u \in H^s(\mathbb{R}).
\]

(In order to find the value of \( a \), it suffices to write that the inequality is true for \( u(x) = \frac{x}{\varepsilon} \), for any \( p > 1 \) and any \( \varepsilon \),\( \beta \varepsilon^{p+2} = \beta^{-\frac{a}{2}} \varepsilon^{-a + a\frac{p}{2}} \), hence \( a = \frac{1}{2} - \frac{2}{p+2} > \frac{1}{2} \).

\[
\int_\mathbb{R} |u(x)|^{p+2} \, dx \leq C \left( \int_\mathbb{R} |u(x)|^2 \, dx \right)^{\frac{p}{2}} \int_\mathbb{R} |u(x)|^{2p+2} \, dx
\]

Again, we use the conservation of the energy and the \( L^2 \)-norm, which together with the above inequality implies...
\[ \| Dv(t) \|_{L^2}^2 \leq 2E(\omega_0) + \frac{C}{\delta + 1} \| Dv(t) \|_{L^2} \| \omega_0 \|_{L^2}^{\sigma + 2}. \]

Since \( \sigma < 2 \), Young's inequality implies
\[ \| Dv(t) \|_{L^2}^2 \leq 2E(\omega_0) + \frac{1}{2} \| Dv(t) \|_{L^2}^2 + C(\delta) \| \omega_0 \|_{L^2}^{2(\sigma + 2)/\sigma - \sigma} \]
\[ \| Dv(t) \|_{L^2}^2 \leq C(\| \omega_0 \|_{H^1}, \sigma) \]
and we deduce again that \( T^*(\omega_0) = + \infty \).

Assume now that \( \sigma \geq 2 \) (and we will assume \( \| \omega_0 \|_{H^1} \) is sufficiently small); the Gagliardo-Nirenberg inequality still applies
\[ \| Dv(t) \|_{L^2}^2 \leq 2E(\omega_0) + \frac{C}{\delta + 1} \| Dv(t) \|_{L^2}^{\sigma + 2} \| \omega_0 \|_{L^2}^{\sigma + 2}. \]

If \( \sigma = 2 \), it is easy to get an \( \| Dv(t) \|_{L^2}^2 \), provided
\[ \| \omega_0 \|_{L^2} \leq (\frac{\delta + 1}{C})^{2/(\sigma + 2)} \]
and again we obtain \( T^*(\omega_0) = + \infty \) in this case.

If \( \sigma > 2 \), let \( y(t) = \| Dv(t) \|_{L^2}^2 \); then the above inequality implies
\[ \frac{d}{dt} y(t) \leq 2E(\omega_0) + \frac{C}{\delta + 1} \| \omega_0 \|_{L^2}^{\sigma + 2} \| Dv(t) \|_{L^2}^2 \]
with \( \beta > 1 \). Let \( f(x) = x - \frac{C}{\delta + 1} \| \omega_0 \|_{L^2}^{\sigma + 2} x^{\beta} - 2E(\omega_0) \)
then \( f(y(t)) \leq 0 \) for all \( t \leq T^*(\omega_0) \). Since \( f'(x) = 1 - C\| \omega_0 \|_{L^2} \) and \( \beta > 0 \), \( f \) has a unique maximum on \( [0, T^*(\omega_0)] \). Moreover, if \( \| \omega_0 \|_{H^1} \) is sufficiently small, then
\[ f(u) = 1 - \frac{C}{\delta + 1} \| \omega_0 \|_{L^2}^{\sigma + 2} - 2E(\omega_0) > 0 \], i.e. \( x^* < 1 \) and
\[ f(y(t)) \leq 0, \| \omega_0 \|_{H^1} \) sufficiently small \( \Rightarrow y(t) \leq x^* \leq 1. \]
Hence, \( y(t) \leq 1, \forall t < T^*(u_0) \) and it follows that \( T^*(u_0) = \infty \).

3. Blow up in finite time (focusing case)

In the remaining cases, i.e. \( \sigma \geq 2 \) and \( \Delta = 1 \), then there actually exist initial data for which \( T^*(u_0) < \infty \). This is the object of the next theorem.

**Theorem 4.** Let \( \Delta = 1 \) and \( \sigma \geq 2 \); let \( u_0 \in H^1(\mathbb{R}) \) with \( E(u_0) < \frac{1}{2} \) then the local solution of (NLS) given in theorem 3 satisfies:

\[
T^*(u_0) < \infty \quad \text{(we say that the solution "blows up" in finite time)}.
\]

The argument of the proof of theorem 4 is formally very simple, but uses the evolution of the quantity

\[
I(t) = \frac{1}{2} \int_{\mathbb{R}} |x|^2 |u(t,x)|^2 dx.
\]

(called the virial identity). In order to make it rigorous, we first need to prove that this quantity is finite if

\[
u \in C([0,T^*]), H^1(\mathbb{R}) \quad \text{is a solution of (NLS)}.
\]

We denote

\[
\Sigma : = \{ u \in L^2(\mathbb{R}), \partial_x u \in L^2(\mathbb{R}), \partial_x u \in L^2(\mathbb{R}) \}.
\]

**Lemma 1.** Let \( u_0 \in \Sigma \), and let \( u \in C([0,T^*]), H^1(\mathbb{R}) \) be the maximal solution of (NLS) given by Theorem 3, with \( u(0) = u_0 \).

Then \( u \in C([0,T^*], \Sigma) \). Moreover, the function \( I \) defined by

\[
I(t) = \frac{1}{2} \int_{\mathbb{R}} |x|^2 |u(t,x)|^2 dx
\]

is \( C \) on \([0,T^*]\) and

\[
\frac{d}{dt} I(t) = - \text{Im} \int_{\mathbb{R}} x \overline{u(t,x)} \partial_x u(t,x) dx.
\]
Proof: In order to justify the computations, we need to use a cut-off argument. Let \( X \in C^\infty(\mathbb{R}) \), with 0 ≤ \( X \) ≤ 1, \( X \equiv 1 \) for \( 1 \leq x \leq 1 \) and \( X \equiv 0 \) for \( |x| \geq 2 \). We set \( X_n(x) = X(x/n) \) for \( n \in \mathbb{N} \). Let \( \mu \in \Sigma \), and let \( u \in C([0, T^*], H^1) \) be the maximal solution of (N15) given by Theorem 3. Then, since \( X_n \) has compact support, \( xX_nu \in C([0, T^*], H^1) \); moreover, for any \( t \) with 0 ≤ \( t \) ≤ \( T^* \), we may compute

\[
\frac{1}{2} \int_R x X_n u (x \mu (x))^2 dx = \Re \langle \partial_x u, x^2 X_n u \rangle_{H^{-1}, H^1} = \Re \langle i \Theta_{x} u \mu (x), x^2 X_n u \rangle_{H^{-1}, H^1}\]

\[
(1) \quad \frac{1}{2} \int_R x X_n u (x u (x))^2 dx = \Re \langle i \Theta_{x} u \mu (x), x^2 X_n u \rangle_{H^{-1}, H^1}
\]

So since \( \|X_n\|_{L^\infty} \leq 1 \), for any \( n \), the first term on the right-hand side above is bounded by

\[
2 \|\Theta_x u\|_{L^2} \|x X_n u\|_{L^2}
\]

by Cauchy-Schwarz inequality. For the second term, we observe that

\[
\|x X_n u(x)\|_{L^\infty} = \|x X(x/n)\|_{L^\infty} \leq C
\]

since \( \text{supp } X' \subset [-2, 2] \). It follows that the second term is also bounded by

\[
2 C \|\Theta_x u\|_{L^2} \|x X_n u\|_{L^2}
\]

Integrating the resulting inequality between 0 and \( T^* \), we find

\[
\frac{d}{dt} \|x X_n u(t)\|_{L^2}^2 - \frac{1}{2} \|x^2 X_n u(t)\|_{L^2}^2 \leq 2(T^{*} T^*) \int_0^t \|\Theta_x u(s)\|_{L^2}^2 \|x X_n u(s)\|_{L^2} ds
\]

Therefore, we can bound the growth of the solution.
for any $t$, with $0 \leq t \leq T$, where $T < T^*$ is fixed. Gronwall's Lemma then implies that
\[ \|x_t X_n u(t)\|_{L^2} \leq C T, \|x_t u\|_{L^2 L^2((0,T); H^1)} \]
for any $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we deduce that $u \in C([0,T], \Sigma)$. The continuity of $u$ is proved in the same way by bounding \( \|x_t X_n u(t) - x_t u(t)\|_{L^2} \) for $s \leq t$ and letting $n$ go to infinity.

Going back to equ. (1), integrating between $0$ and $t < T^*$, and taking the limit of the resulting expression as $n \to \infty$, we find, after noticing that
\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{k \leq N} \sum_{\lambda \in \Delta} (\partial_x u)(x) \partial_x x_n \partial_c x \partial_c x_n \partial_c x dx \partial_c x_n \partial_c x dx = 0, \]
we have
\[ \mathcal{I}(t) = \mathcal{I}(0) + 2 \int_{\mathbb{R}^d} \sum_{k \leq N} \sum_{\lambda \in \Delta} (\partial_x u)(x) \partial_x x_n \partial_c x \partial_c x_n \partial_c x dx \partial_c x_n \partial_c x dx dt. \]
This proves that $I$ is $C^1$ on $[0,T^*]$ and concludes the proof of the lemma. □

Lemma 2. Under the assumptions of Lemma 1, the function $t \mapsto \mathcal{I}(t)$ is $C^2$ on $[0,T^*)$ and
\[ \frac{d^2}{dt^2} \mathcal{I}(t) = 4 \mathcal{E}(\mathcal{E}_0) - 2(5-2) \|x_t u(t)\|_{L^2}^2. \]

Proof (Indication). We only give the formal computations here; in order to justify the computations, one would need to use a cut-off argument, as in the proof of Lemma 1 (and a regularization procedure).
\[
\frac{d}{dt} \text{Im} \int_R \overline{\phi(x)} \frac{\partial}{\partial x} u(t,x) \, dx \\
= \text{Im} \int_R \frac{\partial}{\partial t} \overline{\phi(x)} \, dx + \text{Im} \int_R \overline{\phi(x)} \frac{\partial}{\partial x} u(t,x) \, dx.
\]

We integrate the second term of the right hand side above by parts, and it follows
\[
\frac{d}{dt} \text{Im} \int_R \overline{\phi(x)} \frac{\partial}{\partial t} u(t,x) \, dx \\
= 2 \text{Im} \int_R x \frac{\partial}{\partial x} \overline{\phi(x)} u(t,x) \, dx - \text{Im} \int_R \overline{\phi(x)} \frac{\partial}{\partial x} u(t,x) \, dx.
\]

Using equ. (N5), we obtain
\[
\frac{d}{dt} \text{Im} \int_R \overline{\phi(x)} \frac{\partial}{\partial x} u(t,x) \, dx \\
= 2 \text{Im} \int_R x \left(-i \delta_x \overline{\phi(x)} + i \lambda u(t,x) \overline{\phi(x)} \right) \, dx - \text{Im} \int_R \overline{\phi(x)} \left(i \delta_x^u + i \delta_x \lambda u(t,x) \overline{\phi(x)} \right) \, dx \\
= \frac{1}{2} \int_R \lambda^2 \lambda^2 \overline{\phi(x)} dx - \frac{1}{2} \int_R \lambda^2 \overline{\phi(x)} \overline{\lambda^2} \overline{\phi(x)} dx + \int_R \frac{1}{2} \lambda^2 \overline{\phi(x)} \lambda^2 \overline{\phi(x)} \, dx \\
= 2 \int_R \lambda^2 \lambda^2 \overline{\phi(x)} dx + \frac{1}{2} \int_R \lambda^2 \overline{\phi(x)} \overline{\lambda^2} \overline{\phi(x)} dx \\
= 8 \delta E\left(\overline{\phi(x)} + (2 - \delta) \right) \int_R \lambda^2 \lambda^2 \overline{\phi(x)} dx.
\]

The conclusion follows by using lemma 1 and the conservation of energy. \(\Box\)

\(\text{end of the proof of Theorem. Assume} \ \delta > 2 \ \text{and} \ u \in \Sigma, \ \text{with} \ E(u) < 0; \ \text{then lemma 2 implies that} \ \forall t < \tau, \ \frac{d^2}{dt^2} \|x^2 u(t)\|_L^2 \leq 8 \delta E(u) < 0.\)

Integrating twice in time the above inequality, we obtain
\[
\|x^2 u(t)\|_L^2 \leq 4 \delta E(u) \tau^2 + 4 \text{Im} \int_R x \overline{\phi(x)} \frac{\partial}{\partial x} u(x) \, dx + \|x^2 u_0\|_L^2.
\]
The right hand side of the above inequality goes to $-\infty$ as $t \to T^*$ while the left hand side is positive. We deduce that $T^* < \infty$, necessarily. \[ \square \]

Remarks:

- It is easy to see that for $\sigma > 2$ and $d = 1$, there indeed exist initial data $u_0 \in \Sigma$ such that $E(u_0) < 0$ (exercise).
- In the critical case $\sigma = 2$, using the Gagliardo-Nirenberg inequality 
  \[ \|u\|_{L^{p+2}} \leq C \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}, \] 
  one may check that $E(u_0) < 0$ implies that $\|u_0\|_{L^2}$ is "not too small", which is coherent with the statement of Theorem 3.
- The condition on $u_0$ in order that the solution blows up in finite time may be weakened.
- The question of the behaviour of the solution for $t$ close to $T^*$ ($t < T^*$) is a difficult question; it has been the object of several developments in the recent years in the critical case $\sigma = 2$.