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A semi-discrete scheme for the stochastic nonlinear Schrödinger equation

A. De Bouard[1], **A. Debussche**[2]

[1] CNRS et Université Paris-Sud, UMR 8628, Bât. 425, Université de Paris-Sud, 91405 Orsay Cedex, France
[2] ENS de Cachan, Antenne de Bretagne, Campus de Ker Lann, Av. R. Schuman, 35170 Bruz, France; e-mail: amaud.debussche@bretagne.ens-cachan.fr

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Summary. We study the convergence of a semi-discretized version of a numerical scheme for a stochastic nonlinear Schrödinger equation. The nonlinear term is a power law and the noise is multiplicative with a Stratonovich product. Our scheme is implicit in the deterministic part of the equation as is usual for conservative equations. We also use an implicit discretization of the noise which is better suited to Stratonovich products. We consider a subcritical nonlinearity so that the energy can be used to obtain an *a priori* estimate. However, in the semi discrete case, no Ito formula is available and we have to use a discrete form of this tool. Also, in the course of the proof we need to introduce a cut-off of the diffusion coefficient, which allows to treat the nonlinearity. Then, we prove convergence by a compactness argument. Due to the presence of noise and to the implicit discretization of the noise, this is rather complicated and technical. We finally obtain convergence of the discrete solutions in various topologies.

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1 Introduction

The nonlinear Schrödinger (NLS) equation is one of the basic models for the description of weakly nonlinear dispersive waves, and occurs in many branches of physics : hydrodynamics, plasma physics, nonlinear optics, molecular biology, $\cdots$

Correspondence to: A. Debussche

Recently, more and more attention has been paid to the influence of a Gaussian white noise on the dynamical properties of solutions of this equation (see e.g. [2], [7], [16], [29]).

In [2], [29], for example, a conservative multiplicative NLS equation with a real valued space-time white noise is considered. This equation is two dimensional with a cubic nonlinearity, and may be written as

$$i\partial_t\psi + (\Delta\psi + |\psi|^2\psi) = \dot{\eta}\psi, \quad x \in \mathbb{R}^2, \quad t \geq 0,$$

where ψ is a complex-valued process defined on $\mathbb{R}^+ \times \mathbb{R}^2$, and $\dot{\eta} = \frac{d\eta}{dt}$, with η a real valued Gaussian process with correlation function

$$\mathbb{E}(\eta(t,x)\eta(s,y)) = \delta_{x-y}(s \wedge t).$$

The noise is multiplicative and the product here is a Stratonovich product. Hence, the L^2-norm of the solution is formally conserved. This is related to the fact that $|\psi|^2$ stands for a probability density.

However, from a mathematical point of view, nothing is known concerning the existence of solutions for the initial value problem associated with this equation. The first reason for this absence of result is the lack of spatial smoothness of the noise (the unitary group $e^{it\Delta}$ has no smoothing effect in the usual Sobolev spaces, which are natural spaces to deal with the deterministic equation). The second reason is that an homogeneous noise on $\mathbb{R}^d$ cannot be treated in the context of $L^2(\mathbb{R}^d)$ based Sobolev spaces, even of negative order.

This is why we have considered in [11] a multiplicative NLS equation of the preceding form, with $x \in \mathbb{R}^d$, but in which η is a Wiener process on the space of square integrable functions on $\mathbb{R}^d$, with a covariance operator $\Phi\Phi^*$ which is roughly speaking of finite trace (we actually need slightly more regularity of Φ).

If for example Φ is defined through a real valued kernel k, that is

$$\Phi u(x) = \int_{\mathbb{R}^d} k(x,y)u(y)dy,$$

then one may recover the spatial correlation of η:

$$\mathbb{E}(\eta(t,x)\eta(s,y)) = c(x,y)s \wedge t$$

by the formula

$$c(x,y) = \int_{\mathbb{R}^d} k(x,z)k(y,z)dz.$$

The product in the right hand side of the equation is again a Stratonovich product. We have then proved that in the subcritical case, that is when the nonlinearity $|\psi|^2\psi$ is replaced by $|\psi|^{2\sigma}\psi$ with $\sigma < 2/d$, this equation possesses a global square integrable solution which is unique in a slightly more restricted class, provided that the initial data is given and square integrable. If one assumes more spatial regularity on the noise, then one naturally obtains more regular solutions, by using the same kind of proofs (see [13]).

In [12], we have studied the influence of a Gaussian additive noise (which is still a white noise in time) on the blow up of solutions of the nonlinear Schrödinger equation in the supercritical case $\sigma > 2/d$. We have proved that if the noise is sufficiently spatially correlated, then any initial data immediately leads to a blowing up solution. It is not expected that this kind of behavior occurs for a multiplicative spatially δ-correlated noise : the absence of spatial correlation is instead conjectured to stop the blow up of any solution (see [29] for a formal analysis, and [14] for numerical computations on that subject).

Here, we investigate the convergence of a numerical scheme for a multiplicative NLS equation of the preceding type. The noise is again assumed to be Gaussian, white in time and spatially correlated. The scheme is a semi-discretized version of the one used in [14]. The deterministic part is a Crank-Nicolson type scheme, and the Stratonovich product in the right hand side of the equation is naturally approximated by the product of the increment of the noise with the value of the solution at the mid point. This has the advantage that the L^2-norm is still a conserved quantity for the numerical scheme. It is shown in [14] that this numerical method gives good results in the one dimensional case; we also refer to [14] for details on the numerical implementation. The two dimensional case will appear in a forthcoming paper.

In the absence of noise, the Crank-Nicolson scheme we consider also preserves the energy of the continuous equation. A study of this kind of schemes for deterministic NLS equations can be found in [1], [8], [15] or [30], in which the convergence is proved either by energy methods, or using contraction arguments. In these cases, the existence and uniqueness of the semi-discrete solution requires a smallness condition on the time step, depending on the initial data. In the stochastic case, such a smallness condition on the time step would be random and much too restrictive. For this reason, we do not obtain the uniqueness of the semi-discrete solution, but we prove that for each time step, there exists a measurable selection of semi-discrete solutions, that is there is a semi-discrete solution which is an adapted process; we also prove that the sequence of processes we obtain in this way by varying the time step converges to the solution of the continuous Stratonovich equation as the time step tends to zero.

Note that this problem of non uniqueness of the discrete solution already occurs in the approximation of finite dimensional stochastic differential equations with implicit schemes. Milstein et al. in [26] suggest to use a cut-off of the Gaussian random variables arising in the numerical scheme. This allows them to obtain a unique discrete solution and to derive an order of convergence. This approach could probably be applied in our situation with, however, many additional difficulties, due to the infinite dimension. Moreover, this problem never caused any trouble in the implementation of the method – see [14] – and we chose to analyse the scheme implemented in [14] without any modification.

The convergence result is obtained via a compactness method and a lemma of Gyöngy and Krylov [21]. Hence, the first step is to derive bounds independent of the time step on the semi-discrete solution. Note that if the L^2-norm is still preserved by the semi-discrete stochastic NLS equation, the same is not true for the energy. A bound on the L^2-norm is not sufficient for the use of compactness methods. In the continuous case, a bound on the energy may be obtained by the use of the Ito formula (see [13]). This tool is not available in the semi-discrete case, but we overcome the difficulty using a kind of semi-discrete equivalent of the Ito formula: we inject in the expression of the energy evolution the "integral equation" giving u at time $(\ell + 1)\delta t$ in terms of u at time $\ell\delta t$, where δt is the time step. We use the "integral equation" instead of the original semi-discrete equation to avoid the addition of extra partial differential operators in the remaining terms, which would lead to the impossibility of estimating these terms.

However we cannot avoid the addition of extra nonlinear terms and we have to cut off the equation, as is classical for stochastic partial differential equations. If we truncate the nonlinear term, we loose the conservation of energy for the deterministic equation, and our estimate on the solution. This is the reason why we use instead a truncation of the diffusion coefficient. We then get rid of the cut-off thanks to the uniform bounds on the energy for the continuous equation. Note that this cut-off is a technical tool, and is not used in the implemented scheme.

In this way, we are able to prove that the discretized solution converges to the continuous solution in various topologies. Some of the arguments we have used have been introduced in the series of work [19], [20], [21], [22] on the numerical analysis of stochastic partial differential equations of parabolic type; in particular the idea of using a cut off to obtain a bound on the discrete solutions and the way to get rid of it at the end. However, the present context is much more complicated: it deals with NLS equations, implicit schemes, Stratonovich products, $\cdots$ Many new arguments have been necessary. For instance, the above described idea to overcome the lack of Ito formula in the discrete case, or the particular way to introduce a cut-off. Also, the compactness argument is quite involved.

Recently, numerical analysis of semilinear stochastic partial differential equations in the semi-discrete or fully discrete case has been the object of articles by Printems [28] and Hausenblas [23], [24]. In [23] or [28], only parabolic equations are considered, while [24] generalizes the framework;

however, all these works deal with explicit schemes in the nonlinear and stochastic terms.

Our proof can be extended to the study of a fully discrete scheme, some estimates need to be modified accordingly. However, this requires long and technical computations and would probably make the article very technical. For this reason, we have preferred to start with the semi-discrete case, so that the ideas can be introduced in a simpler context.

Also, we do not give any result on the order of convergence. This would require much more smoothness assumptions on the noise and initial data. Moreover, as is well known, several kind of orders can be defined in the context of stochastic numerical analysis. If the error is estimated pathwise (strong order), the order is very small and in our case it cannot be greater than one half. For the approximation of stochastic differential equations, this problem may be overcome by some correction terms in the scheme (see [25]). In the context of stochastic partial differential equations, it is not clear whether such terms can be designed. When one is interested in the approximation of averaged quantities (weak order), then one typically obtains a better order of convergence (see [32]). However, again such a study seems to be very difficult in the infinite dimensional case.

The paper is organized as follows: in Section 2, we introduce our notations, afterwhat we first recall a local and global existence and uniqueness result proved in [13] for the continuous equation; then we write the semi-discrete scheme and state our main result (convergence theorem). Section 3 is devoted to the proof of the convergence theorem: in Section 3.1 we write the truncated scheme and show the existence of a measurable selection of solutions for both schemes; in Section 3.2 we show an estimate on the truncated discrete solution (using a "discrete Ito formula") and prove the tightness of the sequence in an appropriate function space. Section 3.3 is devoted to the passage to the limit in the discrete equation and in the truncature term, and to the conclusion of the proof. A technical Section 3.4 has been added, where we have gathered the proof of three technical lemmas used in the course of the proof of the convergence theorem.

2 Notations and main result

In general, a norm in a vector space X will be denoted by $|.|_X$ or sometimes $\|.\|_X$ when dealing with operator norms. We use the classical Lebesgue space $L^p(\mathbb{R}^d)$ (of complex valued functions), and the inner product in the real Hilbert space $L^2(\mathbb{R}^d)$ is denoted by $(.,.)$, i.e.

$$(u, v) = \mathrm{Re} \int_{\mathbb{R}^d} u(x)\bar{v}(x)dx$$

for $u, v, \in L^2(\mathbb{R}^d)$. If $s \in \mathbb{R}$, the usual Sobolev space $H^s(\mathbb{R}^d)$ is the space of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ whose Fourier transform $\hat{u}$ satisfies $(1+|\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^d)$. We will denote by $L^2(\mathbb{R}^d, \mathbb{R})$ the subspace of $L^2(\mathbb{R}^d)$ consisting of real valued square integrable functions, and the same for $H^s(\mathbb{R}^d, \mathbb{R})$. We will sometimes use the shorter notations L^p_x or H^s_x to denote respectively $L^p(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$.

If I is an interval of $\mathbb{R}$, X is a Banach space and $1 \le r \le +\infty$, then $L^r(I, X)$ is the space of strongly Lebesgue measurable functions u from I into X such that the function $t \mapsto |u(t)|_X$ is in $L^r(I)$.

Since we use a compactness method, and because we work in the whole $\mathbb{R}^d$ space, we will have to use local Lebesgue and Sobolev spaces. If $p > 1$, the space $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ is the space of complex valued functions defined on $\mathbb{R}^d$ such that for any compact set $K \subset \mathbb{R}^d$, $u \in L^p(K)$. We refer to Section 2 in [10] for a precise definition of the local Sobolev spaces $H^m_{\mathrm{loc}}(\mathbb{R}^d)$ and $H^{-m}_{\mathrm{loc}}(\mathbb{R}^d)$ when m is a positive integer.

Given two separable Hilbert spaces H and $\tilde{H}$, we denote by $\mathcal{L}_2(H, \tilde{H})$ the space of Hilbert-Schmidt operators from H into $\tilde{H}$, endowed with the norm

$$\|\Phi\|^2_{\mathcal{L}_2(H,\tilde{H})} = \mathrm{tr}\Phi^*\Phi = \sum_{k\in\mathbb{N}} |\Phi e_k|^2_{\tilde{H}},$$

where $(e_k)_{k\in\mathbb{N}}$ is any orthonormal basis of H. When $H = L^2(\mathbb{R}^d, \mathbb{R})$ and $\tilde{H} = H^s(\mathbb{R}^d, \mathbb{R})$, $\mathcal{L}_2(H, \tilde{H})$ is simply denoted by $L_2^{0,s}$. Given a Banach space B, we will also consider bounded linear operators from $L^2(\mathbb{R}^d)$ into B, and in order to replace the notion of Hilbert-Schmidt operators, we use in this case, as in [5], [6], the notion of γ-radonifying operators. We denote by $R(L^2, B)$ the space of γ-radonifying operators from $L^2(\mathbb{R}^d, \mathbb{R})$ into B, and we recall (see [6], Proposition 3.1) that the norm in $R(L^2, B)$ may be written as

$$\|\Phi\|_{R(L^2,B)} = \left(\tilde{\mathbb{E}}\left|\sum_{k=1}^{\infty} \gamma_k \Phi e_k\right|^2_B\right)^{1/2}$$

where $(e_k)_{k\in\mathbb{N}}$ is any orthonormal basis of $L^2(\mathbb{R}^d, \mathbb{R})$ and $(\gamma_k)_{k\in\mathbb{N}}$ is any sequence of independent normal real valued random variables on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$.

In all the paper, we assume that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{t\ge 0}$. We also assume that $\hat{W}$ is a cylindrical Wiener process on $L^2(\mathbb{R}^d, \mathbb{R})$ associated with $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\ge 0})$, so that for any orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R}^d, \mathbb{R})$, there is a sequence $(\beta_k)_{k\in\mathbb{N}}$ of real independent Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\ge 0})$ such that

$$\hat{W} = \sum_{k\in\mathbb{N}} \beta_k(t) e_k.$$

We then consider the Wiener process $W = \Phi\hat{W}$, where Φ is at least assumed to be an element of $L_2^{0,1}$ (more precise assumptions on Φ will be stated later). Note that W is a real valued process (when considered as a function of (t, x)).

In all what follows, Δ is the Laplace operator on $\mathbb{R}^d$: $\Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$, $\mathbf{1}_A$ is the characteristic function of the set A. Also, C, C_k, $k \in \mathbb{N}$, will denote various constants, and if it is necessary to precise, the notation $C_k(\cdot)$ means that the constant C_k depends on its arguments only (for instance $C_k(T, m)$ depends on T and m but not on the other parameters or variables).

We consider the multiplicative (NLS) equation

$$idu + (\Delta u + \lambda|u|^{2\sigma}u)dt = u \circ dW \tag{2.1}$$

where $x \in \mathbb{R}^d$, $t \geq 0$, $u(t, x) \in \mathbb{C}$, $\sigma > 0$, $\lambda = \pm 1$ and $\circ$ stands for a Stratonovich product in the right hand side of (2.1). We will actually use the Ito equation equivalent to (2.1). Defining, for $x \in \mathbb{R}^d$, the function

$$F_\Phi(x) = \sum_{k=0}^{\infty} (\Phi e_k(x))^2 \tag{2.2}$$

where $(e_k)_{k\in\mathbb{N}}$ is any orthonormal basis of $L^2(\mathbb{R}^d, \mathbb{R})$, this equivalent Ito equation may be written as

$$idu + (\Delta u + \lambda|u|^{2\sigma}u)dt = udW - \frac{i}{2}uF_\Phi dt. \tag{2.3}$$

It is easily seen using the kernel K associated with Φ – which necessarily exists, since $\Phi \in L_2^{0,1}$ – that the function F_Φ does not depend on the basis $(e_k)_{k\in\mathbb{N}}$. More precisely, we have

$$F_\Phi(x) = |K(x, \cdot)|_{L_y^2}^2.$$

In order to recall the local and global existence theorem proved in [13] in its most general form, we state different assumptions on σ and λ below.

(A1) $d \leq 5$, and $\begin{cases} \sigma > 0 & \text{if } d = 1 \text{ or } 2 \\ 0 < \sigma < 2 & \text{if } d = 3 \\ 1/2 \leq \sigma < 2/(d-2) & \text{if } d = 4 \text{ or } 5, \end{cases}$

(A2) $0 < \sigma < 2/d$ or $\lambda = -1$,

(A3) $d \leq 3$, and $\begin{cases} \sigma > 0 & \text{if } d = 1 \text{ or } 2 \\ 0 < \sigma < 1 & \text{if } d = 3. \end{cases}$

Assumption (A1) corresponds to the local existence theory in $H^1(\mathbb{R}^d)$, and is a little bit more restrictive than in the deterministic case, due to the presence of the stochastic integral. Assumption (A2) allows to obtain global existence using the energy

$$H(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx - \frac{\lambda}{2\sigma+2}\int_{\mathbb{R}^d} |u(x)|^{2\sigma+2} dx. \tag{2.4}$$

Assumption (A3) will be used in the convergence theorem (see Theorem 2.2 and Remark 2.1).

The energy $H(u)$, together with the L^2-norm, is an invariant quantity for the deterministic equation, and allows to bound the $H^1(\mathbb{R}^d)$ norm of the solution under assumption (A2), according to the following Lemma. The proof of this lemma is an immediate consequence of Gagliardo-Nirenberg's inequality (see for example [18])

Lemma 2.1 *Assume that (A2) holds. Then,*

- *if* $\lambda = -1$, $|\nabla u|^2_{L^2_x} \le 2H(u)$
- *if* $\lambda = +1$, *there is a constant* C_σ *depending only on* σ *and* d *such that*

$$|\nabla u|^2_{L^2_x} \le 4H(u) + C_\sigma |u|^{2+\frac{4\sigma}{2-\sigma d}}_{L^2_x}.$$

As was explained in the introduction, due to the lack of regularization of the operator $S(t) = e^{-it\Delta}$, the covariance operator Φ of the Wiener process W needs some spatial smoothness. The assumptions we require on this spatial smoothness also depend on the results we want, and we state them below:

(B1) $\Phi \in L_2^{0,1}$ and if $d \ge 2$, $\Phi \in R(L^2(\mathbb{R}^d), W^{1,\alpha}(\mathbb{R}^d))$ for some $\alpha > 2d$.
(B2) $\Phi \in L_2^{0,s}$ with $s > 1 + d/2$.

The following theorem, which gives the existence and uniqueness of H^1-valued solutions of equation (2.3) is proved in [13]. In the statement of the theorem we use the classical denomination of "admissible pair" to denote any couple of real valued positive numbers (r, p), with $r > 2$ and $2/r = d(1/2 - 1/p)$.

Theorem 2.1 *Assume that (A1) and (B1) hold. Then there is an admissible pair* (r, p) *such that for any* $\mathcal{F}_0$*-measurable* u_0 *with values in* $H^1(\mathbb{R}^d)$*, there is a stopping time* $\tau^*(u_0)$ *and a unique solution* u *of (2.3) starting from* u_0*, with* $u \in C([0, \tau]; H^1(\mathbb{R}^d)) \cap L^r(0, \tau; W^{1,p}(\mathbb{R}^d))$*, a.s, for any stopping time* τ *such that* $\tau < \tau^*(u_0)$ *a.s. Moreover,* $\tau^*(u_0)$ *satisfies*

$$\tau^*(u_0) = +\infty \quad \text{or} \quad \limsup_{t \nearrow \tau^*(u_0)} |u(t)|_{H^1(\mathbb{R}^d)} = +\infty \quad \text{a.s.}$$

If in addition, (A2) holds, then the preceding solution is global, i.e. $\tau^*(u_0) = +\infty$ *a.s. In this case, there is an integer* $k_0(\sigma, d, r)$ *such that for any integer* $k \ge k_0$ *and for any* $u_0 \in L^{(2+\frac{4\sigma}{2-\sigma d})k}(\Omega, H^1(\mathbb{R}^d))$ *(*$u_0 \in L^{2k}(\Omega; H^1(\mathbb{R}^d))$ *is sufficient if* $\lambda = -1$*), the solution* u *is in* $L^{2k}(\Omega, C[0, T_0], H^1(\mathbb{R}^d)) \cap L^1(\Omega, L^r(0, T_0; W^{1,p}(\mathbb{R}^d)))$ *for any* $T_0 > 0$.

Some restrictions arise in Theorem 2.1 compared to the deterministic theory (in which $\sigma < \frac{2}{d-2}$ is allowed in any dimension for local existence). These restrictions are due to the fact that in order to be able to estimate the stochastic integral arising from the term udW, one has to work in $C([0,T]; H^1(\mathbb{R}^d)) \cap L^r(0,T; W^{1,p}(\mathbb{R}^d))$ with $p < \frac{2}{d-1}$ (see [13]), hence possibly with $p < 2\sigma+2$ ($p = 2\sigma + 2$ is used in the deterministic case).

Global existence is obtained thanks to the use of the energy $H(u)$ and Lemma 2.1.

From now on, we assume that Φ and σ satisfy (A3) and (B2). We also assume that we are given a $\mathcal{F}_0$-measurable u_0 in $L^{(2+\frac{4\sigma}{2-\sigma d})k}(\Omega; H^1(\mathbb{R}^d))$, with $k \geq k_0$, k_0 being as in Theorem 2.1.

In all what follows, we consider a fixed positive T_0 and we set for each $n \in \mathbb{N}$, $\delta t = T_0/n$. We also set for ℓ with $0 \leq \ell \leq n-1$:

$$\chi_\ell = \frac{W((\ell+1)\delta t) - W(\ell \delta t)}{\sqrt{\delta t}}$$

and for $a, b > 0$ and real:

$$\left\{ \begin{aligned} & f(a,b) = \tfrac{\lambda}{\sigma+1} \frac{a^{2\sigma+2} - b^{2\sigma+2}}{a^2 - b^2} \quad \text{if} \quad a \neq b \\ & f(a,a) = \lambda a^{2\sigma}. \end{aligned} \right. \tag{2.5}$$

It follows from (B2) that for any integer $\ell \leq n-1$, χ_ℓ is a Gaussian random variable with values in $H^s(\mathbb{R}^d)$ for some $s > 1 + d/2$. Finally, we use the notation $u_{\ell+1/2} = \frac{1}{2}(u_\ell + u_{\ell+1})$ for $0 \leq \ell \leq n-1$. Our scheme is then defined by the following semi-discrete equation, in which ℓ is an integer with $0 \leq \ell \leq n-1$, and u_ℓ is an approximation of $u(\ell \delta t)$:

$$i \frac{u_{\ell+1} - u_\ell}{\delta t} + \Delta u_{\ell+1/2} + f(|u_\ell|, |u_{\ell+1}|) u_{\ell+1/2} = \frac{\chi_\ell}{\sqrt{\delta t}} u_{\ell+1/2}. \tag{2.6}$$

We will say that a process u^n defined on $[0, T_0]$, with values in $H^1(\mathbb{R}^d)$ is a solution of the semi-discrete equation (2.6) if u^n is constant on each time interval $[\ell\delta t, (\ell+1)\delta t)$, equal to u_ℓ on such an interval, and if $(u_\ell)_{0\leq \ell \leq n-1}$ satisfies (2.6) with $\delta t = T_0/n$. In the sequel, we set $u_\ell = u_\ell^n$ to emphasize the dependence of u_ℓ on n.

Equation (2.6) is supplemented with the initial condition

$$u(0) = u_0. \tag{2.7}$$

The discretization of the noise term in the right hand side of (2.6) corresponds to the discretization of the Stratonovich product. It is not difficult to see that the L^2 norm of u^n is constant. For the practical implementation of the scheme in the fully discrete case (see [14]), this implies that at each time step, a random nonlinear equation – whose unknown is $u^n_{\ell+1}$ – has to be solved. However, our experience shows that this creates no difficulties during the simulations. Another choice could be to discretize the Ito form of the equation, but then the L^2 norm of the solution would not be constant.

Note that the energy $H(u)$ is preserved by the deterministic semi-discrete scheme. If u^n is a solution of (2.6)–(2.7) with $W = 0$ (i.e. $\chi_\ell = 0$, $0 \leq \ell \leq n-1$), then $H(u^n_{\ell+1}) = H(u^n_\ell) = H(u_0)$. This is easily seen by multiplying (2.6) (with $W = 0$) by

$$\Delta \bar{u}_{\ell+1/2} + f(|u_\ell|, |u_{\ell+1}|)\bar{u}_{\ell+1/2}$$

and taking the imaginary part.

We may now state our convergence result for the semi-discrete solution of equation (2.6)–(2.7).

Theorem 2.2 *Assume that (A3) and (B2) hold. Then, for each $n \in \mathbb{N}$, there is a solution u^n, a.s. in $L^\infty(0, T_0; H^1(\mathbb{R}^d))$ and adapted to $(\mathcal{F}_t)_{t\in[0,T_0]}$ of the semi-discrete equation (2.6)–(2.7), with $\delta t = \frac{T_0}{n}$. Moreover, the sequence $(u^n)_{n\in\mathbb{N}}$ converges to the solution u starting from u_0 of the continuous equation (2.3), given by Theorem 2.1. The convergence of u^n holds in probability in $L^\infty(0, T, H^r(\mathbb{R}^d))$ and in $L^p(\Omega, L^\infty(0, T; L^2(\mathbb{R}^d)))$ for any $T < T_0$, $r < 1$ and any positive p such that $u_0 \in L^{p'}(\Omega, L^2(\mathbb{R}^d))$ for some $p' > p$.*

Remark 2.1 The supplementary assumption (A3), that is $d \leq 3$ and $\sigma < 1$ if $d = 3$ ensures that the nonlinear term in equation (2.6) is in $L^2(\mathbb{R}^d)$ as soon as u_ℓ and $u_{\ell+1}$ live in $H^1(\mathbb{R}^d)$. This assumption could have been weakened, but the proofs would then be considerably more technical. This point is used in particular to obtain an H^1-bound on the semi-discrete solution u^n by using the energy $H(u)$ (see Section 3.2). Similarly, for the sake of clarity in the proofs, we have chosen to work with the strong assumption (B2) on the noise. However, our arguments can be easily generalized to the weaker assumption that $\Phi \in R(L^2(\mathbb{R}^d), W^{1,\tilde{\alpha}}(\mathbb{R}^d))$ for $\tilde{\alpha}$ large.

The proof of Theorem 2.2 is the object of the next section. As was previously mentioned, a compactness method will be used. The uniqueness of the solution of the continuous equation is necessary to derive the convergence in probability of the original sequence of semi-discrete solutions. Indeed, in order to obtain the convergence of the original sequence, we will make use of the following elementary lemma, which was first used by Gyöngy and Krylov in [21]:

Lemma 2.2 *Let Z_n be a sequence of random elements in a Polish space E equipped with the Borel σ-algebra. Then Z_n converges in probability to an E-valued random element if and only if for every pair of subsequences*

$(Z_{\varphi(n)}, Z_{\psi(n)})$, there is a subsequence of $(Z_{\varphi(n)}, Z_{\psi(n)})$ which converges in law to a random element supported on the diagonal $\{(x, y) \in E\times E,\ x = y\}$.

3 Proof of Theorem 2.2

The proof will be divided into four steps: in the first one, we show the existence of an adapted semi-discrete solution for (2.6)–(2.7), and for a truncated version of (2.6)–(2.7); then, in a second step, we prove the tightness of some sequence related to this latter solution. We proceed with the passage to the limit in the equation, and the conclusion of the proof of Theorem 2.2. In Section 3.4, we gather the proofs of the most technical lemmas used in Sections 3.1 and 3.2, this in order to keep clear the progression of the proof of Theorem 2.2

In the whole Section 3, we assume that σ satisfies (A3), and we set $q = 2(2\sigma + 1) < \frac{2d}{d-2}$.

3.1 Existence of an adapted semi-discrete solution

In order to prove, for a fixed $n \in \mathbb{N}$, the existence of a solution of (2.6)–(2.7), we first fix a family $(\eta_\ell)_{0\le\ell\le n-1}$ of deterministic functions in $H^s(\mathbb{R}^d, \mathbb{R})$, s being the exponent arising in (B2). We also fix $\tilde{u}_\ell \in H^1(\mathbb{R}^d)$, and we show the existence of (at least one) solution $\tilde{u}_{\ell+1} \in H^1(\mathbb{R}^d)$ of

$$(3.1) \qquad i\frac{\tilde{u}_{\ell+1} - \tilde{u}_\ell}{\delta t} + \Delta\tilde{u}_{\ell+1/2} + f(|\tilde{u}_\ell|, |\tilde{u}_{\ell+1}|)\tilde{u}_{\ell+1/2} = \frac{\eta_\ell}{\sqrt{\delta t}}\tilde{u}_{\ell+1/2}.$$

where we have set as before $\tilde{u}_{\ell+1/2} = \frac{1}{2}(\tilde{u}_\ell + \tilde{u}_{\ell+1})$ and f is defined by (2.5).

Lemma 3.1 *Given $\eta_\ell \in H^s(\mathbb{R}^d, \mathbb{R})$ and $\tilde{u}_\ell \in H^1(\mathbb{R}^d)$, (3.1) has at least one solution $\tilde{u}_{\ell+1} \in H^1(\mathbb{R}^d)$.*

Proof. The proof uses a standard Galerkin method, together with a Brouwer fixed point theorem to obtain a finite dimensional approximation of the solution. We then make use of the compactness of the injection $H^1(\mathbb{R}^d) \subset H^r_{\text{loc}}(\mathbb{R}^d)$ for any $r < 1$, after noticing that the following a-priori estimates hold for the solution $\tilde{u}_{\ell+1}$ of (3.1).

Assuming that $\tilde{u}_{\ell+1}$ is an $H^1(\mathbb{R}^d)$ solution of (3.1), and multiplying (3.1) by the complex conjugate of $\tilde{u}_{\ell+1/2}$, integrating over $\mathbb{R}^d$ and taking the imaginary part of the resulting identity yields

$$(3.2) \qquad |\tilde{u}_{\ell+1}|^2_{L^2(\mathbb{R}^d)} = |\tilde{u}_\ell|^2_{L^2(\mathbb{R}^d)}.$$

In the same way, we multiply (3.1) by $-\Delta\bar{\tilde{u}}_{\ell+1/2} - f(|\tilde{u}_\ell|, |\tilde{u}_{\ell+1}|)\bar{\tilde{u}}_{\ell+1/2}$, integrate over $\mathbb{R}^d$ and take the imaginary part of the resulting identity. We obtain:

$$\begin{aligned} H(\tilde{u}_{\ell+1}) - H(\tilde{u}_\ell) &= \sqrt{\delta t}\ \mathrm{Im} \int_{\mathbb{R}^d} \nabla\eta_\ell . \nabla\bar{\tilde{u}}_{\ell+1/2}\tilde{u}_{\ell+1/2} dx \\ &\leq \frac{1}{4}|\nabla\tilde{u}_{\ell+1}|_{L^2}^2 + C(\delta t, |\eta_\ell|_{H^s}, |\tilde{u}_\ell|_{H^1}, |\tilde{u}_{\ell+1}|_{L^2}), \end{aligned}$$

Cauchy-Schwarz inequality and the Sobolev embedding $H^{s-1}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ – recall that $s - 1 > \frac{d}{2}$ - have been used in the last step. Note that this computation can be justified thanks to the finite dimensional approximation of the equation.

It follows then from (3.2) and Lemma 2.1 that

$$|\tilde{u}_{\ell+1}|_{H^1(\mathbb{R}^d)}^2 \leq C(\delta t, |\eta_\ell|_{H^s}, |\tilde{u}_\ell|_{H^1}). \tag{3.3}$$

□

We can now define a multivalued application

$$\Lambda : H^1(\mathbb{R}^d) \times H^s(\mathbb{R}^d, \mathbb{R}) \longrightarrow \mathcal{P}(H^1(\mathbb{R}^d))$$

(where $\mathcal{P}(H^1(\mathbb{R}^d))$ is the set of subsets of $H^1(\mathbb{R}^d)$) such that for each $\tilde{u}_\ell \in H^1(\mathbb{R}^d)$ and $\eta_\ell \in H^s(\mathbb{R}^d, \mathbb{R})$, $\Lambda(\tilde{u}_\ell, \eta_\ell)$ is the set of solutions $\tilde{u}_{\ell+1}$ of (3.1). It is clear, from Lemma 3.1 and equation (3.1) that Λ takes its values into nonempty closed subsets of $H^1(\mathbb{R}^d)$, and that its graph is closed. Hence, from Theorem 3.1 in [4], there is a universally measurable section of Λ, that is there is a univocal application $\kappa : H^1(\mathbb{R}^d) \times H^s(\mathbb{R}^d, \mathbb{R}) \longrightarrow H^1(\mathbb{R}^d)$ such that for any $(u, \eta) \in H^1(\mathbb{R}^d) \times H^s(\mathbb{R}^d, \mathbb{R})$, $\kappa(u, \eta) \in \Lambda(u, \eta)$, and such that κ is measurable when the spaces are endowed with their Borelian σ-algebras.

Now, let

$$\chi_\ell = \frac{W((\ell+1)\delta t) - W(\ell\delta t)}{\sqrt{\delta t}},$$

where W is the Wiener process defined in Section 2, and assume that u_ℓ is a $\mathcal{F}_{\ell\delta t}$-measurable random variable with values in $H^1(\mathbb{R}^d)$ (endowed with the Borelian σ-algebra). Then $u_{\ell+1} = \kappa(u_\ell, \chi_\ell)$ is a $\mathcal{F}_{(\ell+1)\delta t}$- measurable random variable with values in $H^1(\mathbb{R}^d)$. Hence, we have proved the following proposition.

Proposition 3.1 *Let σ and Φ satisfy (A3) and (B2), and let u_0 be $\mathcal{F}_0$-measurable with values in $H^1(\mathbb{R}^d)$, then for each $n \in \mathbb{N}$ and $\delta t = \frac{T_0}{n}$ there is an adapted semi-discrete solution u^n of (2.6) which is almost surely in $L^\infty(0, T_0; H^1(\mathbb{R}^d))$, such that $u^n(0) = u_0$.*

As was pointed out in the introduction, we will have to use a semi-discrete scheme in which the noise has been truncated. For that purpose, we introduce a real valued function $\rho \in C_0^\infty(\mathbb{R})$ such that supp $\rho \subset (-2, 2)$, $\rho(x) = 1$ for

$x \in [-1, 1]$ and $0 \le \rho(x) \le 1$ for $x \in \mathbb{R}$; we then define $\rho_k(x) = \rho(\frac{x}{k})$ for $k \in \mathbb{N}^*$, and we set for $v \in L^q(\mathbb{R}^d)$, $\theta_k(v) = \rho_k(|v|^q_{L^q_x})$; assuming now that $k \in \mathbb{N}^*$ is fixed, we will consider the following truncated equation, giving $u^k_{\ell+1}$ in terms of u^k_ℓ and $\chi_\ell = \frac{1}{\sqrt{\delta t}}(W((\ell+1)\delta t) - W(\ell\delta t))$,

$$\begin{aligned} &i\frac{u^k_{\ell+1} - u^k_\ell}{\delta t} + \Delta u^k_{\ell+1/2} + f(|u^k_\ell|, |u^k_{\ell+1}|)u^k_{\ell+1/2} \\ &= \theta_k(u^k_\ell)\theta_k(u^k_{\ell+1})\frac{\chi_\ell}{\sqrt{\delta t}}u^k_{\ell+1/2}. \end{aligned} \tag{3.4}$$

In this equation, f is given by (2.5), $u^k_{\ell+1/2} = \frac{1}{2}(u^k_\ell + u^k_{\ell+1})$ and $q = 2(2\sigma+1)$.

Using the same arguments as above, we can easily show that there is for each $k \in \mathbb{N}$ a measurable application $\kappa_k : H^1(\mathbb{R}^d) \times H^s(\mathbb{R}^d, \mathbb{R}) \longrightarrow H^1(\mathbb{R}^d)$, such that for any $(u_\ell, \eta_\ell) \in H^1(\mathbb{R}^d) \times H^s(\mathbb{R}^d, \mathbb{R})$, $u_{\ell+1} = \kappa_k(u_\ell, \eta_\ell)$ is a solution of the following equation:

$$\begin{aligned} &i\frac{u_{\ell+1} - u_\ell}{\delta t} + \Delta u_{\ell+1/2} + f(|u_\ell|, |u_{\ell+1}|)u_{\ell+1/2} \\ &= \theta_k(u_\ell)\theta_k(u_{\ell+1})\frac{\eta_\ell}{\sqrt{\delta t}}u_{\ell+1/2}. \end{aligned} \tag{3.5}$$

We then define

$$\begin{aligned} \tilde{\kappa}_k(u, \eta) = {} & \kappa(u, \eta)\mathbf{1}_{\{|u|_{L^q_x} \le k\}}(u, \eta)\mathbf{1}_{\{|\kappa(u,\eta)|_{L^q_x} \le k\}}(u, \eta) \\ & + \kappa_k(u, \eta)\mathbf{1}_{\{|u|_{L^q_x} > k\} \cup \{|\kappa(u,\eta)|_{L^q_x} > k\}}(u, \eta); \end{aligned}$$

this ensures that we will take the same solution of (3.1) and (3.5) as long as it is possible, that is if $u_0 = u^k_0$ and $(\eta_m)_{0 \le m \le \ell}$ are given, and if we define $u_{m+1} = \kappa(u_m, \eta_m)$ and $u^k_{m+1} = \tilde{\kappa}_k(u^k_m, \eta_m)$, then $u_\ell = u^k_\ell$ if $\sup_{m \le \ell} |u_m|_{L^q_x} \le k$; $\tilde{\kappa}_k$ is clearly also measurable, and using the same arguments as before, we easily prove the following proposition.

Proposition 3.2 *Let σ and Φ satisfy (A3) and (B2), and let u_0 be $\mathcal{F}_0$-measurable with values in $H^1(\mathbb{R}^d)$, then for each $n \in \mathbb{N}$, $\delta t = \frac{T_0}{n}$ and each $k \in \mathbb{N}^*$, there is an adapted semi-discrete solution u^n of (2.6) and an adapted semi-discrete solution $u^{n,k}$ of (3.4), which are in $L^\infty(0, T_0; H^1(\mathbb{R}^d))$, such that $u^n(0) = u^{n,k}(0) = u_0$, and such that $u^n(\ell\delta t) = u^{n,k}(\ell\delta t)$ if $\sup_{m \le \ell} |u^n(m\delta t)|_{L^q_x} \le k$.*

Remark 3.1 It may seem more natural to use a cut-off of the form $\rho_k(|u^{n,k}_{\ell+1/2}|^q_{L^q_x})$ to truncate a Stratonovich differential. However, such a term makes the estimates we need on the solution much more complicated to obtain. Anyway, this truncating term is artificial and we get rid of it when passing to the limit (see Section 3.3), so that this has no influence on the final result.

3.2 Tightness

We now briefly show how we can derive some estimates, independent of n – but which depend on k – on the solution $u^{n,k}$ given by Proposition 3.1; these estimates will allow us to obtain the tightness of some sequence related to $u^{n,k}$. We will explain how they may be proved thanks to an "equivalent of the Ito formula" for discrete equations, but we postpone the technical part of the proof until Section 3.4. We now assume that Φ satisfies (B2), σ satisfies (A3) and that u_0 is as in the statement of Theorem 2.2. We also assume that u^n (resp. $u^{n,k}$) is a semi-discrete solution of (2.6) (resp. (3.4)) as given by Proposition 3.2.

To lighten the notations, we denote in what follows by $C(u_0, \cdots)$ a constant which – among other things – depends on $\mathbb{E}(|u_0|_{L^2_x}^{4\sigma/(2-\sigma d)})$ and $\mathbb{E}(H(u_0))$.

We have the following lemma.

Lemma 3.3 *There is a constant $C_k = C(T_0, \|\Phi\|_{L_2^{0,s}}, u_0, k)$, which depends on k but not on n, such that*

$$\mathbb{E}\left(\sup_{t\in[0,T_0]} |u^{n,k}(t)|^2_{H^1(\mathbb{R}^d)}\right) \le C_k$$

and

$$\mathbb{E}\left(\sup_{t\in[0,T_0]} |u^{n,k}(t)|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^d)}\right) \le C_k$$

for any $n \in \mathbb{N}$.

The proof of Lemma 3.3 – which is done in Section 3.4 – is based on the evolution of the energy for the solutions of equation (3.4). Indeed, it is not difficult to show (see the proof of Lemma 3.1) that since θ_k is real valued,

$$|u^{n,k}(t)|^2_{L^2_x} = |u^{n,k}(0)|^2_{L^2_x} = |u_0|^2_{L^2_x} \tag{3.6}$$

for $t \in [0, T_0]$. In the same way, we easily obtain as in the proof of Lemma 3.1, with $\delta t = \frac{T_0}{n}$, and setting $u^{n,k}_\ell = u^{n,k}(\ell\delta t)$, $\ell \le n$,

$$\begin{aligned}&\frac{H(u^{n,k}_{\ell+1}) - H(u^{n,k}_\ell)}{\delta t}\\ &= \frac{1}{\sqrt{\delta t}}\theta_k(u^{n,k}_\ell)\theta_k(u^{n,k}_{\ell+1}) \operatorname{Im}\int_{\mathbb{R}^d} \nabla\chi_\ell.\nabla\bar{u}^{n,k}_{\ell+1/2}u^{n,k}_{\ell+1/2}dx.\end{aligned} \tag{3.7}$$

If instead of the term $\nabla\bar{u}^{n,k}_{\ell+1/2}u^{n,k}_{\ell+1/2}$ in the integral on the right hand side of (3.7), we had a $\mathcal{F}_{\ell\delta t}$-measurable term, then the integral would be a martingale and the right hand side of (3.7) would be much easier to estimate.

However, $(u^{n,k}_{\ell+1/2})_\ell$ is not adapted. Copying the classical proof of the Ito formula consists then in plugging the expression of $u^{n,k}_{\ell+1}$ given by equation (3.4) into the right hand side of (3.7). But each time we do so, we introduce in the right hand side of (3.7) a new Laplace operator, and hence we loose regularity. For that reason, there is no hope of obtaining an estimate in that way.

The idea is then to replace the use of (3.4) by the use of the integral equation associated to (3.4). Indeed, introducing the operator

$$S_{\delta t} = (i - \frac{\delta t}{2}\Delta)(i + \frac{\delta t}{2}\Delta)^{-1} \tag{3.8}$$

and using (3.4) leads to

$$\begin{aligned} u^{n,k}_{\ell+1} &= S_{\delta t} u^{n,k}_\ell - \delta t (i + \frac{\delta t}{2}\Delta)^{-1} f(|u^{n,k}_\ell|, |u^{n,k}_{\ell+1}|) u^{n,k}_{\ell+1/2} \\ &+ \sqrt{\delta t}(i + \frac{\delta t}{2}\Delta)^{-1} \theta_k(u^{n,k}_\ell)\theta_k(u^{n,k}_{\ell+1})\chi_\ell u^{n,k}_{\ell+1/2}. \end{aligned} \tag{3.9}$$

Now, we plug (3.9) in the right hand side of (3.7) each time $u^{n,k}_{\ell+1}$ appears. In doing so, the right hand side of (3.7) is written as the increment of the stochastic integral of an adapted process (with a factor $\frac{1}{\sqrt{\delta t}}$) plus a remaining term in which the factor $\frac{1}{\sqrt{\delta t}}$ has disappeared. Once the identities obtained in this way have been summed over all ℓ between 0 and n, the term corresponding to the stochastic integral is estimated thanks to martingale inequalities. The remaining term is estimated directly, although not so easily due to technical difficulties (see Section 3.4 for details). In particular, the estimate of this remaining term requires the use of the cut-off.

Now, because we will use a compactness method, it is easier to work with time continuous processes; it is then natural to interpolate linearly the values of $u^{n,k}(\ell\delta t)$ for two neighbors ℓ and $\ell + 1$. However, since we need the process to be adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, we do the interpolation in the following way: we define $v^{n,k}(t)$ for $t \in [0, T_0]$ by

$$\begin{cases} v^{n,k}(t) = u_0 \quad \text{if} \quad t \in [0, \delta t] \quad \text{with} \quad \delta t = \frac{T_0}{n} \\ \text{and} \\ v^{n,k}(\alpha\ell\delta t + (1-\alpha)(\ell+1)\delta t) \\ \quad = \alpha u^{n,k}((\ell-1)\delta t) + (1-\alpha)u^{n,k}(\ell\delta t) \\ \text{for} \quad \alpha \in [0,1],\ \ell \in \{1 \cdots, n-1\}. \end{cases} \tag{3.10}$$

We also define $v^n(t)$, $t \in [0, T_0]$ in the same way with $u^{n,k}$ replaced by u^n.

It is clear that defined in this way, $v^{n,k}$ (resp. v^n) is an adapted process, which is continuous on $[0, T_0]$ with values in $H^1(\mathbb{R}^d)$. Moreover, we have immediately

Corollary 3.1 *The estimates of Lemma 3.3 hold with $u^{n,k}$ replaced by $v^{n,k}$ defined by (3.10).*

From now on, we will mainly work with $v^{n,k}$ instead of $u^{n,k}$. Our aim is still to show the tightness of some sequence related to $v^{n,k}$. This is a consequence of the following lemma.

Lemma 3.4 *There are constants α, β, γ, $\delta > 0$ and $C_{j,k} = C_j(T_0, u_0, k, \|\Phi\|_{L_2^{0,s}})$, $j = 1, \cdots, 5$, independent of n such that*

$$\mathbb{E}(|v^{n,k}|^2_{L^{2\sigma+2}(0,T_0;H^1(\mathbb{R}^d))}) \leq C_{1,k} \tag{3.11}$$

$$\mathbb{E}(|v^{n,k}|^{2/2\sigma+1}_{W^{\alpha,2\sigma+2}(0,T_0;H^{-1}(\mathbb{R}^d))}) \leq C_{2,k} \tag{3.12}$$

$$\mathbb{E}(|v^{n,k}|^{2/2\sigma+1}_{C^{\beta}([0,T_0];H^{-1}(\mathbb{R}^d))}) \leq C_{3,k} \tag{3.13}$$

$$\mathbb{E}(||v^{n,k}(\cdot)|^q_{L^q_x}|^{\delta}_{C^{\gamma}([0,T_0])}) \leq C_{4,k}. \tag{3.14}$$

Again, Lemma 3.4 is proved in Section 3.4. The idea, for proving the estimates in Lemma 3.4 involving fractional time derivatives of $v^{n,k}$ is to use that, by (3.4),

$$\begin{aligned}
v^{n,k}(t) &= u_0 + \int_0^t \partial_t v^{n,k}(s)ds \\
&= u_0 + \sum_{\ell=1}^{n-1} \int_0^t \frac{u_\ell^{n,k} - u_{\ell-1}^{n,k}}{\delta t} \mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds \\
&= u_0 + i \sum_{\ell=1}^{n-1} \int_0^t \Big(\Delta u_{\ell-1/2}^{n,k} + f(|u_{\ell-1}^{n,k}|, |u_\ell^{n,k}|)u_{\ell-1/2}^{n,k} \\
&\qquad - \theta_k(u_{\ell-1}^{n,k})\theta_k(u_\ell^{n,k}) \frac{\chi_{\ell-1}}{\sqrt{\delta t}} u_{\ell-1/2}^{n,k} \Big) \mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds;
\end{aligned} \tag{3.15}$$

again, the last term in the right hand side of (3.15) is written as the sum of the stochastic integral of an adapted process plus a remaining term. This remaining term is estimated directly with the use of Corollary 3.1, while the expectation of the square of the stochastic integral is bounded in $W^{\alpha,2\sigma+2}(0, T_0; L^2(\mathbb{R}^d))$ for any $\alpha < 1/2$, thanks to Lemma 2.1 in [17].

We may now state and prove the proposition concerning the tightness of the sequence of semi-discrete solutions. We set, for each $n, k \in \mathbb{N}$

$$\begin{aligned}
z^{n,k} &= (v^{n,k}, |v^{n,k}(\cdot)|^q_{L^q_x}) \\
Z^n &= (z^{n,k})_{k\in\mathbb{N}}
\end{aligned} \tag{3.16}$$

and we define, for $r \geq 0$ the space

$$X^r_{T_0} = L^{2\sigma+2}(0, T_0; H^r_{\mathrm{loc}}(\mathbb{R}^d)) \cap C([0, T_0]; H^{-2}_{\mathrm{loc}}(\mathbb{R}^d)) \times C([0, T_0]; \mathbb{R}). \tag{3.17}$$

We then prove the following proposition (note that we will use Lemma 2.1 to obtain the convergence of v^n in probability), in which W is the Wiener process.

Proposition 3.3 *Let $0 \le r < 1$ and $s' < s$; then for any pair of subsequences $(\varphi(n), \psi(n))_{n\in\mathbb{N}}$, the family of laws $(\mathcal{L}(Z^{\varphi(n)}, Z^{\psi(n)}, W))_{n\in\mathbb{N}}$ is tight in $(X^r_{T_0})^{\mathbb{N}} \times (X^r_{T_0})^{\mathbb{N}} \times C([0, T_0]; H^{s'}_{loc}(\mathbb{R}^d))$.*

Proof of Proposition 3.3 The proof follows from the fact that for any $\alpha < 1/2$, $W \in L^2(\Omega; W^{\alpha,2p}(0, T_0; H^s(\mathbb{R}^d))) \subset L^2(\Omega; C^\beta([0, T_0]; H^s(\mathbb{R}^d)))$ if $0 < \beta < \alpha - 1/2p$, Tychonov Theorem, Lemma 3.4, Tchebychev inequality and the following lemma, which is proved by using a classical compact embedding theorem, Ascoli-Arzela theorem and a diagonal extraction. □

Lemma 3.5 *Let $\alpha, \beta > 0$, $0 \le r < 1$ and let $\delta = (\delta_m)_{m\in\mathbb{N}}$ be a sequence of positive numbers; the set $A(\delta)$ of functions u in $L^{2\sigma+2}(0, T_0; H^1_{loc}(\mathbb{R}^d)) \cap C([0, T_0]; H^{-1}_{loc}(\mathbb{R}^d))$ such that for any $m \ge 1$,*

$$|u|^2_{L^{2\sigma+2}(0,T_0;H^1(B_m))} + |u|^2_{W^{\alpha,2\sigma+2}(0,T_0;H^{-1}(B_m))} + |u|^2_{C^\beta([0,T_0];H^{-1}(B_m))} \le \delta_m$$

is compactly embedded into $L^{2\sigma+2}(0, T_0; H^r_{loc}(\mathbb{R}^d)) \cap C([0, T_0]; H^{-2}_{loc}(\mathbb{R}^d))$, where $B_m = B(0, m)$ is the ball centered at 0 of radius m in $\mathbb{R}^d$.

3.3 Passage to the limit and conclusion

We now fix a pair of subsequences $(\varphi(n), \psi(n))_{n\in\mathbb{N}}$ and positive numbers r, s', with $0 \le r < 1$, r sufficiently close to 1 so that $H^r(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$, $q = 2(2\sigma + 1)$, and $s' < s$ with $s' > 1 + d/2$. We infer from Proposition 3.3, Prokhorov and Skorokhod Theorems that there is a subsequence of $(Z^{\varphi(n)}, Z^{\psi(n)}, W)$ which we still denote by the same letters, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables $(\tilde{Z}^n_1, \tilde{Z}^n_2, \tilde{W}^n)$, $(\tilde{Z}_1, \tilde{Z}_2, \tilde{W})$ with values in

$$(X^r_{T_0})^{\mathbb{N}} \times (X^r_{T_0})^{\mathbb{N}} \times C([0, T_0]; H^{s'}_{loc}(\mathbb{R}^d))$$

such that for any $n \in \mathbb{N}$,

$$\mathcal{L}(\tilde{Z}^n_1, \tilde{Z}^n_2, \tilde{W}^n) = \mathcal{L}(Z^{\varphi(n)}, Z^{\psi(n)}, W)$$

and such that

$$\begin{aligned} &\tilde{Z}^n_j \to \tilde{Z}_j \text{ as } n \to +\infty,\ \tilde{\mathbb{P}} \text{ a.s. in } (X^r_{T_0})^{\mathbb{N}}, \quad \text{for } j = 1, 2, \\ &\tilde{W}^n \to \tilde{W} \text{ as } n \to +\infty,\ \tilde{\mathbb{P}} \text{ a.s. in } C([0, T_0]; H^{s'}_{loc}(\mathbb{R}^d)). \end{aligned}$$

Defining then

$$\tilde{\mathcal{F}}_t = \sigma\left\{\tilde{Z}_j(s), \tilde{W}(s),\ 0 \le s \le t, \quad j = 1, 2\right\}$$

and

$$\tilde{\mathcal{F}}_t^n = \sigma\left\{\tilde{Z}_j^n(s),\, \tilde{W}^n(s),\ 0 \le s \le t,\ \ j = 1, 2\right\},$$

it is easily seen that $\tilde{W}$ and $\tilde{W}^n$ are Wiener processes associated respectively with $(\tilde{\mathcal{F}}_t)_{t\ge 0}$ and $(\tilde{\mathcal{F}}_t^n)_{t\ge 0}$, with covariance operator $\Phi\Phi^*$.

Writing then $\tilde{Z}_j^n = (\tilde{v}_j^{n,k}, \tilde{\gamma}_j^{n,k})_{k\in\mathbb{N}}$ and $\tilde{Z}_j = (\tilde{v}_j^k, \tilde{\gamma}_j^k)_{k\in\mathbb{N}}$, for $j = 1, 2$, it is clear that for each $k \in \mathbb{N}$ and $j = 1, 2$ the function $\tilde{v}_j^{n,k}(t)$ is linear on each time interval $[\ell\delta t, (\ell + 1)\delta t]$, $1 \le \ell \le n - 1$, and is in $L^\infty(0, T_0; H^r_{\mathrm{loc}}(\mathbb{R}^d)) \cap C([0, T_0]; H^{-2}_{\mathrm{loc}}(\mathbb{R}^d))$; hence it is continuous on $[0, T_0]$ with values in $H^r_{\mathrm{loc}}(\mathbb{R}^d)$.

Also, since $\tilde{v}_j^{n,k}$ is adapted with respect to $(\tilde{\mathcal{F}}_t^n)_{t\ge 0}$, it follows that $\tilde{v}_j^{n,k}((\ell+1)\delta t)$ is $\mathcal{F}^n_{\ell\delta t}$-measurable for $\ell = 0, \cdots, n-1$. Also, it follows from the equality of the laws of $(\tilde{v}_1^{n,k}, \tilde{\gamma}_1^{n,k})$ and $(v^{\varphi(n),k}, |v^{\varphi(n),k}|^q_{L^q_x})$, and the equivalent for $(\tilde{v}_2^{n,k}, \tilde{\gamma}_2^{n,k})$, in $X^r_{T_0}$, that

$$\tilde{\gamma}_j^{n,k}(t) = |\tilde{v}_j^{n,k}(t)|^q_{L^q_x},\ \forall\, t \in [0, T_0],\ \ j = 1, 2. \tag{3.18}$$

It follows in particular from the preceding facts and (3.18), that $\tilde{\gamma}_j^{n,k}((\ell+1)\delta t)$ is $\tilde{\mathcal{F}}^n_{\ell\delta t}$-measurable for $\ell = 0, \cdots, n - 1$.

Finally, we infer from (3.15) and the equality of the laws that $\tilde{v}_j^{n,k}(t)$ satisfies

$$\begin{aligned}
\tilde{v}_j^{n,k}(t) = \tilde{v}_0 &+ i \sum_{\ell=1}^{n-1} \int_0^t \Delta\tilde{v}_j^{n,k}((\ell + \tfrac{1}{2})\delta t)\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&+ i \sum_{\ell=1}^{n-1} \int_0^t f\left(|\tilde{v}_j^{n,k}(\ell\delta t)|, |\tilde{v}_j^{n,k}((\ell+1)\delta t)|\right) \tilde{v}_j^{n,k}((\ell + \tfrac{1}{2})\delta t)\\
&\quad \times \mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&- i \sum_{\ell=1}^{n-1} \int_0^t \rho_k(\tilde{\gamma}_j^{n,k}(\ell\delta t))\rho_k(\tilde{\gamma}_j^{n,k}((\ell+1)\delta t)\tilde{v}_j^{n,k}((\ell + \tfrac{1}{2})\delta t)\\
&\quad \times \frac{1}{\delta t}\left(\tilde{W}^n(\ell\delta t) - \tilde{W}^n((\ell-1)\delta t)\right)\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds.
\end{aligned} \tag{3.19}$$

We now let $n \to +\infty$ in equation (3.19), using the fact that $(\tilde{v}_j^{n,k}, \tilde{\gamma}_j^{n,k}) \to (\tilde{v}_j^k, \tilde{\gamma}_j^k)$ $\tilde{\mathbb{P}}$ a.s. in $X^r_{T_0}$.

The passage to the limit in equation (3.19) is rather technical and we only state the lemma giving the limit equation, while we postpone the proof until Section 3.4. The idea for proving this lemma is again to write the last integral in the right hand side of (3.19) as the sum of the stochastic Ito integral with respect to $\tilde{W}^n$ of a discrete adapted process, and of a remaining term.

A part of this remaining term converges to the Ito correction in the continuous equation, while the other part converges to zero. Finally, the stochastic Ito integral converges to the stochastic Ito integral with respect to $\tilde{W}$ of a continuous adapted process.

Lemma 3.6 *For any $k \in \mathbb{N}$ and $j = 1, 2$, $\tilde{v}_j^k$ and $\tilde{\gamma}_j^k$ satisfy the Ito equation*

$$
\text{(3.20)} \quad i d\tilde{v}_j^k + \left(\Delta \tilde{v}_j^k + \lambda |\tilde{v}_j^k|^{2\sigma} \tilde{v}_j^k\right) dt = \rho_k^2(\tilde{\gamma}_j^k)\tilde{v}_j^k d\tilde{W} - \frac{i}{2}\rho_k^4(\tilde{\gamma}_j^k)\tilde{v}_j^k F_\Phi dt
$$

with $F_\Phi(x)$ defined as in (2.2).

Note that $u \mapsto |u(\cdot)|_{L_x^q}$ is not continuous for the topology of $X_{T_0}^r$. This is the reason why we have to consider the couple $(v^{n,k}, |v^{n,k}(\cdot)|_{L_x^q}^q)$. In this way, we have been able to take the limit $n \to \infty$. However, we do not know yet whether $|\tilde{v}_j^k(t)|_{L_x^q}^q = \tilde{\gamma}_j^k$. Next lemma states that this is in fact true.

Lemma 3.7 *For any $k \in \mathbb{N}$, $j = 1, 2$ and for any $t \in [0, T_0]$, $\tilde{v}_j^{n,k}(t)$ converges to $\tilde{v}_j^k(t)$ in $L^2(\Omega; L^q(\mathbb{R}^d))$ as n goes to infinity, and $|\tilde{v}_j^k(t)|_{L^q(\mathbb{R}^d)}^q = \tilde{\gamma}_j^k(t)$ almost surely.*

Proof. Since $\tilde{v}_j^k \in L^2(\Omega; L^1(0, T_0; H^1(\mathbb{R}^d)))$ and satisfies (3.20), it satisfies also the mild form of the equation and lives $\tilde{\mathbb{P}}$-almost surely in $C([0, T_0]; L^2(\mathbb{R}^d))$. Moreover, for a fixed $t \in [0, T_0]$, $\tilde{v}_j^{n,k}(t)$ converges weakly to $\tilde{v}_j^k(t)$ in $L^2(\Omega; H^1(\mathbb{R}^d))$, hence

$$
\text{(3.21)} \qquad \mathbb{E}\left(|\tilde{v}_j^k(t)|_{L^2(\mathbb{R}^d)}^2\right) \leq \liminf_{n \to +\infty} \mathbb{E}\left(|\tilde{v}_j^{n,k}(t)|_{L^2(\mathbb{R}^d)}^2\right).
$$

Now, applying Ito formula to $|\tilde{v}_j^k(t)|_{L^2(\mathbb{R}^d)}^2$, it follows from (3.20) and a standard regularization procedure (see [13]) that

$$
\begin{aligned}
|\tilde{v}_j^k(t)|_{L_x^2}^2 &= |\tilde{v}_0|_{L_x^2}^2 - \int_0^t (\tilde{v}_j^k, \rho_k^4(\tilde{\gamma}_j^k)\tilde{v}_j^k F_\Phi) ds + \sum_{m \in \mathbb{N}} \int_0^t \rho_k^4(\tilde{\gamma}_j^k) |\tilde{v}_j^k \Phi e_m|_{L_x^2}^2 ds \\
&= |\tilde{v}_0|_{L_x^2}^2
\end{aligned}
$$

almost surely. On the other hand, it follows from (3.6) and (3.19) that for $\ell = 1, \cdots, n-1$,

$$
|\tilde{v}_j^{n,k}(\ell \delta t)|_{L^2(\mathbb{R}^d)} = |\tilde{v}_0|_{L^2(\mathbb{R}^d)},
$$

and using the convexity of the L^2 norm and the fact that $\tilde{v}_j^{n,k}$ is piecewise linear,

$$
|\tilde{v}_j^{n,k}(t)|_{L^2(\mathbb{R}^d)} \leq |\tilde{v}_0|_{L^2(\mathbb{R}^d)} = |\tilde{v}_j^k(t)|_{L^2(\mathbb{R}^d)}, \quad \forall t \in [0, T_0]
$$

almost surely. Hence,

$$\limsup_{n\to+\infty} |\tilde{v}_j^{n,k}(t)|_{L^2(\Omega;L^2(\mathbb{R}^d))} \le |\tilde{v}_j^k(t)|_{L^2(\Omega;L^2(\mathbb{R}^d))}$$

and together with (3.21) and the weak convergence of $\tilde{v}_j^{n,k}(t)$ to $\tilde{v}_j^k(t)$ in $L^2(\Omega; L^2(\mathbb{R}^d))$, this implies the strong convergence of this sequence in $L^2(\Omega; L^2(\mathbb{R}^d))$. To prove the strong convergence in $L^2(\Omega; L^q(\mathbb{R}^d))$, we make use of Gagliardo-Nirenberg's inequality:

$$|u|_{L^q(\mathbb{R}^d)} \le C|\nabla u|^{\theta}_{L^2(\mathbb{R}^d)} |u|^{1-\theta}_{L^2(\mathbb{R}^d)}$$

with $\frac{1}{q} = \theta\left(\frac{1}{2} - \frac{1}{d}\right) + \frac{1-\theta}{2}$, from which it follows that for each $t \in [0, T_0]$,

$$\begin{aligned} &|\tilde{v}_j^{n,k}(t) - \tilde{v}_j^k(t)|_{L^2(\Omega;L^q_x)} \\ &\quad \le |\tilde{v}_j^{n,k}(t) - \tilde{v}_j^k(t)|^{1-\theta}_{L^2(\Omega;L^2_x)} \sup_{n\in\mathbb{N}} \left(|\tilde{v}_j^{n,k}(t)|^{\theta}_{L^2(\Omega;H^1_x)} + |\tilde{v}_j^k(t)|^{\theta}_{L^2(\Omega;H^1_x)} \right). \end{aligned}$$

Corollary 3.1 yields then the conclusion. □

It follows from Lemma 3.7 and the continuity of ρ_k that $\tilde{v}_j^k$ satisfies

$$i d\tilde{v}_j^k + \left(\Delta \tilde{v}_j^k + \lambda |\tilde{v}_j^k|^{2\sigma} \tilde{v}_j^k\right) dt = \theta_k^2(\tilde{v}_j^k)\tilde{v}_j^k d\tilde{W} - \frac{i}{2}\theta_k^4(\tilde{v}_j^k)\tilde{v}_j^k F_\Phi dt. \tag{3.22}$$

Now, we would like to conclude that $\tilde{v}_1^k = \tilde{v}_2^k$. However, we do not know whether the solution of (3.22) with initial data $\tilde{v}_0$ is unique or not; indeed, the truncating term in the noise does not allow us to obtain easy estimates on the difference of the stochastic integrals corresponding to two solutions. Thus, we first have to get rid of this truncating term.

Let $\tilde{v}$ be the solution of (2.3) given by Theorem 2.1, with W replaced by $\tilde{W}$ and with initial data $\tilde{v}_0$. We set

$$\tilde{\tau}_k = \inf\left\{t \in [0, T_0],\ |\tilde{v}(t)|^q_{L^q(\mathbb{R}^d)} \ge k\right\};$$

it follows then from the uniqueness part of Theorem 2.1 and (3.22) that

$$\tilde{v}\mathbf{1}_{[0,\tilde{\tau}_k]} = \tilde{v}_j^k \mathbf{1}_{[0,\tilde{\tau}_k]} \text{ for } j = 1, 2. \tag{3.23}$$

We then define

$$\begin{aligned} \tilde{\tau}_j^{n,k} = \sup\Big\{ t = \frac{\ell T_0}{n},\ \ell = 0, \cdots, n \text{ such that} \\ |\tilde{v}_j^{n,k}(s)|^q_{L^q(\mathbb{R}^d)} \le k \text{ for any } s \le t \Big\} \end{aligned}$$

and

$$\tilde{v}_j^{n,\infty}(t) = \begin{cases} \tilde{v}_j^{n,k}(t) \text{ if } t \le \tilde{\tau}_j^{n,k} \text{ for some } k \in \mathbb{N} \\ 0 \text{ if } t \ge \lim_{k\nearrow+\infty} \tilde{\tau}_j^{n,k} \end{cases} \tag{3.24}$$

(note that the definition is consistent since $\tilde{v}_j^{n,k}(t) = \tilde{v}_j^{n,k+1}(t)$ if $t \le \tilde{\tau}_j^{n,k}$).

Lemma 3.8 *$\tilde{v}_j^{n,\infty}$ converges in probability to $\tilde{v}$ in $C([0,T_0];L^q(\mathbb{R}^d))$ as n goes to infinity, for $j=1,2$.*

Proof. First of all, from Lemma 3.3, (3.13) in Lemma 3.4 – which are also satisfied by $\tilde{v}_j^{n,k}$ – and Gagliardo-Nirenberg's inequality, we easily obtain that for a fixed $k\in\mathbb{N}$, $\tilde{v}_j^{n,k}$ and $\tilde{v}_j^{n,\infty}$ are uniformly equicontinuous in time with values in $L^q(\mathbb{R}^d)$. More precisely, for some positive γ, one can find for each positive ε a positive δ such that

$$\mathbb{E}\left(\sup_{|t-s|\le\delta}|\tilde{v}_j^{n,k}(t)-\tilde{v}_j^{n,k}(s)|^{\gamma}_{L^q(\mathbb{R}^d)}\right)\le\varepsilon.$$

Together with Lemma 3.7, this implies without difficulty that $\tilde{v}_j^{n,k}$ converges in probability to $\tilde{v}_j^k$ in $C([0,T_0];L^q(\mathbb{R}^d))$ for any $k\in\mathbb{N}$. Defining then, for a fixed $\varepsilon>0$, with $\varepsilon\le 1$,

$$\tilde{\tau}_j^{\varepsilon,n}(\omega)=\inf\{t\in[0,T_0],\ |\tilde{v}_j^{n,\infty}(t)-\tilde{v}(t)|_{L^q(\mathbb{R}^d)}\ge\varepsilon\}$$

we have

$$\begin{aligned}
&\tilde{\mathbb{P}}\left(\sup_{t\in[0,T_0]}|\tilde{v}_j^{n,\infty}(t)-\tilde{v}(t)|_{L^q(\mathbb{R}^d)}\ge\varepsilon\right)\\
&\quad\le\tilde{\mathbb{P}}\left(\sup_{t\in[0,T_0\wedge\tilde{\tau}_k\wedge\tilde{\tau}_j^{\varepsilon,n}]}|\tilde{v}_j^{n,\infty}(t)-\tilde{v}(t)|_{L^q(\mathbb{R}^d)}\ge\varepsilon\right)\\
&\quad+\tilde{\mathbb{P}}\left(\sup_{t\in[0,T_0]}|\tilde{v}(t)|^q_{L^q(\mathbb{R}^d)}\ge k\right).
\end{aligned}\tag{3.25}$$

Note that for $t\in[0,T_0\wedge\tilde{\tau}_k\wedge\tilde{\tau}_j^{\varepsilon,n}]$, one has

$$\tilde{v}(t)=\tilde{v}_j^k(t)=\tilde{v}_j^{k'}(t),\ \forall\, k'\ge k$$

and

$$|\tilde{v}_j^{n,\infty}(t)|^q_{L^q(\mathbb{R}^d)}\le\left(|\tilde{v}(t)|_{L^q(\mathbb{R}^d)}+\varepsilon\right)^q\le 2^q(k+1).$$

The conclusion of Lemma 3.8 is then implied by (3.25), the fact that $\tilde{v}$ is bounded in probability in $C([0,T_0];H^1(\mathbb{R}^d))\subset C([0,T_0];L^q(\mathbb{R}^d))$, the following Lemma, which states that on $[0,T_0\wedge\tilde{\tau}_k\wedge\tilde{\tau}_j^{\varepsilon,n}]$, one has $\tilde{v}_j^{n,\infty}=\tilde{v}_j^{n,k'}$ for some integer $k'\ge k$, provided that n is sufficiently large, and the convergence of $\tilde{v}_j^{n,k'}$ to $\tilde{v}_j^{k'}$. □

Lemma 3.9 *For a given $k\in\mathbb{N}$, there is a deterministic integer k', which does not depend on n, and a random integer $n_0(k)$, such that for $j=1,2$ and for any $t\in[0,T_0\wedge\tilde{\tau}_k\wedge\tilde{\tau}_j^{\varepsilon,n}]$, one has $\tilde{v}_j^{n,\infty}(t)=\tilde{v}_j^{n,k'}(t)$ if $n\ge n_0(k,\omega)$.*

Proof of Lemma 3.9 Note that the proof of Lemma 3.9 needs some estimates, and is not an immediate consequence of the definition of $\tilde{v}_j^{n,\infty}$. This is because $T_0 \wedge \tilde{\tau}_k \wedge \tilde{\tau}_j^{\varepsilon,n}$ may not coincide with a point $\tilde{\tau}_j^{n,k'}$. The only thing we know is (3.26) below.

Let us fix k, j, n and $\varepsilon \leq 1$, and set $\ell_j^{\varepsilon,n} = \lceil \frac{n(T_0 \wedge \tilde{\tau}_k \wedge \tilde{\tau}_j^{\varepsilon,n})}{T_0} \rceil$, where $\lceil x \rceil$ stands for the smallest integer greater than or equal to x.

On the one hand, we know that

$$|\tilde{v}_j^{n,\infty}((\ell_j^{\varepsilon,n} - 1)\delta t)|^q_{L^q(\mathbb{R}^d)} \leq 2^q(k+1); \tag{3.26}$$

on the other hand, using the definition of $\tilde{v}_j^{n,\infty}$ (see (3.24)), (3.8), (3.9), and setting $\tilde{v}_\ell = \tilde{v}_j^{n,\infty}(\ell\delta t)$, we easily have, with $\ell = \ell_j^{\varepsilon,n}$:

$$\tilde{v}_\ell = S_{\delta t}\tilde{v}_{\ell-1} + \delta t(i + \frac{\delta t}{2}\Delta)^{-1}\left[-f(|\tilde{v}_\ell|, |\tilde{v}_{\ell-1}|)\tilde{v}_{\ell-1/2} + \frac{\tilde{\chi}_\ell}{\sqrt{\delta t}}\tilde{v}_{\ell-1/2}\right];$$

We then apply Hörmander-Mikhlin Theorem (see Section 3.4), make use of the obvious fact that the operator $\sqrt{\delta t}(i + \frac{\delta t}{2}\Delta)^{-1}$ is bounded from $L^2(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$ with a bound that does not depend on $\delta t \leq 1$, and of (3.26) to bound the right hand side of the above equality as follows:

$$\begin{aligned}
|\tilde{v}_\ell|_{L^q_x} &\leq C_1|\tilde{v}_{\ell-1}|_{L^q_x} + \sqrt{\delta t}\Big\{|\sqrt{\delta t}(i + \tfrac{\delta t}{2}\Delta)^{-1} f(|\tilde{v}_\ell|, |\tilde{v}_{\ell-1}|)\tilde{v}_{\ell-1/2}|_{H^1_x} \\
&\qquad + \tfrac{1}{2}|\tilde{\chi}_\ell|_{H^s_x}\left(|\tilde{v}_\ell|_{L^q_x} + |\tilde{v}_{\ell-1}|_{L^q_x}\right)\Big\} \\
&\leq C_1 k^{1/q} + \sqrt{\delta t}\Big\{|f(|\tilde{v}_\ell|, |\tilde{v}_{\ell-1}|)\tilde{v}_{\ell-1/2}|_{L^2_x} \\
&\qquad + \tfrac{1}{2}|\tilde{\chi}_\ell|_{H^s}\left(|\tilde{v}_\ell|_{L^q_x} + |\tilde{v}_{\ell-1}|_{L^q_x}\right)\Big\},
\end{aligned}$$

from which we infer, using (3.26) again, that

$$|\tilde{v}_\ell|_{L^q(\mathbb{R}^d)} + |\tilde{v}_{\ell-1}|_{L^q(\mathbb{R}^d)} \leq 2(C_1 + 1)k^{1/q}$$

provided that δt is sufficiently small, depending on k and ω, that is provided that $n \geq n_0(k, \omega)$. Setting $k' = 2^q(C_1 + 1)^q k$, we then have for $n \geq n_0(k, \omega)$,

$$|\tilde{v}_j^n(\ell_j^{\varepsilon,n}\delta t)|^q_{L^q(\mathbb{R}^d)} \leq k'$$

which proves the lemma. □

We are now able to conclude the proof of Theorem 2.2; indeed, Lemma 3.8 says that $(\tilde{v}_1^{n,\infty}, \tilde{v}_2^{n,\infty})$ converges in law to $(\tilde{v}, \tilde{v})$ in $C([0, T_0]; L^q(\mathbb{R}^d))$. Now, we have

$$\mathcal{L}\left((\tilde{v}_1^{n,k}, \tilde{v}_2^{n,k})_{k\in\mathbb{N}}\right) = \mathcal{L}\left((v^{\varphi(n),k}, v^{\psi(n),k})_{k\in\mathbb{N}}\right).$$

Also, by (3.7), and the conservation of the L^2 norm, there is a deterministic constant C_1, which does not depend on k, such that

$$\begin{aligned} &H(v^{n,k}((\ell+1)\delta t)) - H(v^{n,k}(\ell\delta t)) \\ &\quad \le C_1\sqrt{\delta t}|u_0|_{L^2_x}|\chi_\ell|_{H^s_x}\left(|v^{n,k}(\ell\delta t)|_{H^1_x} + |v^{n,k}((\ell+1)\delta t)|_{H^1_x}\right); \end{aligned}$$

this easily implies, using Lemma 2.1, that

$$\mathbb{E}\left(\sup_{t\in[0,T_0]} |v^{n,k}(t)|^2_{H^1(\mathbb{R}^d)}\right) \le C_2(n) \tag{3.27}$$

where the constant C_2 depends on n, $\|\Phi\|_{L_2^{0,s}}$ and $\mathbb{E}\left(|u_0|^{2+\frac{4\sigma}{2-\sigma d}}_{H^1(\mathbb{R}^d)}\right)$. The same is true with $v^{n,k}$ replaced by v^n. Hence, if

$$\tau^{n,k} = \inf\left\{t\in[0,T_0];\ |v^{n,k}(t)|_{H^1(\mathbb{R}^d)} \ge k\right\},$$

we have $\lim_{k\to+\infty}\tau^{n,k} = T_0$, almost surely. It follows that the laws of $(v^{\varphi(n)}, v^{\psi(n)})$ and $(\tilde v_1^{n,\infty}, \tilde v_2^{n,\infty})$ are equal in $C([0,T]; L^q(\mathbb{R}^d))$ for any $T < T_0$. Applying Lemma 2.2, we derive that for $T < T_0$, v^n converges in probability in $C([0,T]; L^q(\mathbb{R}^d))$ to some v with $\mathcal{L}(v) = \mathcal{L}(\tilde v)$.

Together with the fact that $v^{n,k}$ is bounded in probability in $C([0,T_0]; H^1(\mathbb{R}^d))$ (if k is fixed), this implies that v^n is bounded in probability in $C([0,T]; H^1_x)$. Then, the same arguments as in the proof of Lemma 3.7 show that v^n converges in probability to v in $C([0,T]; L^2(\mathbb{R}^d))$ and by interpolation in $C([0,T]; H^s(\mathbb{R}^d))$ for any $s < 1$. The same arguments as those used in the proof of Lemma 3.6 also say that v is the solution of equation (2.3) starting from u_0. The convergence of u^n to v follows from the uniform equicontinuity of $v^{n,k}$, for a fixed k, and the convergence of v^n. Finally, the conservation of the L^2 norm and the equi-integrability Lemma say that u^n converges to v in $L^p(\Omega; L^\infty(0,T; L^2(\mathbb{R}^d)))$ for any p such that $u_0 \in L^{p'}(\Omega; L^2(\mathbb{R}^d))$ for some $p' > p$. □

3.4 Proof of technical lemmas.

In this subsection, we prove Lemma 3.3, 3.4 and 3.6.

Proof of Lemma 3.3 We first show that

$$\sup_{t\in[0,T_0]} \mathbb{E}\left(|u^{n,k}(t)|^2_{H^1(\mathbb{R}^d)}\right) \le C(T_0, \|\Phi\|_{L_2^{0,s}}, u_0, k). \tag{3.28}$$

As was announced in Section 3.2, we will make use of (3.6)–(3.9). In (almost) all the proof, we omit the indices n and k, since these are fixed. Also, in order to shorten the formulas, we use the notations $u_\ell = u^{n,k}(\ell\delta t)$, $u_{\ell+1/2} = \frac{1}{2}(u_\ell + u_{\ell+1})$, $\theta^\ell = \theta_k(u_\ell)$, $f_\ell = f(|u_\ell|, |u_{\ell+1}|)u_{\ell+1/2}$ and $B_{\delta t} = (i + \frac{\delta t}{2}\Delta)$.

Now, using (3.9), we easily have

$$\begin{aligned}\nabla\chi_\ell.\nabla\bar{u}_{\ell+1/2}u_{\ell+1/2} &= \frac{1}{4}(u_\ell + u_{\ell+1})\nabla\chi_\ell.\nabla(\bar{u}_\ell + \bar{u}_{\ell+1})\\ &= \frac{1}{4}\left(A_\ell^0 + \sqrt{\delta t}A_\ell^1 + \delta t A_\ell^2 + (\delta t)^{3/2}A_\ell^3 + (\delta t)^2 A_\ell^4\right)\end{aligned} \tag{3.29}$$

with

$$\begin{aligned}A_\ell^0 &= (1+S_{\delta t})u_\ell\nabla\chi_\ell.\nabla(1+\bar{S}_{\delta t})\bar{u}_\ell,\\ A_\ell^1 &= B_{\delta t}^{-1}\left(\theta^\ell\theta^{\ell+1}\chi_\ell u_{\ell+1/2}\right)\nabla\chi_\ell.\nabla(1+\bar{S}_{\delta t})\bar{u}_\ell\\ &\quad+(1+S_{\delta t})u_\ell\nabla\chi_\ell.\nabla\bar{B}_{\delta t}^{-1}\left(\theta^\ell\theta^{\ell+1}\chi_\ell\bar{u}_{\ell+1/2}\right),\\ A_\ell^2 &= -B_{\delta t}^{-1}f_\ell\nabla\chi_\ell.\nabla(1+\bar{S}_{\delta t})\bar{u}_\ell - (1+S_{\delta t})u_\ell\nabla\chi_\ell.\nabla\bar{B}_{\delta t}^{-1}f_\ell\\ &\quad+B_{\delta t}^{-1}\left(\theta^\ell\theta^{\ell+1}\chi_\ell u_{\ell+1/2}\right)\nabla\chi_\ell.\nabla\bar{B}_{\delta t}^{-1}\left(\theta^\ell\theta^{\ell+1}\chi_\ell\bar{u}_{\ell+1/2}\right),\\ A_\ell^3 &= -B_{\delta t}^{-1}\left(\theta^\ell\theta^{\ell+1}\chi_\ell u_{\ell+1/2}\right)\nabla\chi_\ell.\nabla\bar{B}_{\delta t}^{-1}\bar{f}_\ell\\ &\quad-B_{\delta t}^{-1}f_\ell\nabla\chi_\ell.\nabla\bar{B}_{\delta t}^{-1}\left(\theta^\ell\theta^{\ell+1}\chi_\ell\bar{u}_{\ell+1/2}\right)\end{aligned}$$

and finally,

$$A_\ell^4 = B_{\delta t}^{-1}f_\ell\nabla\chi_\ell.\nabla\bar{B}_{\delta t}^{-1}\bar{f}_\ell.$$

In order to estimate the right hand side of (3.29), we will use the fact that under our assumptions, we have $u_\ell \in L^q(\mathbb{R}^d)$ for any ℓ, so that $\nabla f_\ell \in L^r(\mathbb{R}^d)$ with $\frac{1}{r} = \frac{2\sigma}{q} + \frac{1}{2} = 1 - \frac{1}{q}$. (Recall that $q = 2(2\sigma+1)$). Moreover

$$|\nabla f_\ell|_{L_x^r} \leq C|u_\ell|_{L_x^q}^{2\sigma}|u_\ell|_{H_x^1}$$

Also, $S_{\delta t}$ and $B_{\delta t}^{-1}$ are Fourier multipliers, obviously bounded on $H^m(\mathbb{R}^d)$ for any $m \in \mathbb{N}$, and it is not difficult to see, using Hörmander-Mikhlin Theorem (see for example [31]), that they are also bounded on $L^p(\mathbb{R}^d)$ for any p with $1 < p < +\infty$. We then deduce from these arguments and the Sobolev embedding $H_x^s \subset L_x^\infty$

$$\begin{aligned}|A_\ell^2|_{L_x^1} &\leq C|\chi_\ell|_{H_x^s}\left(|u_\ell|_{L_x^q}^{2\sigma+1} + |u_{\ell+1}|_{L_x^q}^{2\sigma+1}\right)|u_\ell|_{H_x^1}\\ &\quad+C|\chi_\ell|_{H_x^s}|u_\ell|_{L_x^q}|\nabla f_\ell|_{L_x^r}\\ &\quad+C|\chi_\ell|_{H_x^s}^3|u_{\ell+1/2}|_{H_x^1}|u_{\ell+1/2}|_{L_x^q}\\ &\leq C\left(|u_\ell|_{L_x^q}, |u_{\ell+1}|_{L_x^q}\right)\left(1 + |\chi_\ell|_{H_x^s}^4\right)\left(|u_\ell|_{H_x^1} + |u_{\ell+1}|_{H_x^1}\right).\end{aligned} \tag{3.30}$$

In the same way, we have for $j = 1$ or 3,

$$|A_\ell^j|_{L_x^1} \leq C\left(|u_\ell|_{L_x^q}, |u_{\ell+1}|_{L_x^q}\right)|\chi_\ell|_{H_x^s}^2\left(|u_\ell|_{H_x^1} + |u_{\ell+1}|_{H_x^1}\right). \tag{3.31}$$

Finally, we estimate A_ℓ^4 by noticing that the operator $\sqrt{\delta t}\nabla B_{\delta t}^{-1}$ is also bounded on $L^2(\mathbb{R}^d)$, independently of $\sqrt{\delta t} \le 1$, so that

$$|\sqrt{\delta t} A_\ell^4|_{L^1(\mathbb{R}^d)} \le C|\chi_\ell|_{H^s(\mathbb{R}^d)} \left(|u_\ell|_{L^q(\mathbb{R}^d)}^{2(2\sigma+1)} + |u_{\ell+1}|_{L^q(\mathbb{R}^d)}^{2(2\sigma+1)} \right). \tag{3.32}$$

Note that, up to now, the constant C only depends on σ, the dimension d and $|u_\ell|_{L^q(\mathbb{R}^d)}$, $|u_{\ell+1}|_{L^q(\mathbb{R}^d)}$.

Collecting (3.30)–(3.32), using the fact that $\theta_k(u) = 0$ for $|u|_{L^q(\mathbb{R}^d)} \ge 2k$, that χ_ℓ is a Gaussian random variable, and assuming that $\delta t \le 1$, leads to

$$\begin{aligned}
&\mathbb{E}\left(\theta^\ell \theta^{\ell+1} |A_\ell^1 + \sqrt{\delta t} A_\ell^2 + \delta t A_\ell^3 + (\delta t)^{3/2} A_\ell^4|_{L^1(\mathbb{R}^d)} \right) \\
&\quad \le C_1 \mathbb{E}\left(|u_\ell|_{H^1(\mathbb{R}^d)}^2 + |u_{\ell+1}|_{H^1(\mathbb{R}^d)}^2 \right) + C_2 \left(k, \|\Phi\|_{L_2^{0,s}} \right)
\end{aligned} \tag{3.33}$$

where C_1 is a constant which does not depend on anything (and in particular it does not depend on k). It remains to treat the term coming from A_ℓ^0, in which there is no factor $\sqrt{\delta t}$. The idea, which stems from the proof of Ito formula, is to extract from $\theta^\ell \theta^{\ell+1} A_\ell^0$ a term which has zero expectation, and such that we may estimate the remaining part as the preceding terms. With this aim in view, we write

$$\begin{aligned}
\theta^\ell \theta^{\ell+1} A_0^\ell = {} & \theta^\ell \rho_k(|S_{\delta t} u_\ell|_{L^q(\mathbb{R}^d)}) A_\ell^0 \\
& + \theta^\ell \left(\rho_k(|u_{\ell+1}|_{L^q(\mathbb{R}^d)}^q) - \rho_k(|S_{\delta t} u_\ell|_{L^q(\mathbb{R}^d)}^q) \right) A_\ell^0.
\end{aligned} \tag{3.34}$$

It is clear that the expectation of the first term on the right hand side of (3.34) is zero. For the second term, we write, using (3.9) again,

$$\begin{aligned}
&\left| \rho_k \left(|u_{\ell+1}|_{L_x^q}^q \right) - \rho_k(|S_{\delta t} u_\ell|_{L_x^q}^q) \right| \\
&\quad \le |\rho_k'|_{L^\infty} \left(|u_\ell|_{L_x^q}^{q-1} + |u_{\ell+1}|_{L_x^q}^{q-1} \right) |u_{\ell+1} - S_{\delta t} u_\ell|_{L_x^q} \\
&\quad \le \sqrt{\delta t} |\rho_k'|_{L^\infty} \left(|u_\ell|_{L_x^q}^{q-1} + |u_{\ell+1}|_{L_x^q}^{q-1} \right) \\
&\qquad \times \left(|\sqrt{\delta t} B_{\delta t}^{-1} f_\ell|_{H_x^1} + \frac{1}{2} |\chi_\ell|_{H_x^s} \theta^\ell \theta^{\ell+1} \left(|u_\ell|_{L_x^q} + |u_{\ell+1}|_{L_x^q} \right) \right).
\end{aligned} \tag{3.35}$$

Using again the fact that $\sqrt{\delta t} B_{\delta t}^{-1}$ is bounded from $L^2(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$ independently of $\delta t \le 1$, and the conservation of the L^2 norm (see (3.6)), we deduce from the preceding inequality,

$$\begin{aligned}
&\frac{1}{\sqrt{\delta t}} \left| \theta^\ell \left(\rho_k(|u_{\ell+1}|_{L_x^q}^q) - \rho_k(|S_{\delta t} u_\ell|_{L_x^q}^q) \right) A_\ell^0 \right|_{L^1(\mathbb{R}^d)} \\
&\le |\theta^\ell| C \left(|u_\ell|_{L_x^q}, |u_{\ell+1}|_{L_x^q}, |u_0|_{L_x^2} \right) \left(1 + |\chi_\ell|_{H_x^s}^2 \right) \left(|u_\ell|_{H_x^1} + |u_{\ell+1}|_{H_x^1} \right).
\end{aligned} \tag{3.36}$$

The right hand side above is zero if $|u_\ell|^q_{L^q(\mathbb{R}^d)} \geq 2k$; in the case where both $|u_\ell|^q_{L^q_x}$ and $|u_{\ell+1}|^q_{L^q_x}$ are less than $2k$, it may be bounded above by

$$C(k, u_0)\left(1 + |\chi_\ell|^2_{H^s_x}\right)\left(|u_\ell|_{H^1_x} + |u_{\ell+1}|_{H^1_x}\right).$$

It remains to treat the case where $|u_{\ell+1}|^q_{L^q_x} \geq 2k$, but $|u_\ell|^q_{L^q_x} < 2k$. In this case, the right hand side of (3.4) vanishes, and it is not difficult to see that we have an estimate of the form

$$|u_{\ell+1}|_{L^q_x} \leq C|u_\ell|_{L^q_x} + C\delta t\left(|u_\ell|^{2\sigma+1}_{L^q_x} + |u_{\ell+1}|^{2\sigma+1}_{L^q_x}\right).$$

Hence,

$$|u_{\ell+1}|_{L^q_x}\mathbf{1}_{\{|u_\ell|_{L^q_x}\leq 2k\}} \leq C(k) + C\delta t|u_{\ell+1}|^{2\sigma+1}_{L^q_x}\mathbf{1}_{\{|u_\ell|_{L^q_x}\leq 2k\}}$$

where the constants C are deterministic constants. It follows that

$$|u_{\ell+1}|_{L^q_x}\mathbf{1}_{\{|u_\ell|_{L^q_x}\leq 2k\}} \leq 2C(k) \tag{3.37}$$

provided that $\delta t \leq C_3(k)$, that is provided that $n \geq n_0(k)$ for some integer n_0 which only depends on k.

Collecting (3.7), (3.29), (3.33)–(3.36) and the preceding estimate shows that

$$\begin{aligned}
&\frac{1}{\delta t}\mathbb{E}\left(H(u_{\ell+1}) - H(u_\ell)\right)\\
&\quad\leq C_4\mathbb{E}\left(|u_\ell|^2_{H^1_x} + |u_{\ell+1}|^2_{H^1_x}\right) + C_5\left(k, u_0, \|\Phi\|_{L_2^{0,s}}\right)\\
&\quad\leq 4C_4\mathbb{E}\left(H(u_\ell) + H(u_{\ell+1})\right) + C_6\left(k, u_0, \|\Phi\|_{L_2^{0,s}}\right)
\end{aligned}$$

provided that $n \geq n_0(k)$, where we have used Lemma 2.1 and the conservation of L^2 norm if $\lambda = +1$. We deduce from this last estimate that

$$\begin{aligned}
\mathbb{E}\left(H(u_{\ell+1})\right) &\leq \tfrac{1+4C_4\delta t}{1-4C_4\delta t}\mathbb{E}\left(H(u_\ell)\right) + C_7\left(k, u_0, \|\Phi\|_{L_2^{0,s}}\right)\\
&\leq e^{8C_4\delta t}\mathbb{E}\left(H(u_\ell)\right) + C_7
\end{aligned}$$

provided that $n \geq \max(n_0(k), N_0)$ for some integer N_0; hence,

$$\mathbb{E}\left(H(u_\ell)\right) \leq e^{8C_4 n\delta t}\mathbb{E}\left(H(u_0)\right) + C_8$$

for $n \geq \max(n_0(k), N_0)$ and (3.28) easily follows for n in this range. It remains to treat the case where $n < \max(n_0(k), N_0)$, that is it remains to obtain an estimate on $\sup_{t\in[0,T_0]}\mathbb{E}\left(|u^{n,k}(t)|^2_{H^1_x}\right)$ which may depend on k and n.

Actually, it easily follows from (3.7), Lemma 2.1 and the fact that $\theta_k \leq 1$, that

$$\sup_{t\in[0,T_0]} \mathbb{E}\left(|u^{n,k}(t)|^2_{H^1(\mathbb{R}^d)}\right) \leq C_9\left(n, \|\Phi\|_{L_2^{0,s}}, u_0, T_0\right)$$

with a constant C_9 which does not depend on k, so that (3.28) holds for any n and k.

We will now use a martingale inequality to derive a bound on the expectation of the sup norm in time: coming back to (3.7), (3.29), and using (3.33)–(3.37) yields for $n \geq n_0(k)$

$$\begin{aligned} \frac{1}{\delta t}\left(H(u_{\ell+1}) - H(u_\ell)\right) \leq{} & \frac{\theta^\ell}{4\sqrt{\delta t}}\rho_k(|S_{\delta t}u_\ell|^q_{L^q_x})\mathrm{Im}\int_{\mathbb{R}^d} A^0_\ell dx \\ & + C_4\left(|u_\ell|^2_{H^1_x} + |u_{\ell+1}|^2_{H^1_x}\right) + C_{10}\left(\omega, k, \Phi, u_0\right) \end{aligned} \tag{3.38}$$

where C_4 is the preceding deterministic constant and C_{10} is a positive random variable depending on its arguments, and which satisfies $\mathbb{E}(C_{10}) < +\infty$. Next, we will sum (3.38) from $\ell = 0$ to $\ell = m-1$, with $m \leq n$. We use

$$\begin{aligned} &\sqrt{\delta t}\sum_{\ell=0}^{m-1}\left(\mathrm{Im}\int_{\mathbb{R}^d} A^0_\ell dx\right)\theta^\ell\rho_k(|S_{\delta t}u_\ell|^q_{L^q_x}) \\ &= \mathrm{Im}\sum_{\ell=0}^{m-1}\int_{\mathbb{R}^d}\theta^\ell\rho_k(|S_{\delta t}u_\ell|^q_{L^q_x})\left((1+S_{\delta t})u_\ell\right)\nabla(1+\bar{S}_{\delta t})\bar{u}_\ell \\ &\quad\cdot\nabla\left(W((\ell+1)\delta t) - W(\ell\delta t)\right)dx \\ &= M(m\delta t) \end{aligned}$$

with

$$M(t) = \int_0^t \left(F^{n,k}(s), \nabla dW(s)\right)$$

and

$$\begin{aligned} F^{n,k}(s) = {} & -i\rho_k(|u^{n,k}(s)|^q_{L^q_x})\rho_k(|S_{\delta t}u^{n,k}(s)|^q_{L^q_x}) \\ & \times\left((1+S_{\delta t})u^{n,k}(s)\right)\nabla(1+\bar{S}_{\delta t})\bar{u}^{n,k}(s). \end{aligned}$$

In this way, we obtain

$$H(u_m) \leq H(u_0) + \frac{1}{4}M(m\delta t) + C_4\delta t\sum_{\ell=0}^{n-1}\left(|u_\ell|^2_{H^1_x} + |u_{\ell+1}|^2_{H^1_x}\right) + C_{10}m\delta t. \tag{3.39}$$

It is clear that under our assumptions, M is a square integrable real valued martingale. Using a standard martingale inequality (see Theorem 3.14 in [9], or [27]), and the fact that $\rho_k \leq 1$, we deduce

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t\le T_0}|M(t)|\right)\\
&\quad\le 3\mathbb{E}\left(\left(\int_0^{T_0}|(1+S_{\delta t})u^{n,k}\nabla\Phi|^2_{L_2^{0,0}}|\nabla(1+S_{\delta t})u^{n,k}|^2_{L_x^2}ds\right)^{1/2}\right)\\
&\quad\le C\mathbb{E}\left(\|\Phi\|_{L_2^{0,s}}|u_0|_{L_x^2}\left(\int_0^{T_0}|u^{n,k}(s)|^2_{H_x^1}ds\right)^{1/2}\right)\\
&\quad\le C_{11}\left(\|\Phi\|_{L_2^{0,s}},u_0\right)+C_{12}\sup_{t\in[0,T_0]}\mathbb{E}\left(|u^{n,k}(t)|^2_{H_x^1}\right)
\end{aligned}
$$

where we have used the conservation of L^2 norm again. This last estimate, together with (3.39) and (3.28) yield

$$
\mathbb{E}\left(\sup_{t\in[0,T_0]}H\left(u^{n,k}(t)\right)\right)\le C\left(T_0,\|\Phi\|_{L_2^{0,s}},u_0,k\right).
$$

The conclusion of Lemma 3.3, in the case $n\ge n_0(k)$, follows immediately after the use of Lemma 2.1.

Now, we have already observed that when n is fixed, an estimate of the form

$$
\mathbb{E}\left(\sup_{t\in[0,T_0]}|u^{n,k}(t)|^2_{H^1(\mathbb{R}^d)}\right)\le C\left(T_0,\|\Phi\|_{L_2^{0,s}},u_0,n\right)
$$

holds, with a constant depending on n on the right hand side. Hence the conclusion. □

Proof of Lemma 3.4 Estimate (3.11) follows from Corollary 3.1. In order to prove (3.12), we will use the equation satisfied by $v^{n,k}$, that is (3.15). Again, we omit the indices n and k most of the time, and we use the notations introduced in the proof of Lemma 3.3.

By (3.11), the term

$$
\sum_{\ell=1}^{n-1}\int_0^t\Delta u_{\ell-1/2}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds
$$

in (3.15) is clearly bounded in $L^2(\Omega;W^{1,2\sigma+2}(0,T_0;H^{-1}(\mathbb{R}^d)))$, and the term

$$
\sum_{\ell=1}^{n-1}\int_0^t f_{\ell-1}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds
$$

is bounded in $L^{2/(2\sigma+1)}(\Omega;W^{1,2\sigma+2}(0,T_0;L^2(\mathbb{R}^d)))$ since $H^1(\mathbb{R}^d)\subset L^{2(2\sigma+1)}(\mathbb{R}^d)$.

Now, for the last term in the right hand side of (3.15), we write

$$
\begin{aligned}
&\tfrac{1}{\sqrt{\delta t}}\sum_{\ell=1}^{n-1}\int_0^t \theta^{\ell-1}\theta^{\ell}\chi_{\ell-1}u_{\ell-1/2}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&=\tfrac{1}{\sqrt{\delta t}}\sum_{\ell=1}^{n-1}\int_0^t \theta^{\ell-1}\rho_k(|S_{\delta t}u_{\ell-1}|^q_{L^q_x})\chi_{\ell-1}u_{\ell-1}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&\quad+\tfrac{1}{2\sqrt{\delta t}}\sum_{\ell=1}^{n-1}\int_0^t \theta^{\ell-1}\theta^{\ell}\chi_{\ell-1}(u_\ell-u_{\ell-1})\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&\quad+\tfrac{1}{\sqrt{\delta t}}\sum_{\ell=1}^{n-1}\int_0^t \theta^{\ell-1}\left[\rho_k(|u_\ell|^q_{L^q_x})-\rho_k(|S_{\delta t}u_{\ell-1}|^q_{L^q_x})\right]\\
&\quad\times\chi_{\ell-1}u_{\ell-1}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&=I+II+III.
\end{aligned}
$$

We first have, with $u(t)=u^{n,k}(t)$,

$$
I=\int_0^{t-\delta t}\rho_k(|u(s)|^q_{L^q_x})\rho_k(|S_{\delta t}u(s)|^q_{L^q_x})u(s)dW(s)
$$

and thanks to Lemma 2.1 in [17], for any α with $0\le\alpha<1/2$,

$$
\begin{aligned}
&\mathbb{E}\left(\left|\int_0^{t-\delta t}\rho_k(|u(s)|^q_{L^q_x})\rho_k(|S_{\delta t}u(s)|^q_{L^q_x})u(s)dW(s)\right|^{2\sigma+2}_{W^{\alpha,2\sigma+2}(0,T_0;L^2_x)}\right)\\
&\le C\mathbb{E}\left(\int_0^{T_0}\|\rho_k(|u(s)|^q_{L^q_x})\rho_k(|S_{\delta t}u(s)|^q_{L^q_x})u(s)\Phi\|^{2\sigma+2}_{L_2^{0,0}}ds\right)\\
&\le CT_0\mathbb{E}\left(|u_0|^{2\sigma+2}_{L^2_x}\right)\|\Phi\|^{2\sigma+2}_{L_2^{0,s}}.
\end{aligned}
$$

Hence I is bounded in $L^{2\sigma+2}(\Omega;W^{\alpha,2\sigma+2}(0,T_0;L^2(\mathbb{R}^d)))$ for any α with $0\le\alpha<1/2$.

In order to estimate II, we replace $u_\ell-u_{\ell-1}$ by its expression using equation (3.4); we then obtain

$$
\begin{aligned}
II=-\frac{i}{2}\sum_{\ell=1}^{n-1}\int_0^t&\theta^{\ell-1}\theta^{\ell}\chi_{\ell-1}\\
&\times\left[\sqrt{\delta t}\Delta u_{\ell-1/2}+\sqrt{\delta t}f_{\ell-1}-\theta^{\ell-1}\theta^{\ell}\chi_{\ell-1}u_{\ell-1/2}\right]\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds.
\end{aligned}
$$

The first term is bounded in $L^1(\Omega;W^{1,2\sigma+2}(0,T_0;H^{-1}(\mathbb{R}^d)))$ as follows (assuming $\delta t\le1$)

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{\ell=1}^{n-1}\sqrt{\delta t}\int_0^t \theta^{\ell-1}\theta^\ell\chi_{\ell-1}\Delta u_{\ell-1/2}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\right|_{W^{1,2\sigma+2}(0,T_0;H_x^{-1})}\right)\\
&\le C(T_0)\mathbb{E}\left(\left(\sum_{\ell=1}^{n-1}\delta t|\chi_{\ell-1}\Delta u_{\ell-1/2}|_{H_x^{-1}}^{2\sigma+2}\right)^{1/2\sigma+2}\right)\\
&\le C(T_0)\left(\mathbb{E}\sum_{\ell=1}^{n-1}\delta t|\chi_{\ell-1}|_{H_x^s}^{2(2\sigma+2)}\right)^{1/2(2\sigma+2)}\left(\mathbb{E}\sup_{t\in[0,T_0]}|u(t)|_{H_x^1}^2\right)^{1/2}\\
&\le C(T_0,\|\Phi\|_{L_2^{0,s}},k,u_0)
\end{aligned}$$

by Lemma 3.3 and the Gaussianity of $(\chi_\ell)_{0\le\ell\le n-1}$. In the same way, we have

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{\ell=1}^{n-1}\sqrt{\delta t}\int_0^t \theta^{\ell-1}\theta^\ell\chi_{\ell-1}f_{\ell-1}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\right|^2_{W^{1,2\sigma+2}(0,T_0;L_x^2)}\right)\\
&\le C(T_0)\mathbb{E}\left(\left(\sum_{\ell=1}^{n-1}\delta t\theta^\ell\theta^{\ell-1}|\chi_{\ell-1}f_{\ell-1}|_{L_x^2}^{2\sigma+2}\right)^{1/2\sigma+2}\right)\\
&\le C(T_0)\mathbb{E}\left(\left(\sum_{\ell=1}^{n-1}\delta t|\chi_{\ell-1}|_{H_x^s}^{2\sigma+2}\right)^{1/2\sigma+2}\sup_{\ell\le n-1}\theta^\ell\theta^{\ell-1}\left(|u_\ell|_{L_x^q}^q+|u_{\ell-1}|_{L_x^q}^q\right)\right)\\
&\le C(T_0,\|\Phi\|_{L_2^{0,s}},k).
\end{aligned}$$

The last term in II is bounded similarly in $L^1(\Omega;\, W^{1,2\sigma+2}(0,T_0;\, L^2(\mathbb{R}^d)))$. Hence, II is bounded in $L^1(\Omega;\, W^{1,2\sigma+2}(0,T_0;\, H^{-1}(\mathbb{R}^d)))$. We prove in the same way that III is bounded in $L^1(\Omega;\, W^{1,2\sigma+2}(0,T_0;\, H^{-1}(\mathbb{R}^d)))$, by using (3.35) and the ingredients of the proof of Lemma 3.3 to estimate $\rho_k(|u_\ell|_{L_x^q}^q)-\rho_k(|S_{\delta t}u_{\ell-1}|_{L_x^q}^q)$. Then, (3.12) follows with $0\le\alpha<1/2$. Estimate (3.13) with $0\le\beta<\sigma/2(\sigma+1)$ follows from (3.12) and the embedding $W^{\alpha,2\sigma+2}(0,T_0)\subset C^\beta([0,T_0])$ if $\beta<\alpha-\frac{1}{2\sigma+2}$. It remains to prove (3.14). Note that for $t,s\in[0,T_0]$,

$$\left||v(t)|_{L_x^q}^q-|v(s)|_{L_x^q}^q\right|\le C\sup_{r\in[0,T_0]}|v(r)|_{L_x^q}^{q-1}|v(t)-v(s)|_{L_x^q}.$$

Since by interpolation we have

$$|v(t)-v(s)|_{L_x^q}\le C|v(t)-v(s)|_{H_x^{-1}}^\varepsilon\sup_{t\in[0,T_0]}|v(r)|_{H_x^1}^{1-\varepsilon}$$

with $\frac{1}{q} = \frac{2\varepsilon}{d} + \frac{d-2}{2d}$, it follows that if $\gamma = \varepsilon\beta$,

$$\Big| |v(\cdot)|^q_{L^q_x} \Big|_{C^\gamma([0,T_0])} \le C \sup_{r\in[0,T_0]} |v(r)|^{q-\varepsilon}_{H^1_x} |v|^\varepsilon_{C^\beta(0,T_0;H^{-1}_x)}.$$

Choosing β such that (3.13) holds and $\delta > 0$ such that $(q-\varepsilon)\delta \le 1$ and $\delta\varepsilon \le 2/(2\sigma+1)$, (3.14) is then implied by (3.13) and Lemma 3.3. The proof of Lemma 3.4 is complete. □

The last thing to prove is Lemma 3.6, that is the passage to the limit in equation (3.19).

Proof of Lemma 3.6 In what follows, since j and k are fixed, we omit them in the notations, that is we set $\tilde v^{n,k}_j = \tilde v^n$, $\tilde v^k_j = \tilde v$, $\tilde\gamma^{n,k}_j = \tilde\gamma^n$, $\tilde\gamma^k_j = \tilde\gamma$; moreover, a lower index ℓ means that we take the value of the corresponding quantity at time $\ell\delta t$. We recall that $\tilde v^n \to \tilde v$ as n goes to infinity, almost surely in $L^{2\sigma+2}(0,T_0;H^r_{\rm loc}(\mathbb{R}^d)) \cap C([0,T_0];H^{-2}_{\rm loc}(\mathbb{R}^d))$; hence by the bounds given in Lemma 3.3, $\tilde v^n \to \tilde v$ in $L^p(\Omega;L^{2\sigma+2}(0,T_0;H^r_{\rm loc}(\mathbb{R}^d)))$ for any p with $1 \le p < 2$. We also have $\tilde\gamma^n \to \tilde\gamma$ almost surely in $C([0,T_0])$, hence also in $L^p(\Omega;C([0,T_0]))$ for any $p \ge 1$ since $\tilde\gamma^n \le 1$ almost surely. Finally, $\tilde W^n \to \tilde W$ almost surely in $C([0,T_0];H^{s'}_{\rm loc}(\mathbb{R}^d))$. It follows that

$$\sum_{\ell=1}^{n-1}\int_0^t \Delta\tilde v^n_{\ell+1/2}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds \text{ tends to } \int_0^t \Delta\tilde v(s)ds \text{ almost surely}$$

in $L^\infty(0,T_0;H^{r-2}_{\rm loc}(\mathbb{R}^d))$.

Also, for the third term in the right hand side of (3.19), it easily follows from the embedding $H^r_{\rm loc}(\mathbb{R}^d) \subset L^q_{\rm loc}(\mathbb{R}^d)$, with $q = 2(2\sigma+1)$, that

$$\sum_{\ell=1}^{n-1}\int_0^t f(|\tilde v^n_\ell|,|\tilde v^n_{\ell+1}|)\tilde v^n_{\ell+1/2}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds \text{ tends to } \lambda$$

$$\int_0^t |\tilde v(s)|^{2\sigma}\tilde v(s)ds \text{ almost surely in } L^\infty(0,T_0;L^2_{\rm loc}(\mathbb{R}^d)).$$

Now, as was announced in Section 3.3, in order to find the limit of the last term in the right hand side of (3.19), we have to separate the adapted part, in the time integral, from the remaining part. We thus write, using the equivalent of (3.4) for $\tilde v^n$ which easily follows from (3.18), (3.19):

$$\begin{aligned}
&\sum_{\ell=1}^{n-1}\int_0^t \rho_k(\tilde\gamma^n_\ell)\rho_k(\tilde\gamma^n_{\ell+1})\tilde v^n_{\ell+1/2}\frac{(\tilde W^n_\ell - \tilde W^n_{\ell-1})}{\delta t}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&\quad= \sum_{\ell=1}^{n-1}\int_0^t \rho_k(\tilde\gamma^n_\ell)\rho_k(\tilde\gamma^n_{\ell+1})\tilde v^n_\ell\frac{(\tilde W^n_\ell - \tilde W^n_{\ell-1})}{\delta t}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds
\end{aligned}$$

$$
\begin{aligned}
&-\frac{i}{2}\sum_{\ell=1}^{n-1}\int_0^t \rho_k^2(\tilde\gamma_\ell^n)\rho_k^2(\tilde\gamma_{\ell+1}^n)\tilde v_{\ell+1/2}^n\frac{(\tilde W_\ell^n-\tilde W_{\ell-1}^n)^2}{\delta t}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&+\frac{i}{2}\sum_{\ell=1}^{n-1}\int_0^t \rho_k(\tilde\gamma_\ell^n)\rho_k(\tilde\gamma_{\ell+1}^n)(\tilde W_\ell^n-\tilde W_{\ell-1}^n)\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)\\
&\times\left[\Delta\tilde v_{\ell+1/2}^n+f(|\tilde v_\ell^n|,|\tilde v_{\ell+1}^n|)\tilde v_{\ell+1/2}^n\right]ds\\
&= I+II+III.
\end{aligned}
\tag{3.40}
$$

We first treat the term I, which is actually the most tricky: we write

$$
\begin{aligned}
I&=\sum_{\ell=1}^{n-1}\int_0^t \theta_k(\tilde v_\ell^n)\theta_k(S_{\delta t}\tilde v_\ell^n)\tilde v_\ell^n\frac{(\tilde W_\ell^n-\tilde W_{\ell-1}^n)}{\delta t}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&\quad+\sum_{\ell=1}^{n-1}\int_0^t \theta_k(\tilde v_\ell^n)\left(\theta_k(\tilde v_{\ell+1}^n)-\theta_k(S_{\delta t}\tilde v_\ell^n)\right)\\
&\quad\times\tilde v_\ell^n\frac{(\tilde W_\ell^n-\tilde W_{\ell-1}^n)}{\delta t}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&=I_1+I_2.
\end{aligned}
$$

Concerning I_1, we obtain

$$
\begin{aligned}
I_1&=\sum_{\ell=1}^{n-1}\int_0^t \theta_k^2(\tilde v_\ell^n)\tilde v_\ell^n\frac{(\tilde W_\ell^n-\tilde W_{\ell-1}^n)}{\delta t}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&\quad+\sum_{\ell=1}^{n-1}\int_0^t \theta_k(\tilde v_\ell^n)\left(\theta_k(S_{\delta t}\tilde v_\ell^n)-\theta_k(\tilde v_\ell^n)\right)\frac{(\tilde W_\ell^n-\tilde W_{\ell-1}^n)}{\delta t}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&=I_{1,1}+I_{1,2}.
\end{aligned}
$$

The same regularization procedure as in [3] and the adaptivity of $\theta_k^2(\tilde v_\ell^n)\tilde v_\ell^n$ easily shows that

$$I_{1,1}=\sum_{\ell=1}^{n-1}\int_0^t \theta_k^2(\tilde v_\ell^n)\tilde v_\ell^n\frac{(\tilde W_\ell^n-\tilde W_{\ell-1}^n)}{\delta t}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds \text{ converges}$$

to $\int_0^t \rho_k^2(\tilde\gamma(s))\tilde v(s)dW(s)$ as n goes to infinity, weakly in $L^2(\Omega;\, L^2_{\mathrm{loc}}(\mathbb{R}^d))$ for any $t\in[0,T_0]$.

To treat the term $I_{1,2}$, we write

$$|\theta_k(S_{\delta t}\tilde v_\ell^n)-\theta_k(\tilde v_\ell^n)|=\left(\Lambda_\ell^n,(I-S_{\delta t})\tilde v_\ell^n\right)$$

where $(.,.)$ is the inner product in $L^2(\mathbb{R}^d)$,

$$\Lambda_\ell^n=\int_0^1 \theta_k'\left((1-\lambda)\tilde v_\ell^n+\lambda S_{\delta t}\tilde v_\ell^n\right)d\lambda$$

and

$$\theta_k'(v) = q|v|^{q-2} v \rho_k'(|v|^q_{L^q_x}).$$

Note that Λ^n_ℓ is bounded in $L^{q'}(\mathbb{R}^d)$, with $\frac{1}{q'} + \frac{1}{q} = 1$, by a constant $C(k)$, and that it is an adapted process. Hence, we have for any $t \in [0, T_0]$,

$$\begin{aligned}\mathbb{E}\left(\left|I_{1,2}(t)\right|^2_{L^2_x}\right) &\leq \sum_{\ell=1}^{n-1} \mathbb{E}\left(\theta_k^2(\tilde{v}^n_\ell)\left(\Lambda^n_\ell, (I - S_{\delta t})\tilde{v}^n_\ell\right)^2 |\tilde{v}^n_\ell \Phi|^2_{L_2^{0,0}}\right)\delta t \\ &\leq C(k, \|\Phi\|^2_{L_2^{0,s}}) \sum_{\ell=1}^{n-1} \delta t \mathbb{E}\left(|(I - S_{\delta t})\tilde{v}^n_\ell|^2_{L^q_x} |\tilde{v}^n_\ell|^2_{L^2_x}\right).\end{aligned}$$

Now, $I - S_{\delta t} = \delta t(i + \frac{\delta t}{2}\Delta)^{-1}\Delta$ and

$$|(I - S_{\delta t})\tilde{v}^n_\ell|^2_{L^q_x} \leq C_\varepsilon |(I - S_{\delta t})\tilde{v}^n_\ell|^{2\varepsilon}_{L^2_x} |(I - S_{\delta t})\tilde{v}^n_\ell|^{2(1-\varepsilon)}_{H^1_x}$$

with $\frac{1}{q} = \frac{\varepsilon}{2} + \frac{(1-\varepsilon)(d-2)}{2d}$. Since it is clear that $\sqrt{\delta t}\Delta(i + \frac{\delta t}{2}\Delta)^{-1}$ is bounded from $H^1(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$ independently of $\delta t \leq 1$, we obtain

$$|(I - S_{\delta t})\tilde{v}^n_\ell|^{2\varepsilon}_{L^2(\mathbb{R}^d)} \leq (\delta t)^{\varepsilon/2} C_\varepsilon |\tilde{v}^n_\ell|^\varepsilon_{H^1(\mathbb{R}^d)} |\tilde{v}^n_\ell|^\varepsilon_{L^2(\mathbb{R}^d)};$$

hence

$$\begin{aligned}\mathbb{E}\left(\left|I_{1,2}\right|^2_{L^2(\mathbb{R}^d)}\right) &\leq C(k, \|\Phi\|^2_{L_2^{0,s}}) \sum_{\ell=1}^{n-1} (\delta t)^{1+\varepsilon/2} \mathbb{E}\left(|\tilde{v}_0|^{\varepsilon+2}_{L^2(\mathbb{R}^d)} |\tilde{v}^n_\ell|^{2-\varepsilon}_{H^1(\mathbb{R}^d)}\right) \\ &\leq C(k, \|\Phi\|^2_{L_2^{0,s}}, \tilde{v}_0, T_0)(\delta t)^{\varepsilon/2}\end{aligned}$$

and $\mathbb{E}\left(\left|I_{1,2}(t)\right|^2_{L^2(\mathbb{R}^d)}\right)$ goes to zero as n goes to infinity for any $t \in [0, T_0]$.

We now consider the term I_2: we have

$$\theta_k(\tilde{v}^n_{\ell+1}) - \theta_k(S_{\delta t}\tilde{v}^n_\ell) = \left(\tilde{\Lambda}^n_\ell, \tilde{v}^n_{\ell+1} - S_{\delta t}\tilde{v}^n_\ell\right)$$

with

$$\tilde{\Lambda}^n_\ell = \int_0^1 \theta_k'\left((1-\lambda)S_{\delta t}\tilde{v}^n_\ell + \lambda \tilde{v}^n_{\ell+1}\right) d\lambda.$$

Note that $\tilde{\Lambda}^n_\ell$ is no more adapted, but we still have $\theta_k(\tilde{v}^n_\ell)|\tilde{\Lambda}^n_\ell|_{L^{q'}(\mathbb{R}^d)} \leq C(k)$ for $n \geq n_0(k)$ (see the proof of Lemma 3.3). Also, we make use of the "integral equation" for $\tilde{v}^n$, that is

$$\begin{aligned}\tilde{v}^n_{\ell+1} - S_{\delta t}\tilde{v}^n_\ell = &-\delta t(i + \frac{\delta t}{2}\Delta)^{-1} f_\ell \\ &+\sqrt{\delta t}\, \theta_k(\tilde{v}^n_\ell)\theta_k(\tilde{v}^n_{\ell+1})(i + \frac{\delta t}{2}\Delta)^{-1}(\tilde{\chi}^n_\ell \tilde{v}^n_{\ell+1/2}),\end{aligned}$$

where we have set

$$f_\ell = f(|\tilde{v}_\ell^n|, |\tilde{v}_{\ell+1}^n|)\tilde{v}_{\ell+1/2}^n \quad \text{and} \quad \tilde{\chi}_\ell^n = \frac{\tilde{W}^n(\ell\delta t) - \tilde{W}^n((\ell-1)\delta t)}{\sqrt{\delta t}}.$$

We then write $I_2 = I_{2,1} + I_{2,2}$ with

$$I_{2,1}(t) = -\sum_{\ell=1}^{n-1} \int_0^t \delta t\, \theta_k(\tilde{v}_\ell^n) \left(\tilde{\Lambda}_\ell^n, (i + \frac{\delta t}{2}\Delta)^{-1} f_\ell\right) \\ \times \tilde{v}_\ell^n \frac{(\tilde{W}_\ell^n - \tilde{W}_{\ell-1}^n)}{\delta t} \mathbf{1}_{[\ell\delta t, (\ell+1)\delta t)}(s)ds$$

and

$$I_{2,2}(t) = \sum_{\ell=1}^{n-1} \int_0^t \sqrt{\delta t}\, \theta_k^2(\tilde{v}_\ell^n)\theta_k(\tilde{v}_{\ell+1}^n) \left(\tilde{\Lambda}_\ell^n, (i + \frac{\delta t}{2}\Delta)^{-1}(\tilde{\chi}_\ell^n \tilde{v}_{\ell+1/2}^n)\right) \\ \times \tilde{v}_\ell^n \frac{(\tilde{W}_\ell^n - \tilde{W}_{\ell-1}^n)}{\delta t} \mathbf{1}_{[\ell\delta t, (\ell+1)\delta t)}(s)ds.$$

The term $I_{2,1}$ is treated as follows: we have

$$\begin{aligned} &\sqrt{\delta t} \left|\left(\tilde{\Lambda}_\ell^n, (i + \tfrac{\delta t}{2}\Delta)^{-1} f_\ell\right)\right| \\ &\le \sqrt{\delta t} |\tilde{\Lambda}_\ell^n|_{L_x^{q'}} \left|(i + \tfrac{\delta t}{2}\Delta)^{-1} f_\ell\right|_{L_x^q} \\ &\le \sqrt{\delta t} C_\varepsilon |\tilde{\Lambda}_\ell^n|_{L_x^{q'}} \left|(i + \tfrac{\delta t}{2}\Delta)^{-1} f_\ell\right|_{L_x^2}^\varepsilon \left|(i + \tfrac{\delta t}{2}\Delta)^{-1} f_\ell\right|_{H_x^1}^{1-\varepsilon} \end{aligned}$$

and since $\sqrt{\delta t}(i + \frac{\delta t}{2}\Delta)^{-1}$ is bounded from $L^2(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$ independently of δt, and $(i + \frac{\delta t}{2}\Delta)^{-1}$ is bounded in $L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} \sqrt{\delta t} \left|\left(\tilde{\Lambda}_\ell^n, (i + \tfrac{\delta t}{2}\Delta)^{-1} f_\ell\right)\right| &\le C_\varepsilon (\delta t)^{\varepsilon/2} |\tilde{\Lambda}_\ell^n|_{L_x^{q'}} |f_\ell|_{L_x^2} \\ &\le C_\varepsilon (\delta t)^{\varepsilon/2} |\tilde{\Lambda}_\ell^n|_{L_x^{q'}} \left(|\tilde{v}_\ell^n|_{L_x^q}^{2\sigma+1} + |\tilde{v}_{\ell+1}^n|_{L_x^q}^{2\sigma+1}\right). \end{aligned}$$

Once more, we have

$$\theta_k(\tilde{v}_\ell^n) \left|\tilde{\Lambda}_\ell^n\right|_{L_x^{q'}} \left(|\tilde{v}_\ell^n|_{L_x^q}^{2\sigma+1} + |\tilde{v}_{\ell+1}^n|_{L_x^q}^{2\sigma+1}\right) \le C(k)$$

for $n \ge n_0(k)$ (see the proof of Lemma 3.3), hence for such integers n,

$$\begin{aligned} \mathbb{E}\left(\left|I_{2,1}(t)\right|_{L_x^2}\right) &\le C(k, \varepsilon) \sum_{\ell=1}^{n-1} (\delta t)^{1+\varepsilon/2} \mathbb{E}\left(|\tilde{\chi}_\ell^n|_{H_x^s} |\tilde{v}_\ell^n|_{L_x^2}\right) \\ &\le C(k, \varepsilon, \|\Phi\|_{L_2^{0,s}}^2, \mathbb{E}(|v_0|_{L_x^2}^2), T_0)(\delta t)^{\varepsilon/2} \end{aligned} \tag{3.41}$$

for any $t \in [0, T_0]$ and $I_{2,1}(t)$ tends to zero in $L^1(\Omega; L^2(\mathbb{R}^d))$ for any $t \in [0, T_0]$.

Let us now consider the term $I_{2,2}$. Since

$$(i + \frac{\delta t}{2}\Delta)^{-1} + i = i\frac{\delta t}{2}\Delta(i + \frac{\delta t}{2}\Delta)^{-1},$$

we derive

$$\begin{aligned}&\left(\tilde{\Lambda}_\ell^n, (i + \frac{\delta t}{2}\Delta)^{-1}(\tilde{\chi}_\ell^n \tilde{v}_{\ell+1/2}^n)\right)\\ &\quad = -\left(\tilde{\Lambda}_\ell^n, i\tilde{\chi}_\ell^n \tilde{v}_{\ell+1/2}^n\right) + \frac{\delta t}{2}\left(\tilde{\Lambda}_\ell^n, i\Delta(i + \frac{\delta t}{2}\Delta)^{-1}(\tilde{\chi}_\ell^n \tilde{v}_{\ell+1/2}^n)\right).\end{aligned}$$

In the same way as before, we obtain

$$\begin{aligned}&\left| i\delta t\Delta(i + \tfrac{\delta t}{2}\Delta)^{-1}\tilde{\chi}_\ell^n \tilde{v}_{\ell+1/2}^n \right|_{L_x^q}\\ &\quad \le (\delta t)^{\varepsilon/2} C_\varepsilon \left|\sqrt{\delta t}\,\Delta(i + \tfrac{\delta t}{2}\Delta)^{-1}(\tilde{\chi}_\ell^n \tilde{v}_{\ell+1/2}^n)\right|_{L_x^2}^{\varepsilon} \left|\tilde{\chi}_\ell^n \tilde{v}_{\ell+1/2}^n\right|_{H_x^1}^{1-\varepsilon}\\ &\quad \le (\delta t)^{\varepsilon/2} C_\varepsilon \left(|\tilde{v}_\ell^n|_{H_x^1} + |\tilde{v}_{\ell+1}^n|_{H_x^1}\right)|\tilde{\chi}_\ell^n|_{H_x^s}.\end{aligned}$$

On the other hand, since $(\theta'(\tilde{v}_{\ell+1/2}^n), i\tilde{\chi}_\ell^n \tilde{v}_{\ell+1/2}^n) = 0$ – note that $\tilde{\chi}_\ell^n$ is real valued – we have

$$\left(\tilde{\Lambda}_\ell^n, i\tilde{\chi}_\ell^n \tilde{v}_{\ell+1/2}^n\right) = \left(\tilde{\Lambda}_\ell^n - \theta_k'(\tilde{v}_{\ell+1/2}^n), i\tilde{\chi}_\ell^n \tilde{v}_{\ell+1/2}^n\right)$$

and

$$\begin{aligned}&\left|\tilde{\Lambda}_\ell^n - \theta_k'(\tilde{v}_{\ell+1/2}^n)\right|_{L_x^{q'}}\\ &\quad \le \int_0^1 \left|\theta_k'\left((1-\lambda)S_{\delta t}\tilde{v}_\ell^n + \lambda\tilde{v}_{\ell+1}^n\right) - \theta_k'(\tilde{v}_{\ell+1/2}^n)\right|_{L_x^{q'}} d\lambda\\ &\quad \le C|\rho_k'|_{L^\infty}\left(|\tilde{v}_\ell^n|_{L_x^q}^{q-1} + |\tilde{v}_{\ell+1}^n|_{L_x^q}^{q-1}\right)\left(|\tilde{v}_{\ell+1}^n - S_{\delta t}\tilde{v}_\ell^n|_{L_x^q} + |S_{\delta t}\tilde{v}_\ell^n - \tilde{v}_\ell^n|_{L_x^q}\right).\end{aligned}$$

Using the same kind of estimates as above, it is not difficult to see that

$$\left|\tilde{\Lambda}_\ell^n - \theta_k'(\tilde{v}_{\ell+1/2}^n)\right|_{L_x^{q'}} \le (\delta t)^{\varepsilon/2} C(k, |\tilde{v}_\ell^n|_{L_x^q}, |\tilde{v}_{\ell+1}^n|_{L_x^q})(1 + |\tilde{\chi}_\ell^n|_{H_x^s})|\tilde{v}_\ell^n|_{H_x^1}.$$

Hence, for any $t \in [0, T_0]$,

$$\begin{aligned}&\mathbb{E}\left(\left|I_{2,2}(t)\right|_{L_x^2}\right)\\ &\quad \le \sum_{\ell=1}^{n-1}(\delta t)^{1+\varepsilon/2}\mathbb{E}\Big(C(k, |\tilde{v}_\ell^n|_{L_x^q}, |\tilde{v}_{\ell+1}^n|_{L_x^q})\theta_k^2(\tilde{v}_\ell^n)\theta_k(\tilde{v}_{\ell+1}^n)\\ &\quad\qquad \times (1 + |\tilde{\chi}_\ell^n|_{H_x^s})|\tilde{v}_\ell^n|_{H_x^1}|\tilde{v}_\ell^n|_{L_x^2}\Big)\\ &\quad \le C(k, T_0, v_0, \|\Phi\|_{L_2^{0,s}})\mathbb{E}\left(\int_0^{T_0} |\tilde{v}^n(s)|_{H_x^1}^2 ds\right)(\delta t)^{\varepsilon/2}\end{aligned}$$

and $\lim_{n\to+\infty}\mathbb{E}\left(\left|I_{2,2}(t)\right|_{L^2(\mathbb{R}^d)}\right) = 0$ for any $t \in [0, T_0]$.

This achieves the treatment of I in (3.40). We will not treat the term III, which is easily shown to go to zero in $L^2(\Omega; H^{-1}(\mathbb{R}^d))$ for any $t \in [0, T_0]$. We thus consider II, which will give rise to the Ito correction when passing to the limit. We write:

$$
\begin{aligned}
II = &-\tfrac{i}{2}\sum_{\ell=1}^{n-1}\int_0^t \theta_k^4(\tilde{v}_\ell^n)\tilde{v}_\ell^n \frac{(\tilde{W}_\ell^n - \tilde{W}_{\ell-1}^n)^2}{\delta t}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&-\tfrac{i}{2}\sum_{\ell=1}^{n-1}\int_0^t \theta_k^2(\tilde{v}_\ell^n)\left(\theta_k^2(\tilde{v}_{\ell+1}^n) - \theta_k^2(\tilde{v}_\ell^n)\right)\\
&\times \tilde{v}_\ell^n \frac{(\tilde{W}_\ell^n - \tilde{W}_{\ell-1}^n)^2}{\delta t}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
&-\tfrac{i}{4}\sum_{\ell=1}^{n-1}\int_0^t \theta_k^2(\tilde{v}_\ell^n)\theta_k^2(\tilde{v}_{\ell+1}^n)(\tilde{v}_{\ell+1}^n - \tilde{v}_\ell^n)\\
&\times \frac{(\tilde{W}_\ell^n - \tilde{W}_{\ell-1}^n)^2}{\delta t}\mathbf{1}_{[\ell\delta t,(\ell+1)\delta t)}(s)ds\\
= &II_1 + II_2 + II_3.
\end{aligned}
$$

By the same estimates as above, it can easily be proved that for any $t \in [0, T_0]$, $II_1(t)$ tends to zero in $L^1(\Omega; L^2(\mathbb{R}^d))$ as $n \to +\infty$, and that $II_3(t)$ tends to zero in $L^1(\Omega; H^{-1}(\mathbb{R}^d))$ as $n \to +\infty$. We thus show that $II_1(t)$ converges to

$$-\frac{i}{2}\int_0^t \tilde{\gamma}^4(s)\tilde{v}(s)F_\Phi ds$$

in $L^1(\Omega; L^2_{\mathrm{loc}}(\mathbb{R}^d))$ as $n \to +\infty$, where $F_\Phi(x) = \sum_{k=0}^\infty (\Phi e_k)^2(x)$, $(e_k)_{k\in\mathbb{N}}$ being any complete orthonormal system in $L^2(\mathbb{R}^d, \mathbb{R})$. Hence, we have to estimate

$$
\begin{aligned}
&\mathbb{E}\left|\sum_{\ell=1}^{n-1}\left\{\theta_k^4(\tilde{v}_\ell^n)\tilde{v}_\ell^n(\tilde{W}_\ell^n - \tilde{W}_{\ell-1}^n)^2 - \int_{(\ell-1)\delta t}^{\ell\delta t}\tilde{\gamma}^4(s)\tilde{v}(s)F_\Phi ds\right\}\right|_{L^2(B_R)}\\
&\le \mathbb{E}\left|\sum_{\ell=1}^{n-1}\left\{\theta_k^4(\tilde{v}_\ell^n)\tilde{v}_\ell^n\left[(\tilde{W}_\ell^n - \tilde{W}_{\ell-1}^n)^2 - F_\Phi\delta t\right]\right\}\right|_{L^2(B_R)}\\
&\quad + \mathbb{E}\left|\sum_{\ell=1}^{n-1}\int_{(\ell-1)\delta t}^{\ell\delta t}\left(\theta_k^4(\tilde{v}_\ell^n)\tilde{v}_\ell^n - \tilde{\gamma}^4(s)\tilde{v}(s)\right)F_\Phi ds\right|_{L^2(B_R)}.
\end{aligned}
$$

The second term in the right hand side above converges to zero, since $|F_\Phi|_{H^s(\mathbb{R}^d)} \le \|\Phi\|^2_{L_2^{0,s}}$ and $\tilde{v}^n \to \tilde{v}$ as $n \to +\infty$ in $L^{3/2}(\Omega; L^2(0, T; L^2(B_R)))$ while $\theta_k(\tilde{v}^n) \to \tilde{\gamma}$ as $n \to +\infty$ in $L^p(\Omega; C([0, T_0]))$ for any $p \ge 1$. On the other hand, thanks to the Cauchy-Schwarz inequality in the expectation, the independence property of the family $(\tilde{W}^n_\ell - \tilde{W}^n_{\ell-1})_\ell$ and the fact that $\tilde{v}^n_\ell$ is $\mathcal{F}_{(\ell-1)\delta t}$ measurable, the square of the first term is bounded above by

$$\sum_{\ell=1}^{n-1} \mathbb{E}\left(\left|\theta_k^4(\tilde{v}^n_\ell)\tilde{v}^n_\ell\left[(\tilde{W}^n_\ell - \tilde{W}^n_{\ell-1})^2 - F_\Phi \delta t\right]\right|^2_{L^2(B_R)}\right)$$
$$\leq C\mathbb{E}(|v_0|^2_{L^2}) \sum_{\ell=1}^{n-1} \mathbb{E}\left(\left|(\tilde{W}^n_\ell - \tilde{W}^n_{\ell-1})^2 - F_\Phi \delta t\right|^2_{H^s}\right)$$

and this last term tends to zero as n goes to infinity.

Collecting all the terms leads to the fact that $\tilde{v}$ satisfies equation (3.20) and ends the proof of Lemma 3.6. □

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