THE KORTEWEG-DE VRIES EQUATION WITH MULTIPLICATIVE HOMOGENEOUS NOISE

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Abstract. We prove the global existence and uniqueness of solutions both in the energy space and in the space of square integrable functions for a Korteweg-de Vries equation with noise. The noise is multiplicative, white in time, and is the muliplication by the solution of a homogeneous noise in the space variable.

1. Introduction and statement of the results

The aim of the paper is to prove the global existence and uniqueness of strong solutions for a Korteweg-de Vries equation with noise, which may be written in Itô form as

$$du + (\partial_x^3 u + \frac{1}{2}\partial_x(u^2))dt = u\phi dW \tag{1.1}$$

where u is a random process defined on $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, W is a cylindrical Wiener process on $L^2(\mathbb{R})$ and ϕ is a convolution operator on $L^2(\mathbb{R})$ defined by

$$\phi f(x) = \int_{\mathbb{R}} k(x-y)f(y)dy, \text{ for } f \in L^2(\mathbb{R})$$

where the convolution kernel k is an $H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ function of $x \in \mathbb{R}$. Here $H^1(\mathbb{R})$ is the usual Sobolev space of square integrable functions of the space variable x, having their first order derivative in $L^2(\mathbb{R})$. Considering a complete orthonormal system $(e_i)_{i\in\mathbb{N}}$ in $L^2(\mathbb{R})$, we may alternatively write W as

$$W(t,x) = \sum_{i\in\mathbb{N}} \beta_i(t)\phi e_i(x), \tag{1.2}$$

$(\beta_i)_{i\in\mathbb{N}}$ being an independent family of real valued Brownian motions. Hence, the correlation function of the process ϕW is

$$\mathbb{E}(\phi W(t,x)\phi W(s,y)) = c(x-y)(s \wedge t), \quad x, y \in \mathbb{R}, \quad s, t > 0,$$

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where

$$c(z) = \int_{\mathbb{R}} k(z+u)k(u)du.$$

The existence and uniqueness of solutions for stochastic KdV equations of the type (1.1) but with an additive noise have been studied in [4], [7], [8]. Here we extend those results to equation (1.1), that is the multiplicative case with homogeneous noise.

Note that an equation of this form, but with an additional weak dissipation has been considered in [13]. Indeed, in this latter case where a dissipative term is added, such a noise may be viewed as a perturbation of the dissipation. Although our existence and uniqueness results would easily extend to the case where weak dissipation is added, the dissipative term is of no help in the existence proof, so we prefer stating the result for equation (1.1).

Assuming $k \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ will allow us to prove the global existence and uniqueness of solutions to equation (1.1) in the energy space $H^1(\mathbb{R})$, that is in the space where both quantities

$$m(u) = \frac{1}{2}\int_{\mathbb{R}} u^2(x)dx \tag{1.3}$$

and

$$H(u) = \frac{1}{2}\int_{\mathbb{R}} (\partial_x u)^2 dx - \frac{1}{6}\int_{\mathbb{R}} u^3 dx \tag{1.4}$$

are well defined. Note that m and H are conserved quantities for the equation without noise, that is

$$\partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x(u^2) = 0. \tag{1.5}$$

It is important to solve equation (1.1) in the energy space, indeed most of the studies on the qualitative behavior of the solutions are done in this space. One of our aim in the future is to analyse the qualitative influence of a noise on a soliton solution of the deterministic equation, as we did in the additive case in [6], and this requires the use of the hamiltonian (1.4). However, our method of construction of solutions easily extends to treat the case of a kernel $k \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, obtaining global existence and uniqueness in $L^2(\mathbb{R})$. It seems difficult to get a result with less regularity.

It may be noted also that the use of a noise of the form given in (1.1) naturally brings some localization in the noisy part of the equation, at least in the limit where the amplitude of the noise goes to zero, and when the initial state is a solitary wave – or soliton – solution of the deterministic equation, that is a well localized solution which propagates with a constant shape and velocity. This localization in the noise was a missing ingredient in the study of the influence of an additive noise on the propagation of a soliton (see [6]).

The precise existence result is the following, and the method we use to prove it closely follows the method in [7].

Theorem 1.1. *Assume that the kernel k of the noise satisfies $k \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, $s = 0$ or 1. Then for any u_0 in $H^s(\mathbb{R})$, there is a unique adapted solution u with paths almost surely in $C(\mathbb{R}^+; H^s(\mathbb{R}))$ of equation (1.1). Moreover, $u \in L^2(\Omega; C(\mathbb{R}^+; L^2(\mathbb{R})))$.*

As in [7, 8], we use the functional framework introduced by Bourgain to study dispersive equations. Following [3], [14], [15], for $s, b \in \mathbb{R}$, $X_{b,s}$ denotes the space of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^2)$ for which the norm

$$\|f\|_{X_{b,s}} = \left(\int\!\!\int_{\mathbb{R}^2} (1+|\xi|)^{2s} (1+|\tau-\xi^3|)^{2b} |\hat{f}(\tau,\xi)|^2 d\tau d\xi \right)^{1/2}$$

is finite, where $\hat{f}(\tau,\xi)$ stands for the space-time Fourier transform of $f(t,x)$. In the same way, we set for $b, s_1, s_2 \in \mathbb{R}$,

$$\|f\|_{\widetilde{X}_{b,s_1,s_2}} = \left(\int\!\!\int_{\mathbb{R}^2} |\xi|^{2s_2} (1+|\xi|)^{2s_1} (1+|\tau-\xi^3|)^{2b} |\hat{f}(\tau,\xi)|^2 d\tau d\xi \right)^{1/2}$$

and

$$\widetilde{X}_{b,s_1,s_2} = \left\{ f \in \mathcal{S}'(\mathbb{R}^2),\ \|f\|_{\widetilde{X}_{b,s_1,s_2}} < +\infty \right\}.$$

Note that the use of the space $\widetilde{X}_{b,s_1,s_2}$ is necessary here. Indeed, since we work with stochastic equations driven by white in time noises, we cannot require too much time regularity, and we have to choose $0 < b < 1/2$. But then the bilinear estimate which allows to treat the KdV equation is not true in the space $X_{b,s}$ as was already mentioned for the additive case in [7].

For $T \geq 0$, we also introduce the spaces $X^T_{b,s}$ and $\widetilde{X}^T_{b,s_1,s_2}$ of restrictions to $[0,T]$ of functions in $X_{b,s}$ and $\widetilde{X}_{b,s_1,s_2}$. They are endowed with

$$\|f\|_{X^T_{b,s}} = \inf \left\{ \|\widetilde{f}\|_{X_{b,s}},\ \widetilde{f} \in X_{b,s} \text{ and } f|_{[0,T]} = \widetilde{f}|_{[0,T]} \right\}$$

$$\|f\|_{\widetilde{X}^T_{b,s_1,s_2}} = \inf \left\{ \|\widetilde{f}\|_{\widetilde{X}_{b,s_1,s_2}},\ \widetilde{f} \in \widetilde{X}_{b,s_1,s_2} \text{ and } f|_{[0,T]} = \widetilde{f}|_{[0,T]} \right\}.$$

Because equation (1.1) is a multiplicative equation with a nonlinear deterministic part, we have to consider first a cut-off version of this equation (see Section 2). As we make use of the functional framework defined above, the cut-off will arise as a function of the norm of the solution of the type $\|\cdot\|_{X^t_{b,s}}$. Moreover, this function of the norm must be a regular function, in order to allow us to use a fixed point argument (i.e. in order that our mapping is a contraction mapping, see Section 2). The fact that the functional spaces we consider are nonlocal spaces in the time variable then brings a lot of technical difficulties, concerning points that would be obvious if we were dealing with more classical function spaces (see e.g. the proof of Lemma 2.1).

The paper is organized as follows: Section 2 is devoted to the proof of several preliminary lemmas and propositions, which once brought together lead quite easily to the proof of global existence and uniqueness for the cut-off version of the equation – or to the local existence and uniqueness for equation (1.1). In Section 3 we prove that the solutions of equation (1.1) are global in time, by using estimates on the moments of the L^2-norm of the solution. Again, due to the spaces we consider for the local existence, the globalization argument is not obvious.

2. Preliminaries and existence for a truncated equation

As is usual, we introduce the mild form of the stochastic Korteweg-de Vries equation (1.1). We denote by $U(t) = e^{-t\partial_x^3}$ the unitary group on $L^2(\mathbb{R})$ generated by the linear equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0.$$

Using Fourier transform, we have $\mathcal{F}(U(t)v)(\xi) = e^{it\xi^3}\mathcal{F}(v)(\xi)$. We then rewrite (1.1) as follows

$$u(t) = U(t)u_0 - \frac{1}{2}\int_0^t U(t-r)\partial_x(u^2(r))dr + \int_0^t U(t-r)(u(r)\phi dW(r)),\ t \geq 0. \tag{2.1}$$

The $X_{b,s}$ and $\widetilde{X}_{b,s_1,s_2}$ norms defined in the introduction have the nice property that they are increasing with T. However, it is more convenient to work with other norms, given by the multiplication by the function $\mathbb{1}_{[0,T]}$. In the case we consider here, that is $0 \leq b < 1/2$, we can prove the following result, stating that the two norms are equivalent.

Lemma 2.1. *Let $s \geq 0$ and $0 \leq b < 1/2$, then there exist two constants C_1, C_2 depending on b but not on T such that for any $f \in X_{b,s}$*

$$C_1\|f\|_{X^T_{b,s}} \leq \|\mathbb{1}_{[0,T]}(t)f\|_{X_{b,s}} \leq C_2\|f\|_{X^T_{b,s}}.$$

Proof. The first inequality is clear and in fact we may choose $C_1 = 1$. For the other inequality, let us set $g(t) = \mathbb{1}_{[0,T]}(t)U(-t)f(t)$ so that

$$\begin{aligned}\|\mathbb{1}_{[0,T]}(t)f\|^2_{X_{b,s}} &= \iint_{\mathbb{R}^2}(1+|\xi|)^{2s}(1+|\tau|)^{2b}|\hat{g}(\tau,\xi)|^2 d\tau d\xi \\ &= \iint_{\mathbb{R}^2}(1+|\xi|)^{2s}\|(\mathcal{F}_x g)(\cdot,\xi)\|^2_{H^b_t}d\xi.\end{aligned}$$

The result follows from the following inequality

$$\|\mathbb{1}_{[0,T]}h\|_{H^b(\mathbb{R})} \leq C\|h\|_{H^b(\mathbb{R})},\ h \in H^b(\mathbb{R}),$$

which holds for a constant $C \geq 0$ depending on $0 < b < 1/2$. To prove this, we use the following equivalent norm on $H^b(\mathbb{R})$ (see for instance [1]):

$$\|h\|^2_{H^b(\mathbb{R})} = \iint_{\mathbb{R}^2}\frac{|h(t)-h(r)|^2}{|t-r|^{1+2b}}dtdr + \|h\|^2_{L^2(\mathbb{R})}.$$

Clearly, $\|\mathbb{1}_{[0,T]}h\|^2_{L^2(\mathbb{R})} \leq \|h\|^2_{L^2(\mathbb{R})}$. Moreover

$$\begin{aligned}&\iint_{\mathbb{R}^2}\frac{\left|\mathbb{1}_{[0,T]}(t)h(t) - \mathbb{1}_{[0,T]}(r)h(r)\right|^2}{|t-r|^{1+2b}}drdt = 2\iint_{r<t}\frac{\left|\mathbb{1}_{[0,T]}(t)h(t) - \mathbb{1}_{[0,T]}(r)h(r)\right|^2}{|t-r|^{1+2b}}drdt \\ &= 2\int_0^T\int_0^t\frac{|h(t)-h(r)|^2}{|t-r|^{1+2b}}drdt + 2\int_0^T\int_{-\infty}^0\frac{|h(t)|^2}{|t-r|^{1+2b}}drdt + 2\int_T^\infty\int_0^T\frac{|h(r)|^2}{|t-r|^{1+2b}}drdt \\ &= I + II + III.\end{aligned}$$

The first term I is less than $\|h\|_{H^b(\mathbb{R})}$. The second and third terms are equal to

$$II = \frac{1}{b}\int_0^T |t|^{-2b}|h(t)|^2dt, \; III = \frac{1}{b}\int_0^T |T-r|^{-2b}|h(r)|^2dr.$$

Both are bounded by $C\|h\|_{H^b(0,T)}$. To see this, we note that it is obvious for $b=0$ and results from Hardy inequality for $b=1$ when $H^b(0,T)$ is replaced by $H^1_0(0,T)$. The result follows by interpolation since, for $0 \le b < 1/2$, $H^b(0,T) = H^b_0(0,T)$. □

We now define $Y^T_{b,0} = X^T_{b,0} \cap \widetilde{X}^T_{b,0,-3/8}$ endowed with the norm

$$\|f\|_{Y^T_{b,0}} = \max\{\|f\|_{X^T_{b,0}},\ \|f\|_{\widetilde{X}^T_{b,0,-3/8}}\}.$$

We also use the space $Y^T_{b,1} = X^T_{b,1} \cap \widetilde{X}^T_{b,1,-3/8}$ with a similar definition of its norm. From now on and thanks to Lemma 2.1, we will use the definition $\|v\|^T_{Y_{b,s}} = \|\mathbb{1}_{[0,T]}v\|_{Y_{b,s}}$ each time we take a norm in $Y^T_{b,s}$ with $0 \le b < 1/2$.

For $u_0 \in L^2(\Omega; H^1(\mathbb{R}))$, we set

$$z(t) = U(t)u_0, \text{ and } v(t) = u(t) - z(t).$$

Then (2.1) is rewritten as

$$\begin{aligned} v(t) = \; & -\tfrac{1}{2}\int_0^t U(t-r)\left[\partial_x(v^2(r)) + \partial_x(z^2(r)) + 2\partial_x(z(r)v(r))\right]dr \\ & + \int_0^t U(t-r)((z(r)+v(r))\phi dW(r)),\ t \ge 0. \end{aligned} \tag{2.2}$$

Let θ be a cut-off function – $\theta(x) = 0$ for $x \ge 2$, $\theta(x) = 1$ for $0 \le x \le 1$, with $\theta \in C_0^\infty(\mathbb{R}^+)$ – and let $\theta_R = \theta(\frac{\cdot}{R})$; we consider the cut-off version of (2.2) written for $R > 0$ as:

$$\begin{aligned} v_R(t) = \; & -\frac{1}{2}\int_0^t U(t-r)\left[\theta_R^2\left(\|v_R\|_{Y^r_{b,0}}\right)\partial_x(v_R^2(r))\right]dr \\ & -\int_0^t U(t-r)\left[\theta_R\left(\|v_R\|_{Y^r_{b,0}}\right)\partial_x(z(r)v_R(r))\right]dr \\ & -\frac{1}{2}\int_0^t U(t-r)\left[\partial_x(z^2(r))\right]dr \\ & + \int_0^t U(t-r)((z(r)+v_R(r))\phi dW(r)),\ t \ge 0. \end{aligned} \tag{2.3}$$

We find v_R as a fixed point of the mapping $\mathcal{T}_R$, $\mathcal{T}_R v_R$ being defined by the right hand side above. Note that the cut-off is made in the L^2 in space norm, even for the H^1 result. We will choose $0 < b < 1/2$ and $1/2 < c$.

We use the following Lemma.

Lemma 2.2. *For any $0 \le b < 1/2$, $R > 0$, $v \in Y_{b,1}^T$, there exists $C(R)$ such that*

$$\left\| \theta_R \left(\|v\|_{Y_{b,0}^t} \right) v(t) \right\|_{Y_{b,0}^T} \le C(R)$$

and, for $s = 0$ or 1, there is a positive constant C, independent of R, such that

$$\left\| \theta_R \left(\|v\|_{Y_{b,0}^t} \right) v(t) \right\|_{Y_{b,s}^T} \le C \left\| v(t) \right\|_{Y_{b,s}^T}.$$

Proof. We use arguments similar to the proof of Lemma 2.1. Let $w(t) = U(-t)v(t)$ then, using the same norm on $H^b(\mathbb{R})$ as in Lemma 2.1,

$$\left\| \theta_R \left(\|v\|_{Y_{b,0}^t} \right) v(t) \right\|_{X_{b,0}^T}^2 \le C \int_{\mathbb{R}} \left\| \theta_R \left(\|v\|_{Y_{b,0}^t} \right) (\mathcal{F}_x w)(t, \xi) \right\|_{H_t^b([0,T])}^2 d\xi.$$

The L^2 part of the H^b norm above is easily estimated, while the other part is bounded above by

$$\begin{aligned}
&C \int_{\mathbb{R}} \int_0^T \int_0^t \theta_R^2 \left(\|v\|_{Y_{b,0}^t} \right) \frac{|(\mathcal{F}_x w)(t,\xi) - (\mathcal{F}_x w)(r,\xi)|^2}{|t-r|^{1+2b}} dr dt d\xi \\
&+ C \int_{\mathbb{R}} \int_0^T \int_0^t \left(\theta_R \left(\|v\|_{Y_{b,0}^t} \right) - \theta_R \left(\|v\|_{Y_{b,0}^r} \right) \right)^2 \frac{|(\mathcal{F}_x w)(r,\xi)|^2}{|t-r|^{1+2b}} dr dt d\xi \\
&= I + II.
\end{aligned}$$

Next, we define $\tau_R = \inf\{t \ge 0,\ \|v\|_{Y_{b,0}^t} \ge 2R\}$; then $\theta_R \left(\|v\|_{Y_{b,0}^t} \right) = 0$ for $t \ge \tau_R$ and

$$\begin{aligned}
I \quad &\le C \int_{\mathbb{R}} \int_0^{\tau_R} \int_0^t \frac{|(\mathcal{F}_x w)(t,\xi) - (\mathcal{F}_x w)(r,\xi)|^2}{|t-r|^{1+2b}} dr dt d\xi \\
&\le C \int_{\mathbb{R}} \|(\mathcal{F}_x w)(\cdot, \xi)\|_{H^b(0,\tau_R)}^2 d\xi \le C \|v\|_{X_{b,0}^{\tau_R}} \le 2CR.
\end{aligned}$$

In order to estimate II, we use the fact that for $r < t$,

$$\begin{aligned}
&\left(\theta_R \left(\|v\|_{Y_{b,0}^t} \right) - \theta_R \left(\|v\|_{Y_{b,0}^r} \right) \right)^2 \le \frac{C}{R^2} |\theta'|_{L^\infty}^2 |\|v\|_{Y_{b,0}^t} - \|v\|_{Y_{b,0}^r}|^2 \\
&\le \frac{C}{R^2} |\|\mathbb{1}_{[0,t]} v\|_{Y_{b,0}} - \|\mathbb{1}_{[0,r]} v\|_{Y_{b,0}}|^2 \le \frac{C}{R^2} \|\mathbb{1}_{[r,t]} v\|_{Y_{b,0}}^2 \\
&\le \frac{C}{R^2} \int_{\mathbb{R}} (1 + |\eta|^{-3/4}) \|\mathbb{1}_{[r,t]} (\mathcal{F}_x w)(., \eta)\|_{H^b}^2 d\eta.
\end{aligned}$$

We leave to the reader the estimate of the contribution to II of the L^2 part of the H^b norm above; indeed, it follows the same line as the estimate of the remaining contribution, which is bounded above by

$$\frac{C}{R^2}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\eta|^{-3/4})\int_0^{\tau_R}\int_0^t\int_r^t\int_r^{\sigma_2}\frac{|(\mathcal{F}_x w)(\sigma_2,\eta)-(\mathcal{F}_x w)(\sigma_1,\eta)|^2}{|\sigma_2-\sigma_1|^{1+2b}}d\sigma_1 d\sigma_2$$
$$\times\frac{|(\mathcal{F}_x w)(r,\xi)|^2}{|t-r|^{1+2b}}drdtd\eta d\xi$$
$$\leq\frac{C}{R^2}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\eta|^{-3/4})\int_0^{\tau_R}\int_0^{\sigma_2}\int_0^{\sigma_1}\left(\int_{\sigma_2}^{\tau_R}\frac{dt}{|t-r|^{1+2b}}\right)|(\mathcal{F}_x w)(r,\xi)|^2 dr$$
$$\times\frac{|(\mathcal{F}_x w)(\sigma_2,\eta)-(\mathcal{F}_x w)(\sigma_1,\eta)|^2}{|\sigma_2-\sigma_1|^{1+2b}}d\sigma_1 d\sigma_2 d\eta d\xi$$

where we have inverted the integrals in the time variables; this last term is in turn bounded above by

$$\frac{C}{R^2}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\eta|^{-3/4})\int_0^{\tau_R}\int_0^{\sigma_2}\left(\int_0^{\sigma_1}|\sigma_1-r|^{-2b}|(\mathcal{F}_x w)(r,\xi)|^2 dr\right)$$
$$\times\frac{|(\mathcal{F}_x w)(\sigma_2,\eta)-(\mathcal{F}_x w)(\sigma_1,\eta)|^2}{|\sigma_1-\sigma_2|^{1+2b}}d\sigma_1 d\sigma_2 d\eta d_\xi,$$

by the fact that $|\tau_R-r|^{-2b}\leq|\sigma_2-r|^{-2b}\leq|\sigma_1-r|^{-2b}$ for $0\leq r\leq\sigma_1\leq\sigma_2\leq t\leq\tau_R$.

Using then the same arguments as in the proof of Lemma 2.1, we finally get

$$\begin{aligned}II &\leq \frac{C}{R^2}\int_{\mathbb{R}}\|(\mathcal{F}_x w)(.,\xi)\|^2_{H^b(0,\tau_R)}d\xi\int_{\mathbb{R}}(1+|\eta|^{-3/4})\|(\mathcal{F}_x w)(.,\eta)\|^2_{H^b(0,\tau_R)}d\eta\\ &\leq \frac{C}{R^2}\|v\|^2_{Y^{\tau_R}_{b,0}}\|v\|^2_{X^{\tau_R}_{b,0}}.\end{aligned}$$

This, together with the estimate of I implies the first inequality of the Lemma for the $X_{b,0}$ part of the $Y_{b,0}$ norm; the $\widetilde{X}_{b,0,-3/8}$ part, and the second inequality of the Lemma are proved in the same way. □

Next results state the estimates on the bilinear term appearing in (2.3).

Proposition 2.3. *Let $a>0$, $0<b<1/2<c<1$, with $b+c>1$ and a,b,c sufficiently close to $1/2$, then for any $v\in Y^T_{b,s}$, $z\in X^T_{c,s}$, $s=0$ or 1, we have*

$$\|\partial_x(v^2)\|_{Y^T_{-a,s}}\leq C\|v\|_{Y^T_{b,0}}\|v\|_{Y^T_{b,s}},$$

$$\|\partial_x(vz)\|_{Y^T_{-a,s}}\leq C\left(\|v\|_{Y^T_{b,0}}\|z\|_{X^T_{c,s}}+\|v\|_{Y^T_{b,s}}\|z\|_{X^T_{c,0}}\right),$$

and

$$\|\partial_x(z^2)\|_{Y^T_{-a,s}}\leq C\|z\|_{X^T_{c,s}}\|z\|_{X^T_{c,0}}.$$

Proof. These estimates are proved in [7], Proposition 2.2 and 2.3, for $s=0$. These are easily extended to $s=1$. It suffices to add a factor $1+|\xi|$ in the expression

$$\begin{aligned}&|\langle f,\partial_x(gh)\rangle|\\ &=\left|\int_\tau\int_\xi \xi\hat f(\tau,\xi)\int_{\tau_1}\int_{\xi_1}\overline{\hat g(\tau-\tau_1,\xi-\xi_1)\hat{h}(\tau_1,\xi_1)}d\tau_1 d\xi_1\,d\tau d\xi\right|,\end{aligned}$$

and to use the fact that $1+|\xi|\le(1+|\xi-\xi_1|)+(1+|\xi_1|)$. □

Remark 2.4. *It does not seem possible to get rid of the homogeneous Sobolev space, i.e. of $\widetilde{X}_{b,0,-3/8}$, to get the result of Proposition 2.3, when $b<1/2$, even in the case $s=1$; indeed, a careful reading of the proof of Proposition 2.2 in [7] shows that the additional factor $|\xi|^{-3/4}$ induced by the use of $\widetilde{X}_{b,0,-3/8}$ is necessary in a region of the integral where $|\xi_1|\ll|\xi|$, so that $|\xi-\xi_1|\sim|\xi|$, and with moreover $|\xi|\le 1$; hence the supplementary factor $(1+|\xi-\xi_1|)(1+|\xi_1|)/(1+|\xi|)$ is of no help there.*

It remains to derive the estimates on the stochastic integrals in (2.3). In order to be able to globalize the solutions in Section 3, we will need estimates on all the moments of the stochastic integrals.

Proposition 2.5. *Let $m\in\mathbb{N}$, $s=0$ or 1, and $v\in L^{2m}(\Omega,X^T_{b,s})$; then for any $0\le b\le 1/2$,*

$$\begin{aligned}\mathbb{E}\left(\left\|\int_0^t U(t-r)\left[v(r)\phi dW(r)\right]\right\|^{2m}_{X^T_{b,s}}\right)&\le C\|k\|^{2m}_{H^s(\mathbb{R})}\mathbb{E}\left(\|v\|^{2m}_{X^T_{0,s}}\right)\\ &\le CT^{bm}\|k\|^{2m}_{H^s(\mathbb{R})}\mathbb{E}\left(\|v\|^{2m}_{X^T_{b,s}}\right).\end{aligned}$$

Moreover

$$\begin{aligned}\mathbb{E}\left(\left\|\int_0^t U(t-r)\left[v(r)\phi dW(r)\right]\right\|^{2m}_{\widetilde{X}^T_{b,s-3/8}}\right)&\le C\left(\|k\|^{2m}_{H^s(\mathbb{R})}+\|k\|^{2m}_{L^1(\mathbb{R})}\right)\mathbb{E}\left(\|v\|^{2m}_{X^T_{0,s}}\right)\\ &\le CT^{bm}\left(\|k\|^{2m}_{H^s(\mathbb{R})}+\|k\|^{2m}_{L^1(\mathbb{R})}\right)\mathbb{E}\left(\|v\|^{2m}_{X^T_{b,s}}\right).\end{aligned}$$

Proof. We prove the result for $s=1$, the proof is exactly the same for $s=0$. We set $w(t)=\mathbb{1}_{[0,T]}(t)\int_0^t U(t-r)[v(r)\phi dW(r)]$. Let

$$g(t)=\mathbb{1}_{[0,T]}(t)\int_0^t U(-r)\left[v(r)\phi dW(r)\right],$$

then $w(t)=U(t)g(t)$, $t\ge 0$. We have

$$\mathbb{E}\left(\|w\|^{2m}_{X_{b,1}}\right)=\mathbb{E}\left(\left(\iint_{\mathbb{R}^2}(1+|\xi|)^2(1+|\tau|)^{2b}|\hat g(\tau,\xi)|^2 d\tau d\xi\right)^m\right)$$

Choosing Brownian motions $(\beta_k)_{k\in\mathbb{N}}$, defined on $\mathbb{R}$, we have

$$\begin{aligned}
(\mathcal{F}_x g)(t,\xi) &= \sum_{k=0}^{\infty} \mathbb{1}_{[0,T]}(t) \int_0^t e^{ir\xi^3} \mathcal{F}_x\left(v(r)\phi e_k\right)(\xi) d\beta_k(r) \\
&= \sum_{k=0}^{\infty} \mathbb{1}_{[0,T]}(t) \int_{-\infty}^t \mathbb{1}_{[0,T]}(r) e^{ir\xi^3} \mathcal{F}_x\left(v(r)\phi e_k\right)(\xi) d\beta_k(r).
\end{aligned}$$

It follows

$$\begin{aligned}
\hat{g}(\tau,\xi) &= \sum_{k=0}^{\infty} \int_{\mathbb{R}} \mathbb{1}_{[0,T]}(t) \int_{-\infty}^t \mathbb{1}_{[0,T]}(r) e^{ir\xi^3} \mathcal{F}_x\left(v(r)\phi e_k\right)(\xi) d\beta_k(r) e^{-i\tau t} dt \\
&= \sum_{k=0}^{\infty} \int_{\mathbb{R}} \mathbb{1}_{[0,T]}(r) e^{ir\xi^3} \mathcal{F}_x\left(v(r)\phi e_k\right)(\xi) \left(\int_r^{\infty} \mathbb{1}_{[0,T]}(t) e^{-i\tau t} dt \right) d\beta_k(r).
\end{aligned}$$

By Burkholder inequality, we deduce:

$$\begin{aligned}
&\mathbb{E}\left(\left(\iint_{\mathbb{R}^2} (1+|\xi|)^2 (1+|\tau|)^{2b} |\hat{g}(\tau,\xi)|^2 d\tau d\xi \right)^m \right) \\
&\le C_m \mathbb{E}\Bigg(\Bigg(\sum_{k=0}^{\infty} \iiint_{\mathbb{R}^3} (1+|\xi|)^2 (1+|\tau|)^{2b} \mathbb{1}_{[0,T]}(r) \left| \mathcal{F}_x\left(v(r)\phi e_k\right)(\xi) \right|^2 \\
&\qquad\qquad \times \left| \int_r^{\infty} \mathbb{1}_{[0,T]}(t) e^{-i\tau t} dt \right|^2 dr d\xi d\tau \Bigg)^m \Bigg)
\end{aligned}$$

It easy to see that

$$\left| \int_r^{\infty} \mathbb{1}_{[0,T]}(t) e^{-i\tau t} dt \right|^2 \le \min\{T^2,\ 2\tau^{-2}\}.$$

Therefore, using Lemma 2.6 below,

$$\begin{aligned}
&\mathbb{E}\left(\left(\iint_{\mathbb{R}^2} (1+|\xi|)^2 (1+|\tau|)^{2b} |\hat{g}(\tau,\xi)|^2 d\tau d\xi \right)^m \right) \\
&\le C \mathbb{E}\Bigg(\Bigg(\iiiint_{\mathbb{R}^4} (1+|\xi|)^2 (1+|\tau|)^{2b} \min\{T^2,\ 2\tau^{-2}\} \mathbb{1}_{[0,T]}(r) \left| (\mathcal{F}_x v(r))(\xi+\eta) \right|^2 \\
&\qquad\qquad \times |\hat{k}(\eta)|^2 d\eta dr d\xi d\tau \Bigg)^m \Bigg)
\end{aligned}$$

which in turn we bound from above, using the unitarity of $U(t)$ in L^2 and in H^1, by

$$C\mathbb{E}\Bigg(\Bigg(\iiint_{\mathbb{R}^3}\left[(1+|\xi+\eta|)^2+(1+|\eta|^2)\right]\mathbb{1}_{[0,T]}(r)\,|(\mathcal{F}_x v(r))(\xi+\eta)|^2 \times|\hat{k}(\eta)|^2 d\eta dr d\xi\Bigg)^m\Bigg)$$

$$\le C\left[\|k\|^{2m}_{L^2(\mathbb{R})}\mathbb{E}\left(\|\mathbb{1}_{[0,T]}v\|^{2m}_{X_{0,1}}\right)+\|k\|^{2m}_{H^1(\mathbb{R})}\mathbb{E}\left(\|\mathbb{1}_{[0,T]}v\|^{2m}_{X_{0,0}}\right)\right]$$

$$\le C\left[\|k\|^{2m}_{L^2(\mathbb{R})}\mathbb{E}\left(\|v\|^{2m}_{X^T_{0,1}}\right)+\|k\|^{2m}_{H^1(\mathbb{R})}\mathbb{E}\left(\|v\|^{2m}_{X^T_{0,0}}\right)\right].$$

For the second statement, we proceed similarly. However, the extra $|\xi|^{-3/4}$ implies that a special treatment of the integral for $|\xi|\le 1$. On the region $|\xi|\ge 1$, we simply use $|\xi|^{-3/4}\le 1$. The following estimate is thus sufficient to conclude.

$$\mathbb{E}\Bigg(\Bigg(\int_{|\xi|\le 1}\iint_{\mathbb{R}^2}(1+|\xi|)^2|\xi|^{-3/4}\mathbb{1}_{[0,T]}(r)\,|(\mathcal{F}_x v(r))(\xi+\eta)|^2\,|\hat{k}(\eta)|^2 d\eta dr d\xi\Bigg)^m\Bigg)$$

$$\le C\mathbb{E}\Bigg(\Bigg(\iint_{\mathbb{R}^2}\left(\int_{|\xi|\le 1}|\xi|^{-3/4}|\hat{k}(\eta-\xi)|^2 d\xi\right)\mathbb{1}_{[0,T]}(r)\,|\mathcal{F}_x(v(r))(\eta)|^2\,d\eta dr\Bigg)^m\Bigg)$$

$$\le C\|\hat{k}\|_{L^\infty(\mathbb{R})}\mathbb{E}\Big(\|v\|^{2m}_{X^T_{0,0}}\Big)$$

$$\le C\|k\|_{L^1(\mathbb{R})}\mathbb{E}\Big(\|v\|^{2m}_{X^T_{0,0}}\Big).$$

□

We now give the Lemma used in the above proof.

Lemma 2.6. *Let $v\in X_{0,0}$, then for any complete orthonormal system $(e_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R})$, we have*

$$\sum_{k=0}^{\infty}|\mathcal{F}_x\left(v(r)\phi e_k\right)(\xi)|^2=\int_{\mathbb{R}}|\mathcal{F}_x(v(r))(\xi+\eta)|^2\,|\hat{k}(\eta)|^2 d\eta.$$

Proof. We have

$$\mathcal{F}_x\left(v(r)\phi e_k\right)(\xi)=\mathcal{F}_x\left(v(r,x)\langle k(x-y),e_k(y)\rangle_{L^2_y}\right)(\xi)=\langle\mathcal{F}_x\left(v(r,x)k(x-y)\right)(\xi),e_k(y)\rangle_{L^2_y}.$$

Therefore, by Parseval identity,

$$\sum_{k=0}^{\infty}|\mathcal{F}_x\left(v(r)\phi e_k\right)(\xi)|^2=\|\mathcal{F}_x\left(v(r,x)k(x-y)\right)(\xi)\|^2_{L^2_y},$$

and by Plancherel theorem and an easy computation

$$\sum_{k=0}^{\infty}|\mathcal{F}_x\left(v(r)\phi e_k\right)(\xi)|^2=\|\mathcal{F}_{x,y}\left(v(r,x)k(x-y)\right)(\xi,\eta)\|^2_{L^2_\eta}=\|\mathcal{F}_x(v(r))(\xi+.)\bar{\hat{k}}\|^2_{L^2},$$

which gives the conclusion. □

The proof of the next proposition is left to the reader. It only makes use of classical arguments and ideas similar to those at the end of the proof of Proposition 2.5.

Proposition 2.7. *Let $s = 0$ or 1. For any $T_0 > 0$, any stopping time τ and any predictable process $v \in L^2(\Omega; C([0, T_0 \wedge \tau]; H^s(\mathbb{R})))$, $\int_0^\cdot U(\cdot - r)[\phi v(r) dW(r)]$ has continuous paths with values in $H^s(\mathbb{R}) \cap \dot{H}^{-3/8}(\mathbb{R})$ and for any integer m, there is a constant C_m with*

$$\mathbb{E}\left(\sup_{t\le T_0\wedge\tau}\left\|\int_0^t U(t-r)[\phi v(r)dW(r)]\right\|^{2m}_{H^s\cap\dot{H}^{-3/8}}\right)\le C_m\mathbb{E}\left(\sup_{t\le T_0\wedge\tau}\|v(t)\|^{2m}_{H^s}\right).$$

We are now able to prove the following existence theorem for the truncated equation.

Theorem 2.8. *Let $s = 0$ or 1 and assume that the convolution kernel of the operator ϕ satisfies $k \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$; then for any u_0 in $H^s(\mathbb{R})$, equation (2.3) with $z(t) = U(t)u_0$ has a unique solution $v_R \in Y^{T_0}_{b,s}$, for any b with $0 < b < 1/2$, and any $T_0 \ge 0$. Moreover $v_R \in L^2(\Omega; C([0, T_0]; H^s(\mathbb{R})))$.*

Proof. We use a fixed point argument on equation (2.3). The following lemma, whose first and third estimates were proved in [7], while the second one can be proved in the same way, is useful.

Lemma 2.9.

- *Let $u_0 \in H^s(\mathbb{R})$, $s = 0$ or 1. For any $T > 0$ and $c > 1/2$, $z = U(\cdot)u_0 \in X^T_{c,s}$ and*

$$\|z\|_{X^T_{c,s}} \le C(T)\|u_0\|_{H^s(\mathbb{R})}.$$

- *For any $u_0 \in H^s(\mathbb{R}) \cap \dot{H}^{-3/8}(\mathbb{R})$, and any b with $0 \le b < 1/2$, $z = U(\cdot)u_0 \in Y^T_{b,s}$ and*

$$\|z\|_{Y^T_{b,s}} \le C(T)(\|u_0\|_{H^s(\mathbb{R})} + \|u_0\|_{\dot{H}^{-3/8}(\mathbb{R})}).$$

- *For any $a, b \in (0, 1)$ with $a + b \le 1$, and any $f \in Y^T_{-a,s}$, $\int_0^\cdot U(\cdot - r)f(r)dr \in Y^T_{b,s}$ and*

$$\left\|\int_0^\cdot U(\cdot-r)f(r)dr\right\|_{Y^T_{b,s}} \le CT^{1-(a+b)}\|f\|_{Y^T_{-a,s}}.$$

We first assume that the hypothesis of Theorem 2.8 hold with $s = 0$. Let us fix a, b, c as in Proposition 2.3, with $a + b < 1$. We fix T_0 and take $T \le T_0$. Let $v_1, v_2 \in Y^T_{b,0}$, T being also fixed. We set $\tilde{v}_i(t) = \theta_R\left(|v_i|_{Y^t_{b,0}}\right) v_i(t)$, $i = 1, 2$. Then, recalling that $\mathcal{T}_R v_R$ is defined by the right hand side of (2.3), we have

$$\begin{aligned}
&\mathbb{E}\left(\|\mathcal{T}_R v_1 - \mathcal{T}_R v_2\|^2_{Y^T_{b,0}}\right)\\
&\leq C\mathbb{E}\left(\left\|\int_0^t U(t-r)\partial_x\left[(\tilde{v}_1(r))^2 - (\tilde{v}_2(r))^2\right]dr\right\|^2_{Y^T_{b,0}}\right)\\
&+C\mathbb{E}\left(\left\|\int_0^t U(t-r)\partial_x\left[(\tilde{v}_1(r)-\tilde{v}_2(r))\,z(r)\right]dr\right\|^2_{Y^T_{b,0}}\right)\\
&+C\mathbb{E}\left(\left\|\int_0^t U(t-r)\left[(v_1(r)-v_2(r))\,\phi dW(r)\right]\right\|^2_{Y^T_{b,0}}\right)
\end{aligned}$$

which, by Lemma 2.9 and Proposition 2.5, applied with $m=1$, is bounded above by

$$\begin{aligned}
&CT^{2(1-(a+b))}\mathbb{E}\left(\left\|\partial_x\left((\tilde{v}_1)^2-(\tilde{v}_2)^2\right)\right\|^2_{Y^T_{-a,0}}\right) + CT^{2(1-(a+b))}\mathbb{E}\left(\|\partial_x\left((\tilde{v}_1-\tilde{v}_2)\,z\right)\|^2_{Y^T_{-a,0}}\right)\\
&+CT^b\mathbb{E}\left(\|v_1-v_2\|^2_{X^T_{b,0}}\right).
\end{aligned}$$

By Proposition 2.3, it follows

$$\begin{aligned}
&\mathbb{E}\left(\|\mathcal{T}_R v_1 - \mathcal{T}_R v_2\|^2_{Y^T_{b,0}}\right)\\
&\leq CT^{2(1-(a+b))}\left\{\mathbb{E}\left(\|\tilde{v}_1-\tilde{v}_2\|^2_{Y^T_{b,0}}\|\tilde{v}_1+\tilde{v}_2\|^2_{Y^T_{b,0}}\right) + \mathbb{E}\left(\|\tilde{v}_1-\tilde{v}_2\|^2_{Y^T_{b,0}}\right)\|z\|^2_{X^T_{c,0}}\right\}\\
&+CT^b\mathbb{E}\left(\|v_1-v_2\|^2_{X^T_{b,0}}\right).
\end{aligned}$$

By Lemma 2.2,

$$\|\tilde{v}_1+\tilde{v}_2\|^2_{Y^T_{b,0}} \leq C(R).$$

Moreover, it is not difficult to use the arguments of the proof of Lemma 2.2 and prove

$$\|\tilde{v}_1-\tilde{v}_2\|^2_{Y^T_{b,0}} \leq C(R)\,\|v_1-v_2\|^2_{Y^T_{b,0}}.$$

We deduce that for some $\alpha>0$,

$$\mathbb{E}\left(\|\mathcal{T}_R v_1 - \mathcal{T}_R v_2\|^2_{Y^T_{b,0}}\right) \leq C(R,T_0,\|u_0\|_{L^2(\mathbb{R})})T^\alpha\mathbb{E}\,\|v_1-v_2\|^2_{Y^T_{b,0}}.$$

Thus, $\mathcal{T}_R$ has a unique fixed point $v_R\in L^2(\Omega;Y^T_{b,0})$ for $T\leq T_*$ where T_* is chosen such that

$$C(R,T_0,\|u_0\|_{L^2(\mathbb{R})})T_*^\alpha \leq 1/2.$$

Moreover, using arguments similar to the proof of Proposition 2.5, it can be seen that $\int_0^t U(-r)\left[(z(r)+v_R(r))\,\phi dW(r)\right]$ is a square integrable martingale in $L^2(\mathbb{R})\cap\dot{H}^{-3/8}(\mathbb{R})$. Since

$(U(t))_{t\in\mathbb{R}}$ is strongly continuous on $L^2(\mathbb{R})\cap \dot{H}^{-3/8}(\mathbb{R})$, we deduce that

$$\int_0^t U(t-r)\left[(z(r)+v_R(r))\,\phi dW(r)\right]\in L^2(\Omega;C([0,T_*];L^2(\mathbb{R})\cap\dot{H}^{-3/8}(\mathbb{R}))).$$

Using then Lemma 2.9 with $b>1/2$ and similar estimates as above, we deduce that v_R is also in $L^2(\Omega;C([0,T_*];L^2(\mathbb{R})\cap\dot{H}^{-3/8}(\mathbb{R})))$.

Then, we construct a solution in $[T_*,2T_*]$. First, we write equation (2.3) with $t\ge T_*$ in the form

$$\begin{aligned} v_R(T_*+t) = \; & U(t)v_R(T_*)-\frac{1}{2}\int_0^t U(t-r)\left[\theta_R^2\left(\|v_R\|_{Y_{b,0}^{T_*+r}}\right)\partial_x(v_R^2(T_*+r))\right]dr \\ & -\int_0^t U(t-r)\left[\theta_R\left(\|v_R\|_{Y_{b,0}^{T_*+r}}\right)\partial_x(z(T_*+r)v_R(T_*+r))\right]dr \\ & -\frac{1}{2}\int_0^t U(t-r)\left[\partial_x(z^2(T_*+r))\right]dr \\ & +\int_0^t U(t-r)((z(T_*+r)+v_R(T_*+r))\phi dW(r)),\; t\ge 0. \end{aligned}\tag{2.4}$$

Since $v_R(T_*)\in L^2(\Omega;L^2(\mathbb{R})\cap\dot{H}^{-3/8}(\mathbb{R}))$, the first term is in $L^2(\Omega;Y_{b,0}^{T_*})$. It is then easily seen that v_R can be found on $[T_*,2T_*]$ as a fixed point in $L^2(\Omega;Y_{b,0}^{T_*})$ in the same way as on the interval $[0,T_*]$.

Iterating this, we get a solution on $[0,T_0]$ which is in fact in $L^2(\Omega;Y_{b,0}^{T_0})$ and also in $L^2(\Omega;C([0,T_0];L^2(\mathbb{R})\cap\dot{H}^{-3/8}(\mathbb{R})))$. This proves the result for $s=0$.

Now suppose that the assumptions hold with $s=1$. Let $v\in L^2(\Omega;Y_{b,1}^T)$; we have, setting $\tilde{v}(t)=\theta_R(\|v\|_{Y_{b,0}^t})v(t)$, and using Lemma 2.9,

$$\begin{aligned} \|\mathcal{T}_R v\|_{Y_{b,1}^T} \;& \le CT^{1-(a+b)}\left[\|\tilde{v}\partial_x\tilde{v}\|_{Y_{-a,1}^T}+\|\partial_x(\tilde{v}z)\|_{Y_{-a,1}^T}+\left\|\partial_x\left(z^2\right)\right\|_{Y_{-a,1}^T}\right] \\ & +\left\|\int_0^t U(t-r)\left[z(r)\phi dW(r)\right]\right\|_{Y_{b,1}^T}+\left\|\int_0^t U(t-r)\left[v(r)\phi dW(r)\right]\right\|_{Y_{b,1}^T}, \end{aligned}$$

so that by Proposition 2.3,

$$\begin{aligned} \|\mathcal{T}_R v\|_{Y_{b,1}^T} \;& \le CT^{1-(a+b)}\left[\|\tilde{v}\|_{Y_{b,0}^T}\|\tilde{v}\|_{Y_{b,1}^T}+\|\tilde{v}\|_{Y_{b,0}^T}\|z\|_{X_{c,1}^T}+\|\tilde{v}\|_{Y_{b,1}^T}\|z\|_{X_{c,0}^T}+\|z\|_{X_{c,0}^T}\|z\|_{X_{c,1}^T}\right] \\ & +\left\|\int_0^t U(t-r)\left[z(r)\phi dW(r)\right]\right\|_{Y_{b,1}^T}+\left\|\int_0^t U(t-r)\left[v(r)\phi dW(r)\right]\right\|_{Y_{b,1}^T}. \end{aligned}$$

We then make use of Lemma 2.2 and Lemma 2.9 to get

$$\begin{aligned}
\|\mathcal{T}_R v\|_{Y_{b,1}^T} &\leq CT^{1-(a+b)}\left(\|v\|_{Y_{b,1}^T}+\|z\|_{X_{c,1}^{T_0}}\right)\left(C(R)+\|z\|_{X_{c,0}^{T_0}}\right)\\
&\quad+\left\|\int_0^t U(t-r)\left[z(r)\phi dW(r)\right]\right\|_{Y_{b,1}^T}+\left\|\int_0^t U(t-r)\left[v(r)\phi dW(r)\right]\right\|_{Y_{b,1}^T}\\
&\leq C(R,T_0,\|u_0\|_{L^2(\mathbb{R})})T^{1-(a+b)}\left(\|v\|_{Y_{b,1}^T}+\|u_0\|_{H^1(\mathbb{R})}\right)\\
&\quad+\left\|\int_0^t U(t-r)\left[z(r)\phi dW(r)\right]\right\|_{Y_{b,1}^T}+\left\|\int_0^t U(t-r)\left[v(r)\phi dW(r)\right]\right\|_{Y_{b,1}^T}.
\end{aligned}$$

Thus, by Proposition 2.5 and Lemma 2.9,

$$\mathbb{E}\left(\|\mathcal{T}_R v\|^2_{Y_{b,1}^T}\right) \leq \left[C(R,T_0,\|u_0\|_{L^2(\mathbb{R})})T^{2(1-(a+b))}+CT^b\right]\left(\mathbb{E}\left(\|v\|^2_{Y_{b,1}^T}\right)+\|u_0\|^2_{H^1(\mathbb{R})}\right).$$

This shows that $\mathcal{T}_R$ maps $L^2(\Omega;Y_{b,1}^T)$ into itself. Moreover, the ball in $L^2(\Omega;Y_{b,1}^T)$ of radius R_0 is invariant by $\mathcal{T}_R$ if $T\leq T_{**}$ such that $C(R,T_0,\|u_0\|_{L^2(\mathbb{R})})T_{**}^{2(1-(a+b))}+CT_{**}^b\leq 1/2$ and $R_0\geq\|u_0\|_{H^1(\mathbb{R})}$.

Choosing $T_*\leq T_{**}$ in the construction of the solution v_R of (2.3) in L^2, it follows that the solution v_R is in $L^2(\Omega;Y_{b,1}^{T_*})$. We then use similar arguments as above to prove that $v_R\in L^2(\Omega;C([0,T_*];H^1(\mathbb{R}))$ and

$$\begin{aligned}
\mathbb{E}\left(\sup_{[0,T_*]}\|v_R\|^2_{H^1(\mathbb{R})}\right) &\leq CT_*^{2(1-(a+\tilde{b}))}\left(\mathbb{E}\Big(\|v_R\|^2_{Y_{b,1}^{T_*}}\Big)+\|u_0\|^2_{H^1(\mathbb{R})}\right)\left(C(R)+\|u_0\|_{L^2(\mathbb{R})}\right)^2\\
&\quad+\mathbb{E}\left(\sup_{[0,T_*]}\left\|\int_0^t U(t-r)\left[(z(r)+v_R(r))\phi dW(r)\right]\right\|^2_{H^1(\mathbb{R})}\right)\\
&\leq CT_*^{2(1-(a+\tilde{b}))}\left(\mathbb{E}\Big(\|v_R\|^2_{Y_{b,1}^{T_*}}\Big)+\|u_0\|^2_{H^1(\mathbb{R})}\right)\left(C(R)+\|u_0\|_{L^2(\mathbb{R})}\right)^2\\
&\quad+CT_*\mathbb{E}\left(\sup_{[0,T_*]}\|v_R\|^2_{H^1(\mathbb{R})}\right)+CT_*\|u_0\|^2_{H^1(\mathbb{R})},
\end{aligned}$$

with $\tilde{b}>1/2$ and $a+\tilde{b}<1$. Choosing a smaller T_* if necessary, we deduce

$$\mathbb{E}\left(\sup_{[0,T_*]}\|v_R\|^2_{H^1(\mathbb{R})}\right)\leq R_0^2,$$

if $R_0\geq\|u_0\|_{H^1(\mathbb{R})}$. On $[T_*,2T_*]$, we use equation (2.4) and obtain by similar arguments

$$\mathbb{E}\left(\|v_R(T_*+\cdot)\|^2_{Y_{b,1}^{T_*}}\right)\leq C(T_*)\mathbb{E}\left(\|v_R(T_*)\|^2_{H^1(\mathbb{R})\cap\dot{H}^{-3/8}(\mathbb{R})}\right)+\|u_0\|^2_{H^1(\mathbb{R})},$$

and

$$\mathbb{E}\left(\sup_{[T_*,2T_*]}\|v_R\|^2_{H^1(\mathbb{R})}\right)\le C\mathbb{E}\left(\|v_R(T_*)\|^2_{H^1(\mathbb{R})\cap\dot H^{-3/8}(\mathbb{R})}\right)+R_0^2.$$

We know from the L^2 contruction that $v_R\in L^2(\Omega;C([0,T_0];\dot H^{-3/8}(\mathbb{R})))$, therefore

$$\mathbb{E}\left(\sup_{[T_*,2T_*]}\|v_R\|^2_{H^1(\mathbb{R})}\right)\le C\mathbb{E}\left(\|v_R(T_*)\|^2_{H^1(\mathbb{R})}\right)+R_0^2+\mathbb{E}\left(\sup_{[0,T_0]}\|v_R\|^2_{\dot H^{-3/8}(\mathbb{R})}\right).$$

It is now easy to iterate this argument and deduce that the solution v_R is in $L^2(\Omega;Y^{T_0}_{b,1})$ and also in $L^2(\Omega;C([0,T_0];H^1(\mathbb{R})))$. □

Theorem 2.8 gives the following local in time existence result for the non truncated equation.

Corollary 2.10. *Let $s=0$ or 1 and assume that $k\in H^s(\mathbb{R})\cap L^1(\mathbb{R})$; then for any $u_0\in H^s(\mathbb{R})$, there is a stopping time $\tau^*(u_0,\omega)$ a.s. positive, such that (2.1) has an adapted solution u, defined a.s. on $[0,\tau^*(u_0)[$, unique in some class, and with paths a.s. in $C([0,\tau^*(u_0)[;H^s(\mathbb{R}))$. If $s=1$, the uniqueness holds among solutions with paths in $C([0,\tau^*(u_0)[;H^1(\mathbb{R}))$ a.s; moreover, the stopping time $\tau^*(u_0)$ satisfies*

$$\tau^*(u_0)=+\infty\quad\text{or}\quad\limsup_{t\nearrow\tau^*(u_0)}\|u-U(\cdot)u_0\|_{Y^t_{b,0}}=+\infty,\ a.s.$$

Remark 2.11. *Let us explain what we mean by a solution on the random interval $[0,\tau^*(u_0)[$. This means that u is defined on $[0,\tau^*(u_0)[$ and is an adapted process such that for any stopping time $\tau<\tau^*(u_0)$ the following holds on $[0,\tau]$:*

$$u(t)=U(t)u_0-\frac{1}{2}\int_0^t U(t-r)\partial_x(u^2(r))dr+\int_0^{t\wedge\tau}U(t-r)(u(r)\phi dW(r)).$$

Proof. Let $z(t)=U(t)u_0$ and let $v_R\in Y^{T_0}_{b,s}$ for any $b<1/2$ and any $T_0>0$ be the solution of (2.3) given by Theorem 2.8. We then set $\tau_R=\inf\{t\ge 0,\ \|v_R\|_{Y^t_{b,0}}\ge R\}$; for $t\in[0,\tau_R]$, we have $\theta_R(\|v_R\|_{Y^t_{b,0}})=1$, hence v_R is a solution of (2.2) on $[0,\tau_R]$. It is not difficult to see that τ_R is non decreasing in R and that $v_{R+1}=v_R$ on $[0,\tau_R]$. Hence we may define u on $[0,\tau^*(u_0)[$ with $\tau^*(u_0)=\lim_{R\to\infty}\tau_R$ by setting $u(t)=v_R(t)+z(t)$ for $t\in[0,\tau_R]$ and u is then a solution of (2.1) on $[0,\tau^*(u_0)[$. The uniqueness for u holds in the class $z+Y^{\tau_R}_{b,0}$ for any R and it is not difficult to see that any solution u with paths in $C([0,\tau^*(u_0)[;H^1(\mathbb{R}))$ is in this class. The last property of the lemma is an immediate consequence of the definition of $\tau^*(u_0)$. □

3. Global existence

As already seen, Theorem 2.8 gives a local in time existence result for the equation without cut-off. In the present section, we end the proof of Theorem 1.1 by showing that those solutions are globally defined in time. To that aim we need an estimate on $\|v\|_{Y^T_{b,0}}$. We will use the following result.

Proposition 3.1. *Assume that $k\in L^2(\mathbb{R})$. Let $u\in C([0,\tau]);L^2(\mathbb{R}))$ be a solution of equation (2.1) with $u_0\in L^2(\mathbb{R})$, where τ is a stopping time. Then, for any $m\ge 1$, $u\in L^{2m}(\Omega;C([0,\tau];L^2(\mathbb{R})))$*

and for any $T > 0$

$$\mathbb{E}\left(\sup_{t\in[0,\tau\wedge T]} |u(t)|^{2m}_{L^2(\mathbb{R})}\right) \le C(T, \|u_0\|_{L^2(\mathbb{R})}, m).$$

Proof. The result is a straightforward consequence of Ito formula. We prove it for $m = 2$. For $m \le 2$ it then follows from Hölder inequality. For $m \ge 2$, the proof is similar.

We apply Ito formula to $M(u) = \|u\|^2_{L^2(\mathbb{R})}$ and obtain after a regularization argument and easy computations (see [4] for more details in the case of an additive noise or [5] for the case of the stochastic Schrödinger equation):

$$M(u(\tau\wedge r)) = M(u_0) + 2\sum_{k=0}^{\infty}\int_0^{\tau\wedge r}\int_{\mathbb{R}} u^2(\sigma,x)\phi e_k(x)dxd\beta_k(\sigma) + \|k\|^2_{L^2(\mathbb{R})}\int_0^{\tau\wedge r} M(u(\sigma))d\sigma.$$

We take the square of this identity and deduce:

$$\begin{aligned}
&\mathbb{E}\left(\sup_{r\in[0,\tau\wedge T]} M^2(u(r))\right)\\
&\le 2M^2(u_0) + 4\mathbb{E}\left(\sup_{r\in[0,\tau\wedge T]}\left|\sum_{k=0}^{\infty}\int_0^{r\wedge T}\int_{\mathbb{R}} u^2(\sigma,x)\phi e_k(x)dxd\beta_k(\sigma)\right|^2\right)\\
&+2\|k\|^4_{L^2(\mathbb{R})}\mathbb{E}\left(\left(\int_0^{\tau\wedge T} M(u(\sigma))d\sigma\right)^2\right)\\
&\le 2M^2(u_0) + \left(4\|k\|^2_{L^2(\mathbb{R})} + 2T\|k\|^4_{L^2(\mathbb{R})}\right)\mathbb{E}\left(\int_0^{\tau\wedge T} M^2(u(r))dr\right),
\end{aligned}$$

thanks to Burkholder and Hölder inequalities, and to Lemma 2.6. The result follows from Gronwall Lemma. □

Let v_R be the solution given by Theorem 2.8, let $T_0 > 0$ be fixed, and let $\tau_R = \inf\{t \in [0,T_0],\ \|v_R\|_{Y^t_{b,0}} \ge R\}$; then on $[0,\tau_R]$, v_R+z is a solution to (2.1) which is a.s. in $C([0,\tau_R];L^2(\mathbb{R}))$ and Proposition 3.1 applies:

$$\mathbb{E}\left(\sup_{t\in[0,\tau_R\wedge T_0]} |v_R(t)+z(t)|^{2m}_{L^2(\mathbb{R})}\right) \le C(T_0, \|u_0\|_{L^2(\mathbb{R})}, m). \tag{3.1}$$

We now show that this implies an estimate on the $Y^T_{b,0}$ norm of v_R.

Lemma 3.2. *Let v_R be the solution of the truncated equation* (2.3)*, then there exists a constant $C(T_0, \|u_0\|_{L^2(\mathbb{R})})$ independent of R such that*

$$\mathbb{E}\left(\|v_R\|_{Y^{\tau_R}_{b,0}}\right) \le C(T_0, \|u_0\|_{L^2(\mathbb{R})}).$$

Proof.

Step 1: Using similar arguments as in the beginning of the proof of Theorem 2.8, taking into account the fact that v_R satisfies equation (2.3), we prove using Lemma 2.2 that for $T_0 \geq T \geq 0$,

$$\begin{aligned}
\|v_R\|_{Y_{b,0}^{T\wedge\tau_R}} &\leq CT^{1-(a+b)}\left[\|v_R\|^2_{Y_{b,0}^{T\wedge\tau_R}} + \|z\|^2_{X_{c,0}^{T_0}}\right] + \left\|\int_0^t U(t-r)\left[u_R(r)\phi dW(r)\right]\right\|_{Y_{b,0}^{T_0}} \\
&\leq CT^{1-(a+b)}\left[\|v_R\|^2_{Y_{b,0}^{T\wedge\tau_R}} + C(T_0)\|u_0\|^2_{L^2(\mathbb{R})}\right] + \left\|\int_0^t U(t-r)\left[u_R(r)\phi dW(r)\right]\right\|_{Y_{b,0}^{T_0}},
\end{aligned}$$

with $u_R(t) = v_R(t) + U(t)u_0$. We set

$$K_1 = K_1(\omega) = CT_0^{1-(a+b)}C(T_0)\|u_R\|^2_{C([0,T_0];L^2(\mathbb{R}))} + \left\|\int_0^t U(t-r)\left[u_R\phi dW(r)\right]\right\|_{Y_{b,0}^{T_0}},$$

then

$$CT^{1-(a+b)}\|v_R\|^2_{Y_{b,0}^{T\wedge\tau_R}} - \|v_R\|_{Y_{b,0}^{T\wedge\tau_R}} + K_1 \geq 0.$$

Therefore, if we choose $T = T(\omega)$ such that $T^{1-(a+b)} = \dfrac{3}{16CK_1}$, we have

$$\|v_R\|_{Y_{b,0}^{T\wedge\tau_R}} \leq 2K_1.$$

Note indeed that $v_R(0) = 0$ and that $\|v_R\||Y_{b,0}^t$ is a continuous function of t. Similarly, for any $k \geq 0$, we define

$$v_R^k(t) = u_R(t) - U(t-kT)u_R(kT), \quad t \in [kT,(k+1)T],$$

with $T = T(\omega)$ chosen above. Then the same argument shows that

$$\left\|v_R^k\right\|_{Y_{b,0}^{[kT\wedge\tau_R,(k+1)T\wedge\tau_R]}} \leq 2K_1,$$

where we use the space $Y_{b,0}^{[T_1,T_2]}$ whose definition is exactly the same as $Y_{b,0}^T$ but $[0,T]$ is replaced by $[T_1,T_2]$.

Step 2: Since u_R is a solution of (2.1) on $[0,\tau_R]$, we may write for any $t \in [0,\tau_R]$, $u_R(t)$ as

$$\begin{aligned}
u_R(t) &= U(t)u_0 - \tfrac{1}{2}\int_0^t U(t-r)\partial_x(u_R^2(r))dr + \int_0^t U(t-r)\left[u_R(r)\phi dW(r)\right] \\
&= U(t)u_0 - \frac{1}{2}\sum_{k=0}^{k_t}\int_{kT}^{(k+1)T\wedge t} U(t-r)\partial_x\left[\left(v_R^k(r) + U(r-kT)u_R(kT)\right)^2\right]dr \\
&+ \int_0^t U(t-r)\left[u_R(r)\phi dW(r)\right],
\end{aligned}$$

where k_t is the integer part of t/T. Using this decomposition and the unitarity of $U(\sigma)$ in $\dot{H}^{-3/8}$ for any σ, we deduce that for any $t \in [0, \tau_R]$,

$$
\begin{aligned}
&\|u_R(t) - U(t)u_0\|_{\dot{H}^{-3/8}(\mathbb{R})} \\
&\leq \frac{1}{2}\sum_{k=0}^{k_t} \left\| \int_{kT}^{(k+1)T\wedge t} U(t-r)\partial_x \left[\left(v_R^k(r) + U(r-kT)u_R(kT) \right)^2 \right] dr \right\|_{\dot{H}^{-3/8}(\mathbb{R})} \\
&+ \left\| \int_0^t U(t-r) \left[u_R(r)\phi dW(r) \right] \right\|_{\dot{H}^{-3/8}(\mathbb{R})} \\
&\leq \frac{1}{2}\sum_{k=0}^{k_t} \left\| \int_{kT}^{(k+1)T\wedge t} U((k+1)T \wedge t - r)\partial_x \left[\left(v_R^k(r) + U(r-kT)u_R(kT) \right)^2 \right] dr \right\|_{\dot{H}^{-3/8}(\mathbb{R})} \\
&+ \left\| \int_0^t U(t-r) \left[u_R(r)\phi dW(r) \right] \right\|_{\dot{H}^{-3/8}(\mathbb{R})}.
\end{aligned}
$$

Now, suppose that a is fixed with $0 < a < 1/2$ in such a way that Proposition 2.3 holds, and set $\tilde{b} = 1 - a$, so that $\tilde{b} > 1/2$. Then, using the fact that $Y_{\tilde{b},0}^{[T_1,T_2]} \subset C([[T_1, T_2]; L^2(\mathbb{R}) \cap \dot{H}^{-3/8}(\mathbb{R}))$ for any positive T_1, T_2, we have for $t \in [0, \tau_R]$ and $k = 0, \ldots, k_t$,

$$
\begin{aligned}
&\left\| \int_{kT}^{(k+1)T\wedge t} U((k+1)T \wedge t - r)\partial_x \left[\left(v_R^k(r) + U(r-kT)u_R(kT) \right)^2 \right] dr \right\|_{\dot{H}^{-3/8}(\mathbb{R})} \\
&\leq \left\| \int_{kT}^{\cdot} U(\cdot - r)\partial_x \left[\left(v_R^k(r) + U(r-kT)u_R(kT) \right)^2 \right] dr \right\|_{C([kT\wedge\tau_R,(k+1)T\wedge\tau_R];\dot{H}^{-3/8}(\mathbb{R}))} \\
&\leq C \left\| \int_{kT}^{\cdot} U(\cdot - r)\partial_x \left[\left(v_R^k(r) + U(r-kT)u_R(kT) \right)^2 \right] dr \right\|_{Y_{\tilde{b},0}^{[kT\wedge\tau_R,(k+1)T\wedge\tau_R]}}.
\end{aligned}
$$

By Lemma 2.9, the above term is majorized for each $k \in \{0, \cdots, k_t\}$ by

$$
\begin{aligned}
&CT^{1-(a+\tilde{b})} \left\| \partial_x \left[\left(v_R^k + U(\cdot - kT)u_R(kT) \right)^2 \right] \right\|_{Y_{-a,0}^{[kT\wedge\tau_R,(k+1)T\wedge\tau_R]}} \\
&\leq C \left\{ \|v_R^k\|^2_{Y_{b,0}^{[kT\wedge\tau_R,(k+1)T\wedge\tau_R]}} + \|u_R(kT)\|^2_{L^2(\mathbb{R})} \right\},
\end{aligned}
$$

by Proposition 2.3 and Lemma 2.9 again, since $a + \tilde{b} = 1$. By the result of step 1, we obtain

$$
\|u_R(t) - U(t)u_0\|_{\dot{H}^{-3/8}(\mathbb{R})} \leq K_2, \quad t \in [0, \tau_R],
$$

with

$$
K_2 = \frac{1}{2}CT_0T^{-1}\left[4K_1^2 + \|u_R\|^2_{C([0,T_0];L^2(\mathbb{R}))}\right] + \left\| \int_0^{\cdot} U(\cdot - r)\left[u_R(r)\phi dW(r)\right] \right\|_{C([0,T_0];\dot{H}^{-3/8}(\mathbb{R})}.
$$

Step 3: By Lemma 2.9, and the unitarity of $U(kT)$ on $L^2(\mathbb{R})\cap\dot H^{-3/8}(\mathbb{R})$ we have

$$\begin{aligned}\|U(t-kT)u_R(kT)-U(t)u_0\|_{Y_{b,0}^{[kT\wedge\tau_R,(k+1)T\wedge\tau_R]}} &\le C\,\|U(-kT)u_R(kT)-u_0\|_{L^2(\mathbb{R})\cap\dot H^{-3/8}(\mathbb{R})}\\ &\le C\,\|u_R(kT)-U(kT)u_0\|_{L^2(\mathbb{R})\cap\dot H^{-3/8}(\mathbb{R})}.\end{aligned}$$

Therefore, using step 2,

$$\|U(\cdot-kT)u_R(kT)-U(\cdot)u_0\|_{Y_{b,0}^{[kT\wedge\tau_R,(k+1)T\wedge\tau_R]}}\le K_3=C\left(K_2+2\,\|u_R\|_{C([0,T_0];L^2(\mathbb{R}))}\right).$$

Finally, for $t\in[kT\wedge\tau_R,(k+1)T\wedge\tau_R]$, we have

$$v_R(t)=v_R^k(t)+U(t-kT)u_R(kT)-U(t)u_0,$$

and we may write, k_0 being the integer part of T_0/T,

$$\begin{aligned}\|v_R\|_{Y_{b,0}^{\tau_R}} &\le \sum_{k=0}^{k_0}\|v_R\|_{Y_{b,0}^{[kT\wedge\tau_R,(k+1)T\wedge\tau_R]}}\\ &\le \sum_{k=0}^{k_0}\left\|v_R^k\right\|_{Y_{b,0}^{[kT\wedge\tau_R,(k+1)T\wedge\tau_R]}}+\|U(\cdot-kT)u(kT)-U(\cdot)u_0\|_{Y_{b,0}^{[kT\wedge\tau_R,(k+1)T\wedge\tau_R]}}\\ &\le \left(\frac{T_0}{T}+1\right)(2K_1+K_3).\end{aligned}$$

Note that T^{-1} is proportional to $K_1^{1/(1-(a+b))}$; by Proposition 3.1 and Proposition 2.7, K_1 and K_3 have all moments finite, and it follows

$$\mathbb{E}\left(\|v_R\|_{Y_{b,0}^{\tau_R}}\right)\le c(T_0,\|u_0\|_{L^2(\mathbb{R})})$$

which concludes the proof of Lemma 3.2. □

It is now straightforward to achieve the proof of Theorem 1.1. Indeed, due to Corollary 2.10, it suffices to see that $\limsup_{R\to\infty}\tau_R\wedge T_0=T_0$ in probability as R goes to infinity. But this is an easy consequence of Markov inequality and Lemma 3.2, since

$$\mathbb{P}(\tau_R<T_0)=\mathbb{P}\left(\|v_R\|_{Y_{b,0}^{T_0\wedge\tau_R}}\ge R\right).$$

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