Nonlinear dispersive PDEs and applications in optics (2013-2014)

Aim = modeling, mathematical and numerical analysis of equations arising in nonlinear optics, i.e., description of the propagation of light through matter, when nonlinear effects have to be taken into account (i.e., nonlinear response of the medium).

We will be in particular interested in the propagation of light in optical fibers, especially on long-distance propagation (several thousands of km, e.g., transoceanic propagation).

Optical fiber = fiber made of glass or plastic, which plays the role of a wave guide in the propagation of light.

\[
\begin{array}{c}
E \\
\rightarrow \\
\text{propagation}
\end{array}
\]

Of course, this has very important applications, in particular in communications. First experiments in optical fibres ±1960; strong development since ±1990 due to

- the development of "lasers" (which allowed the use of quasi-monochromatic incident light wave)
- the development of engineering techniques, and in particular of optical amplifiers, which allowed to compensate for loss in the signal on very long distances.

Consequence: limiting factors in the propagation are not due to dispersion, but to chromatic dispersion. Mathematically, this means that equations modelling the propagation are not dissipative, but dispersive equations, taking account of nonlinear response of the medium, the typical model is the nonlinear Schrödinger equation
for the "envelop" of the electric field:

(E) \[ \frac{\partial E}{\partial t} - \frac{1}{\varepsilon} \frac{\partial^2 E}{\partial x^2} + \omega_0 \mu_0 \varepsilon_0 E = 0 \]

for \((\varphi, t) \in \mathbb{R}^2\), \(E(\varphi, t) = \varepsilon(t - k_0 \varphi)\), \(\varepsilon = \varepsilon_0 \varepsilon_1\). \(\varepsilon\) is a small parameter, and \(k_0\) is a constant, called "group velocity dispersion" (all the other constants will be explained later).

**Dispersion:** Consider an equation of the form \(Lu = \mu\), where \(U = U(t, x) \in \mathbb{R}\) or \(\mathbb{C}\), \(t \in \mathbb{R}^+\), \(x \in \mathbb{R}\) (or \(\mathbb{R}^d\)), and \(L\) is a differential or pseudo-differential operator with real coefficients defined by \(Lu(\psi) = \int \psi' \mu(\varphi) \psi(\varphi) d\varphi\), with \(\mu\) a real-valued function and \(\hat{\mu}(\varphi) = \int e^{-i\varphi \cdot x} \mu(x) dx\) is the Fourier transform in space of \(\mu\).

Looking for a plane wave solution \(U(t, x) = e^{i(\varphi_0 \cdot x - \omega_0 t)}\) of the equation, one obtains the "dispersion relation" \(G(\omega_0, \varphi_0) = 0\), which here has the form \(\omega_0 = \Omega(\varphi) = -P(\varphi)\). Indeed, if \(u(x) = e^{i\varphi_0 \cdot x}\), for some fixed \(\varphi_0 \in \mathbb{R}^d\), then \(\hat{u}(\varphi_0) = \delta_{\varphi_0}\), where \(\delta_{\varphi_0}\) is the "delta function", i.e., the measure defined by \([\delta_{\varphi_0}, \psi] = \psi(\varphi_0)\) for any \(\psi \in C^\infty(\mathbb{R}^d)\); hence \(\hat{u}(t, \varphi) = e^{i\varphi \cdot x} e^{-i\varphi_0 \cdot x} \delta_{\varphi_0}\) and \(u(t, x) = e^{i(\varphi_0 \cdot x + \varphi \cdot (t - x))}\), hence \(\omega(\varphi) = -P(\varphi)\).

**Def:** The equation \(\Delta \mu = \mu\) is said "dispersive" if \(\omega(\varphi)\) is real and \(\text{det} \left( \begin{array}{cc} \delta_{\varphi_0} & \delta_{\varphi_0} \\ \delta_{\varphi_0} & \delta_{\varphi_0} \end{array} \right) \text{det} \delta_{\varphi_0} \neq 0\). In space
dimension $d$ ($d=1$), we simply have $w''(t) \neq 0$. The definition means that the "group velocity" $\mathbf{v}_G$ really depends on the wave vector $k$, i.e., wave vectors travel with different velocities (it follows that "wave packets" have a tendency to disperse).

**Examples:**

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} - \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u = 0, \forall x \in \mathbb{R}$$

- Wave equation: $w(t) = \pm c \xi \Rightarrow w''(t) = 0$, not dispersive
- Written using spatial Fourier transform as
  $$\left( \frac{\partial^2}{\partial t^2} + (1+c^2 \xi^2) \right) \hat{u} = \left( \frac{\partial}{\partial t} - i \sqrt{1+c^2 \xi^2} \right) \left( \frac{\partial}{\partial t} + i \sqrt{1+c^2 \xi^2} \right) \hat{u} = 0 \Rightarrow w(t) = \pm \sqrt{1+c^2 \xi^2}, \text{ dispersive since } w''(t) \neq 0, \text{ but not "strongly" dispersive, since } w'(t) \text{ is bounded (and } w''(t) \to \infty \text{ as } |\xi| \to \infty).$$
- $i \chi u + \beta \chi^3 u + \beta \chi x u = 0, x \in \mathbb{R}$; Airy equation (linearization of the Korteweg-de Vries equation); $w(t) = 1/3 - \beta \xi^2$, dispersive (note that $w''(t) \to \infty$ as $|\xi| \to \infty$)
- $i \delta u + \beta \delta x u = 0, x \in \mathbb{R}$; Schrödinger equation; $w(t) = \pm \xi^2$, dispersive; the equation may also be written in dimension $d = 1$ as $\Delta u + \beta u = 0$; still dispersive with the above definition.

Nonlinear model examples:

Most classical examples: Korteweg-de Vries (KdV) and nonlinear Schrödinger (NLS) equations, obtained as asymptotic models from basic equations of physics (Maxwell equ. in optics, Euler equ. in fluid mechanics) assuming that there is a small parameter...
KdV equation (Hajj, Boussinesq, 1872)
\[ \partial_t u + u \partial_x u + \partial_x^3 u + \partial_x (u^2) = 0, \; u \in \mathbb{R}, \; t > 0 \]

Describes evolution of the surface elevation of water under some specific assumptions (long waves, small amplitude, small depth, no transverse effects)

Benjamin-Ono (1967)
\[ \partial_t u + \partial_x \partial_x^2 u + \partial_x \Theta^{-1} (u) + \partial_x (u^2) = 0, \; u \in \mathbb{R}, \; t > 0 \]

\[ \Theta = \text{Hilbert transform in space, defined by} \]
\[ \tilde{\Theta} f (\xi) = i \text{sgn}(\xi) \hat{f} (\xi), \quad \text{sgn}(\xi) = \frac{1}{i} \text{ for } \xi 
eq 0 \]

Again, model for water waves (long waves); internal waves in stratified fluids.

Alternative equation for KdV: BBSN (or RLW - regularized long wave equ):
\[ \partial_t u + \partial_x^2 u + \partial_x (u^2) = 0, \; u \in \mathbb{R}, \; t > 0 \]
may be written as \[ \partial_t u = - (1 + \partial_x^2)^{-1} \partial_x (u^2 + u) \]
with \[ (1 + \partial_x^2)^{-1} u (\xi) = \frac{\hat{u} (\xi)}{1 + \xi^2} \]; hence \[ p (\xi) = - w (\xi) = \int \frac{\xi^2}{1 + \xi^2} \]; the group velocity \( w (\xi) \) is bounded \& small dispersion (compared to KdV, the equation has no smoothing effect, even though the two equations are equivalent models for small \( \xi \))

NLS equation:
\[ i \partial_t u + \partial_x (u + \partial_x u) \partial_x u = 0, \; u \in \mathbb{R}^d, \; t > 0, \; \lambda (t,x) \in C, \; \lambda = \pm 1, \; a > 0. \] This equ. is a model in plasma physics, nonlinear optics, water surface waves; usually, \( \tau = 1 \) (KdV equation)

Another model of the same type: Darcy-Steinnes
\[ \partial_t u + \partial_x^2 u + \partial_x u = X (u^2 u + bu \partial_x u) \]
\[ \partial_x p + m \partial_x u = \partial_x (u^2 u) \]
with \( \delta_t = x^2 = 1, m, \beta \in \mathbb{R} \), \((x, y) \in \mathbb{R}^2\), \(u(t, x) \in \mathbb{C}\), \(v(t, x) \in \mathbb{R}\); all signs are possible except \( \delta = -1 \) and \( m < 0 \); when \( \delta = 1 \) and \( m > 0 \), the system may be written as a single (NLS) equation with a "nonlocal" term:

\[
\frac{i \delta_t u + \delta_x u}{\partial_t} = X \left( \partial_x^2 u \right) + \beta u E(u)
\]

where \( E(u) = \Delta_m \partial_x^2 u \), \( \Delta_m = \partial_x^2 + m \partial_y^2 \), i.e., \( E(u) \partial_x^2 u = \frac{\partial_x^2 u}{\partial_x^2 + m \partial_y^2} \).

If \( \gamma = (s_1, s_2) \), again, model for \((\gamma, \partial)\) surface waves

**Common properties**

- **Conservative equations** (conservation of energy, charge (or mass), ...): For the (NLS) equation, \( E(u) = \frac{1}{2} \int |u(\cdot)\partial_x u|^2 \partial_t^2 + \beta \int u E(u) |u|^2 \partial_t^2 \ dx \)
- **Existence of solitary wave solutions**, e.g., traveling waves:
  \( u(t, x) = \psi(x - ct) \) or stationary states: \( u(t, x) = e^{i \omega} \psi(x) \), or a combination of both.

For the \( 1 \)D cubic (NLS) equation, solitary waves are called "solitons." Solitons are used in optical fibers to code for bits of information (soliton = 1, absence of soliton = 0); actually, solitons are replaced by pulses of light; propagation over very long distances (several thousands of km) is explained by the stability of the solitons; hence dispersion in optical fibers is compensated thanks to nonlinearity.

Another solution to compensate dispersion: "dispersion-managed fiber." Concatenation of fibers with alternating signs of dispersion

\[
\begin{align*}
\pm \cdots & \\
& \Rightarrow \quad \delta \quad \varepsilon \\
& \varepsilon = \varepsilon^2
\end{align*}
\]
The study of the corresponding models will be the object of the second part of the course.

Outline of the course:

I - Modeling in nonlinear optics: formal derivation of the (NLS) equation from Maxwell equ., for propagation of an electromagnetic wave in an optical fiber

II - Mathematical study of the (1-D) (NLS) equation

III - Numerical analysis (finite differences, splitting methods)

IV - Mathematical analysis of models in dispersion-managed fibers (long time behavior: effective equations; existence of stationary states; regularity and decay of stationary states)

Some references:

I - Modeling:
A. Newell, J. Moloney, Nonlinear optics
G. Agrawal, Nonlinear fiber optics

II - Mathematical analysis of NLS:
T. Cazenave, Semilinear Schrödinger equ., Courant Institute
P.L. Sulem, C. Sulem, The nonlinear Schrödinger equ., self focusing and wave collapse

III - Numerical analysis: