Mathematics

A Qualitative Approach to Inverse Scattering for Anisotropic Media

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Research supported by grants from AFOSR and NSF

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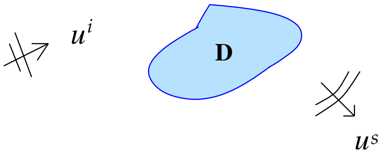
Scattering by an Inhomogeneous Media



$$
\begin{aligned}
\Delta u^s + k^2 u^s &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D} \\
\nabla \cdot A \nabla u + k^2 n u &= 0 && \text{in } D \\
u &= u^s + u^i && \text{in } \partial D \\
\nu \cdot A \nabla u &= \nu \cdot \nabla (u^s + u^i) && \text{in } \partial D \\
\lim_{r \to \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) &= 0
\end{aligned}
$$

The matrix valued function A with $C^1(D)$ entries and $n \in L^\infty(D)$ are such that $\Re(A) \geq \alpha > 0$, $\Im(A) \leq 0$, $\Re(n) > 0$ and $\Im(n) \geq 0$. Here k is the wave number and is proportional to the frequency ω, u^i is the incident wave and S is the unit sphere.

$$\nabla \cdot A \nabla u^s + k^2 n u^s = \nabla \cdot (I - A) \nabla u^i + k^2 (1 - n) u^i \qquad \text{in } \mathbb{R}^3.$$

A, n are extended by I, 1 respectively and $u^s := u - u^i$ in D.

Far Field Operator

Scattering Data

$u_\infty(\hat{x}, d, k)$, for $d \in S_i \subset S$, $\hat{x} \in S_m \subset S$ and (possibly) $k \in [k_1, \ k_2]$.

The far field operator $F : L^2(S) \to L^2(S)$ is defined by

$$(Fg)(\hat{x}) := \int\limits_S u_\infty(\hat{x}, d, k) g(d) ds_d.$$

- Fg is the far field pattern of the scattered field corresponding to the incident field

$$v_g(x) := \int_S e^{ikx \cdot d} g(d) ds_d$$

  (known as a Herglotz wave function).

- F is related to the scattering operator $\mathcal{S}$ by

$$\mathcal{S} = I + \frac{ik}{2\pi} F$$

Far Field Operator

Theorem

$F : L^2(S) \to L^2(S)$ is injective and has dense range if and only if there does not exist a nontrivial solution to the transmission eigenvalue problem

$$
\begin{aligned}
\Delta v + k^2 v &= 0 && \text{in} \quad D\\
\nabla \cdot A \nabla w + k^2 n w &= 0 && \text{in} \quad D\\
w &= v && \text{on} \quad \partial D\\
\nu \cdot A \nabla w &= \nu \cdot \nabla v && \text{on} \quad \partial D
\end{aligned}
$$

such that $v := v_g$ is a Herglotz wave function.

Values of $k \in \mathbb{C}$ for which the transmission eigenvalue problem has non trivial solution are called transmission eigenvalues.

Transmission eigenvalues relates to non-scattering frequencies.

Qualitative Methods for the Support

- The linear sampling method has been widely used for various inverse scattering problems, limited aperture data etc.

  CAKONI-COLTON (2014), *A Qualitative Approach to Inverse Scattering Theory, Springer.*

- Factorization methods is mathematically rigorous for exact data and justified for noisy data.

  A. KIRSCH AND N. GRINBERG (2008), *The Factorization Method for Inverse Problems, Oxford University Press*.

- The generalized linear sampling method.

  AUDIBERT - HADDAR (2014) - *Inverse Problems.*

All these method explore the (linear) far field operator to construct an indicator function for the support D of the inhomogeneity

CAKONI - HADDAR (2012) - Transmission Eigenvalues, *Inside Out, MSRI*.

Transmission Eigenvalue Problem

Having determined the support D without knowing anything about the material properties we would like to get some information about the constitutive parameters A and n.

For this we appeal to the transmission eigenvalue problem for $v \in H^1(D)$ and $w \in H^1(D)$ such that

$$
\begin{aligned}
\Delta v + k^2 v &= 0 && \text{in} \quad D \\
\nabla \cdot A \nabla w + k^2 n w &= 0 && \text{in} \quad D \\
w &= v && \text{on} \quad \partial D \\
\nu \cdot A \nabla w &= \nu \cdot \nabla v && \text{on} \quad \partial D
\end{aligned}
$$

Related Questions

- Connect transmission eigenvalues to A and n.
- Determine transmission eigenvalues from scattering data?

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Analysis of the Interior Transmission Problem and the Transmission Eigenvalue Problem

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Interior Transmission Problem

From now on to introduce our ideas we assume that both A and n are real valued.

The interior transmission problem reads: Find $v \in H^1(D)$ and $w \in H^1(D)$ such that

$$\begin{aligned} \Delta v + k^2 v &= \ell_1 && \text{in} \quad D \\ \nabla \cdot A \nabla w + k^2 n w &= \ell_2 && \text{in} \quad D \\ w - v &= 0 && \text{on} \quad \partial D \\ \nu \cdot A \nabla w - \nu \cdot \nabla v &= h && \text{on} \quad \partial D \end{aligned}$$

for $\ell_1 \in L^2(D)$, $\ell_2 \in L^2(D)$ and $h \in H^{-1/2}(\partial D)$.

Notations

$$a_{min} := \inf_{x\in D} \inf_{|\xi|=1} \xi \cdot A(x)\, \xi > 0, \quad \text{and} \quad a_{max} := \sup_{x\in D} \sup_{|\xi|=1} \xi \cdot A(x)\, \xi < \infty.$$

$$n_{min} := \inf_{x\in D} n(x) > 0 \quad \text{and} \quad n_{min} := \sup_{x\in D} n(x) < \infty.$$

Consider a δ-neighborhood $\mathcal{N}$ of the boundary ∂D

$$\mathcal{N} := \{x \in D : \ \text{dist}(x, \partial D) < \delta\}$$

$$a_{\star} := \inf_{x\in \mathcal{N}} \inf_{|\xi|=1} \xi \cdot A(x)\, \xi > 0 \quad \text{and} \quad a^{\star} := \sup_{x\in \mathcal{N}} \sup_{|\xi|=1} \xi \cdot A(x) \xi < \infty$$
$$n_{\star} := \inf_{x\in \mathcal{N}} n(x) > 0 \quad \text{and} \quad n^{\star} := \sup_{x\in \mathcal{N}} n(x) < \infty.$$

Modified Interior Transmission Problem

The modified transmission eigenvalue problem

$$\begin{aligned} \Delta v - \kappa^2 v &= \ell_1 && \text{in} \quad D \\ \nabla \cdot A \nabla w - \kappa^2 n_0 w &= \ell_2 && \text{in} \quad D \\ w - v &= 0 && \text{on} \quad \partial D \\ \nu \cdot A \nabla w - \nu \cdot \nabla v &= h && \text{on} \quad \partial D \end{aligned}$$

for some choice of $\kappa > 0$ and $n_0 > 0$ is a compact perturbation of the interior transmission problem in

$$\mathbf{H}(D) := \left\{ (w, v) \in H^1(D) \times H^1(D) : \; w - v \in H_0^1(D) \right\}.$$

In variational form

$$\int_D A \nabla w \cdot \nabla \overline{w}' \, dx - \int_D \nabla v \cdot \nabla \overline{v}' \, dx + \kappa^2 \int_D n_0 w \, \overline{w}' \, dx - \kappa^2 \int_D v \, \overline{v}' \, dx$$

$$= \int_{\partial D} h \overline{w'} \, ds + \int_D \ell_1 \overline{v'} \, dx - \int_D \ell_2 \overline{w'} \, dx, \quad \text{for all} \quad (w', v') \in \mathbf{H}(D).$$

Modified Interior Transmission Problem

Assume that either $a^\star < 1$ and choose $n_0 < 1$, or $a_\star > 1$ and choose $n_0 > 1$. Then for $\kappa > 0$ large enough the sesquilinear form

$$a((w,v),(w',v')) :=$$

$$\int_D A\nabla w \cdot \nabla \overline{w}'\, dx - \int_D \nabla v \cdot \nabla \overline{v}'\, dx + \kappa^2 \int_D n_0 w\, \overline{w}'\, dx - \kappa^2 \int_D v\, \overline{v}'\, dx$$

is T-coercive, i.e. $a^T((w,v),(w',v')) := a((w,v),\mathbf{T}(w',v'))$ is coercive with the isomorphism $\mathbf{T} : \mathbf{H}(D) \to \mathbf{H}(D)$ defined by

$$\mathbf{T} : (w,v) \mapsto (w - 2\chi v, -v) \quad \text{or} \quad \mathbf{T} : (w,v) \mapsto (w, -v + 2\chi w),$$

respectively, where χ is C^∞ cut off function supported in $\overline{\mathcal{N}}$.

Proof on the board

Bonnet-Ben Dhia - Chesnel, Lucas - Haddar (2011) - *C. R. Math. Acad. Sci. Paris*

Transmission Eigenvalue Problem

If either $a^\star < 1$ or $a_\star > 1$ then the interior transmission problem is well posed provided that $k \in \mathbb{C}$ is not a transmission eigenvalue.

Under the above assumptions, to show discreteness of transmission eigenvalues it suffices to find one $k \in \mathbb{C}$ that is not a transmission eigenvalue.

If either $a^\star < 1$ and $n^\star < 1$, or $a_\star > 1$ and $n_\star > 1$ then the set of transmission eigenvalues is discrete in $\mathbb{C}$ with $+\infty$ as the only possible accumulation point.

If either $a_{max} < 1$ or $a_{min} > 1$, and $\int_D (n-1)dx \neq 0$, then the set of transmission eigenvalues is discrete in $\mathbb{C}$ with $+\infty$ as the only possible accumulation point.

Transmission Eigenvalue Problem: $n \equiv 1$ case.

The transmission eigenvalue problem for $n \equiv 1$ can be written for $\mathbf{w} = A\nabla w \in L^2(D)$, $\mathbf{v} = \nabla v \in L^2(D)$ and $N := A^{-1}$ as

$$\begin{aligned}
\nabla(\nabla \cdot \mathbf{v}) + k^2\mathbf{v} = 0 \qquad & \text{in} \quad D \\
\nabla(\nabla \cdot \mathbf{w}) + k^2 N\mathbf{w} = 0 \qquad & \text{in} \quad D \\
\nu \cdot \mathbf{w} = \nu \cdot \mathbf{v} \qquad & \text{on} \quad \partial D \\
\nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{v} \qquad & \text{on} \quad \partial D
\end{aligned}$$

with $\mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$ where

$$\begin{aligned}
H_0(\mathrm{div}, D) : &= \left\{\mathbf{u} \in L^2(D)^2,\ \nabla \cdot \mathbf{u} \in L^2(D),\ \nu \cdot \mathbf{u} = 0 \text{ on } \partial D\right\} \\
\mathcal{H}_0(D) : &= \left\{\mathbf{u} \in H_0(\mathrm{div}, D) :\ \nabla \cdot \mathbf{u} \in H_0^1(D)\right\}.
\end{aligned}$$

which for $\mathbf{u} := \mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$ is equivalent to

$$\int_D (N - I)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + k^2\mathbf{u}\right)\cdot\left(\nabla\nabla \cdot \overline{\mathbf{u}'} + k^2 N\,\overline{\mathbf{u}'}\right)\, dx = 0, \quad \forall\, \mathbf{u}' \in \mathcal{H}_0(D).$$

Transmission Eigenvalue Problem

At this point we assume that either $a_{max} < 1$ or $a_{min} > 1$.
and consider only $k > 0$.

Take $a_{max} < 1$ which implies that $\xi \cdot (N-I)^{-1}\xi \geq \alpha|\xi|^2$, $\alpha = \dfrac{a_{max}}{1-a_{max}}$.

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}') := \left((N-I)^{-1}\left(\nabla\nabla\cdot\mathbf{u} + k^2\mathbf{u}\right), \left(\nabla\nabla\cdot\mathbf{u}' + k^2\mathbf{u}'\right)\right)_D + k^4\left(\mathbf{u},\mathbf{u}'\right)_D,$$

$$\mathcal{B}(\mathbf{u},\mathbf{u}') := \left(\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{u}'\right)_D.$$

Here $(\cdot,\cdot)_D$ denotes the $L^2(D)$-inner product.

The eigenvalue problem becomes

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}') - k^2\mathcal{B}(\mathbf{u},\mathbf{u}') = 0 \qquad \textit{or} \qquad \mathbb{A}_k\mathbf{u} - k^2\mathbb{B}\mathbf{u} = 0$$

$(\mathbb{A}_k\mathbf{u},\mathbf{u}')_{\mathcal{H}_0(D)} = \mathcal{A}_k(\mathbf{u},\mathbf{u}')$ and $(\mathbb{B}\mathbf{u},\mathbf{u}')_{\mathcal{H}_0(D)} = \mathcal{B}(\mathbf{u},\mathbf{u}')$.

Transmission Eigenvalue Problem

$$\begin{aligned}\mathcal{A}_k(\mathbf{u},\mathbf{u}) - k^2\mathcal{B}(\mathbf{u},\mathbf{u}) &\geq \left(\alpha - \frac{\alpha^2}{\epsilon}\right)\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)} + (1+\alpha-\epsilon)k^2\|\mathbf{u}\|^2_{L^2(D)} \\ &- k^2\frac{1}{\lambda_1(D)}\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)}\end{aligned}$$

hence from the Poincaré inequality

$$\|\nabla\cdot\mathbf{u}\|^2_{L^2(D)} \leq \frac{1}{\lambda_1(D)}\|\nabla\nabla\cdot\mathbf{u}\|^2_{L^2(D)}$$

there are no transmission eigenvalues if $k^2 < \alpha/(1+\alpha)\lambda_1(D)$ where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.

A Faber-Krahn type inequality for TE – All transmission eigenvalues satisfy

$$k^2 > \lambda_1(D)a_{max}$$

Existence of Real Transmission Eigenvalues

- The mapping $k \to \mathbb{A}_k$ is continuous from $(0, +\infty)$ to the set of self-adjoint coercive operators from $\mathcal{H}_0(D) \to \mathcal{H}_0(D)$.
- $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is self-adjoint, compact and non-negative.

There exists an increasing sequence of eigenvalues $\lambda_j(k)_{j\geq 1}$ of the generalized eigenvalue problem

$$\mathbb{A}_k u - \lambda(k)\mathbb{B}u = 0 \quad \text{in} \;\; \mathcal{H}_0(D)$$

such that

$$\lambda_j(k) = \min_{W\subset\mathcal{U}_j} \max_{u\in W\setminus\{0\}} \frac{(\mathbb{A}_k u, u)}{(\mathbb{B}u, u)}$$

where $\mathcal{U}_j$ denotes the set of all j-dimensional subspaces W of $\mathcal{H}_0(D)$, $W \cap N(\mathbb{B}) = \{0\}$

Then k is a transmission eigenvalue if and only if satisfies

$$\lambda_j(k) = k^2$$

Existence of Real Transmission Eigenvalues

Max-min principle for $\lambda_j(\tau)$ implies that if there exists $k_0 > 0$ and $k_1 > 0$ such that

- $\mathbb{A}_{k_0} - k_0^2\mathbb{B}$ is positive on $\mathcal{H}_0(D)$,
- $\mathbb{A}_{k_1} - k_1^2\mathbb{B}$ is non positive on a m dimensional subspace of $\mathcal{H}_0(D)$

then each $\lambda_j(k) = k^2$ for $j = 1, \ldots, m$, has at least one solution in $[k_0, k_1]$, i.e. there exists m transmission eigenvalues counting multiplicity within the interval $[k_0,\, k_1]$.

It is now obvious that determining such constants k_0 and k_1 provides the existence of transmission eigenvalues as well as the desired isoperimetric inequalities.

Existence of Real Transmission Eigenvalues

Theorem (Cakoni-Gintides-Haddar)

Assume that $a_{max} < 1$. Then, there exists an infinite discrete set of real transmission eigenvalues k_j accumulating at $+\infty$. Furthermore

$$k_j(a_{min}, B_1) \leq k_j(a_{min}, D) \leq k_j(A(x), D) \leq k_j(a_{max}, D) \leq k_j(a_{min}, B_2)$$

where $B_2 \subset D \subset B_1$.

If $A := aI$, $1 \neq a > 0$ is constant, the first transmission eigenvalue uniquely determines the constant index of refraction.

Similar results can be obtained for the case when $a_{min} > 1$.

Transmission Eigenvalues: $n \neq 1$ case

The analysis of the existence of real transmission eigenvalues when $n \neq 1$ is more complicated and restrictive.

CAKONI-KIRSCH (2010) - *Int. J. Comput. Sci. Math.*

- If the contrasts $A - I$ and $n - 1$ have the same fixed sign, then there exists an infinite discrete set of real transmission eigenvalues accumulating at $+\infty$.
- If the contrasts $A - I$ and $n - 1$ have the opposite fixed sign, then there exits at least one real transmission eigenvalue providing that n is small enough.

HARRIS-CAKONI-SUN (2014) - *Inverse Problems*

Assume that there is $D_0 \subset D$ (void) where $A = I$ and $n = 1$, otherwise in A and n satisfy the above assumption. Then there exists at least one real transmission eigenvalue provided that D_0 is sufficiently small and this eigenvalue is depends monotonically increasing on the void size.

Transmission Eigenvalues: $n \neq 1$ case

Set $u = w - v \in H_0^1(D)$. Find $v = v_u$ by solving a Neuman type problem: For every $\psi \in H^1(D)$

$$\int_D (A - I)\nabla v \cdot \nabla \overline{\psi} - k^2(n-1)v\overline{\psi}\, dx = \int_D A\nabla u \cdot \nabla \overline{\psi} - k^2 n u \overline{\psi}\, dx.$$

Having $u \to v_u$, we require that $v := v_u$ satisfies $\Delta v + k^2 v = 0$.

Thus we define $\mathbb{L}_k : H_0^1(D) \to H_0^1(D)$

$$(\mathbb{L}_k u, \phi)_{H_0^1(D)} = \int_D \nabla v_u \cdot \nabla \overline{\phi} - k^2 v_u \cdot \overline{\phi}\, dx, \qquad \phi \in H_0^1(D).$$

Then the transmission eigenvalue problem is equivalent to

$$\mathbb{L}_k u = 0 \qquad \text{in} \qquad H_0^1(D) \qquad \text{which can be written}$$

$$(\mathbb{I} + \mathbb{L}_0^{-1/2}\mathbb{C}_k\mathbb{L}_0^{-1/2})u = 0 \qquad \text{in} \qquad H_0^1(D)$$

$\mathbb{L}_0$ self-adjoint positive definite and $\mathbb{C}_k$ self-adjoint compact.

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Determination of Transmission Eigenvalues from Scattering Data

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Determination of Transmission Eigenvalues

First approach is based on the Linear Sampling Method

CAKONI-COLTON-HADDAR (2010) *C. R. Math. Acad. Sci. Paris*

The linear sampling method explores the far field equation

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z, k), \quad \text{for} \quad g \in L^2(S), \quad z \in D, \quad k \in [k_0,\, k_1]$$

As you know "solutions" to this equations are such that the Herglotz function $v_z := v_g(x) = \int_S e^{ikx\cdot d} g(d)\, ds$ and w_z solve

$$\begin{array}{rll}
\Delta v_z + k^2 v_z = 0 & \text{in} & D \\
\nabla \cdot A \nabla w_z + k^2 n w_z = 0 & \text{in} & D \\
w_z - v_z = \Phi(\cdot, z) & \text{on} & \partial D \\
\nu \cdot A \nabla w_z - \nu \cdot \nabla v_z = \nu \cdot \nabla \Phi(\cdot, z) & \text{on} & \partial D
\end{array}$$

Determination of Transmission Eigenvalues

$$Fg = Bv_g$$

with the compact operator $B : \left\{\Delta v + k^2 v = 0, v \in H^1(D)\right\} \to L^2(S)$

$$B : u^i \mapsto u^s_\infty, \text{ with } \nabla \cdot A\nabla u^s + k^2 n u^s = \nabla \cdot (I - A)\nabla u^i + k^2(1-n)u^i.$$

Hence we have that

$$Bv_z = \Phi_\infty(\hat{x}, z, k).$$

- If k is not a transmission eigenvalue there exists a sequence of $g^z_\epsilon \in L^2(D)$ such that

  $$\|Fg^z_\epsilon - \Phi_\infty(\cdot, z, k)\|_{L^2(S)} \to 0 \qquad \epsilon \to 0$$

  and the Herglotz function $v_{g^z_\epsilon} \to v_z$ in $H^1(D)$
- If k is a transmission eigenvalue and g^z_ϵ as above, $v_{g^z_\epsilon}$ can not be bounded in $H^1(D)$ norm as $\epsilon \to 0$, for almost all $z \in D$.

  Proof on the board

Determination of Transmission Eigenvalues

Can the same be said about the Tikhonov regularized solution g_δ^z of the far field equation with noisy far field operator F^δ, i.e. the unique minimizer g_δ^z of

$$\|F^\delta g_\delta^z - \Phi_\infty(\cdot, z)\|^2_{L^2(S)} + \epsilon \|g_\delta^z\|^2_{L^2(S)}$$

where ϵ is the Tikhonov regularization parameter?

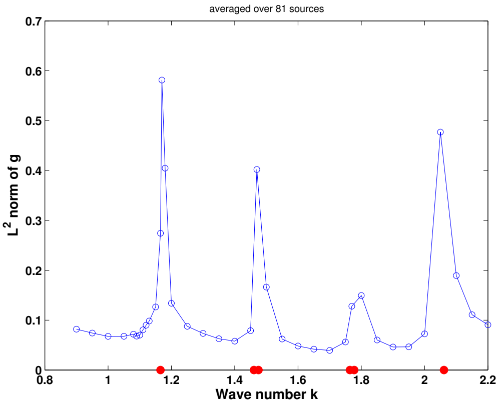
If F has dense range it is easy to show that

$$\lim_{\delta \to 0} \|F^\delta g_\delta^z - \Phi(\cdot, z)\|_{L^2(S)} = 0.$$

- Thus for almost all $z \in D$, if k is a transmission eigenvalue $\lim_{\delta \to 0} \|v_{g_\delta^z}\|_{H^1(D)} = \infty$.
- If k is not a transmission eigenvalue $\lim_{\delta \to 0} \|v_{g_\delta^z}\|_{H^1(D)}$ exists.

The proof of the latter involves the factorization method (Arens 2004)

Computation of Transmission Eigenvalues



D square 2×2, $A = I$ and $n = 16$. The far field equation is solved for several source points z inside D using 42 incoming directions and measurements. Red dots indicate exact eigenvalues.

Inside-Outside Duality

Characterize the transmission eigenvalues k from the behavior of the eigenvalues of the far field operator $F_k : L^2(S) \to L^2(S)$

$$(F_k g)(\hat{x}) := \int_S u_\infty(\hat{x}, d, k) g(d) ds_d$$

- KIRSCH-LECHLEITER (2013) - *Inverse Problems*
- LECHLEITER-PETERS (2015) - *Com. Math. Sci.*

Essential is a symmetric factorization of the far field operator

$$F_k = H_k \mathbf{T}_k H_k^*$$

where (loosely) $H_k : L^2(S) \to \mathcal{X}_k(D)$ is such that H_k^* has dense range, $\mathbf{T}_k : \mathcal{X}_k(D) \to \mathcal{X}_k(D)$ is compact perturbation of a coercive operator and its imaginary part satisfies a sign condition.

Inside-Outside Duality

Assume that $A = I$ and either $n > 1$ or $n < 1$, or $n = 1$ and either $A > I$ or $A < I$. We call q the contrast, i.e. $q = n - 1$ or $q = I - A$.

Facts on the compact operator F_k (recall $\mathcal{S}_k = I + \frac{ik}{2\pi} F_k$).

- For real A and n, F_k is normal, i.e. $F_k F_k^* = F_k^* F_k$. Thus, $\mathcal{S}_k$ is unitary, i.e. $\mathcal{S}_k \mathcal{S}_k^* = \mathcal{S}_k^* \mathcal{S}_k = I$.
- As such F_k has an infinite number of eigenvalues $\lambda_j(k)$ accumulating to 0: they lie on the circle in $\mathbb{C}$

$$|\lambda|^2 - \frac{4\pi}{k} \Im(\lambda) = 0.$$

- For k not a transmission eigenvalue, as $j \to \infty$, $\lambda_j(k)/|\lambda_j(k)| \to -1$ if $q > 0$ and $\lambda_j(k)/|\lambda_j(k)| \to 1$ if $q < 0$.
- Fix $q > 0$, then the smallest phase eigenvalue $\lambda_*(k)$ is well defined, i.e.

$$\vartheta_*(k) := \min \left\{ \vartheta_j(k) \in [0, \pi) : \text{ where } \lambda_j(k) = r_j(k) e^{i\vartheta_j(k)} \right\}.$$

Inside-Outside Duality

Inside-Outside Duality (Kirsch, Lechleiter, Peters)

- If $q > 0$, and
$$\lim_{k_0-\epsilon<k\nearrow k_0} \vartheta_*(k) = 0$$
and
$$\lim_{k_0+\epsilon>k\searrow k_0} \vartheta_*(k) = 0$$
for small enough $\epsilon > 0$. Then $k_0 > 0$ is a transmission eigenvalue.

  For $q < 0$ the above hold if the limits are π.
- The converse hods true for at least the first eigenvalue provided that the contrast q is perturbation of a sufficiently large or small constant.

TE and Non-desctructive Testing

For a given (unknown) anisotropic media A, we find an isotropic homogenous media a_0 that has the first transmission eigenvalue the same as the (measured) first transmission eigenvalue for the anisotropic media. Monotonicity properties gives that this a_0 is between A_{max} and A_{min}.

Numerical Example: We consider $D := [-1, 1] \times [-1, 1]$ and fix $n = 2$

A	τ_1	Predicted a_0
diag(5.5,6.5)	1.9657	5.95
diad(5,7)	1.9696	5.79
diag(6,6.5)	1.9591	6.24
diag(6,7)	1.9547	6.45

TE and Non-desctructive Testing

$D := [-1,1]^2$, $A = \text{diag}(5,6)$, $n = 2$, void $D_0 := B_\epsilon(0)$, $A_0 = I$, $n_0 = 1$

HARRIS-CAKONI-SUN (2014) - *Inverse Problems*

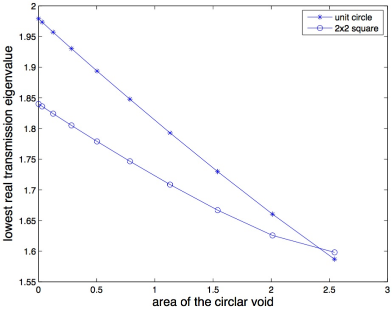


Figure 5. Graph of first transmission eigenvalue k_1 v.s. the size of a (large) circular void for $A = diag(5,6)$ and $n = 2$, and D the unit circle and the square $[-1, 1]\times[-1, 1]$.

Table 4. First TEV for various void sizes computed by the FEM

ϵ	0.2	0.19	0.18	0.17	0.16	0.15	0.14	0.13	0.12	0.11	0.1
Circle	9.53	9.27	9.02	8.77	8.54	8.31	8.08	7.86	7.64	7.43	7.22
Square	7.76	7.57	7.39	7.21	7.04	6.87	6.70	6.53	6.37	6.21	6.05

Spectral Analysis of Transmission Eigenvalue Problem

Where in the complex plane do transmission eigenvalues lie?

- HITRIK-KRUPCHYK-OLA-PAIVARINTA (2011) - *Math. Research Letters*

Comprehensive spectral theory for transmission eigenvalue problem for isotropic media.

- ROBBIANO (2013) - *Inverse Problems*

Comprehensive spectral theory with Weyl asymptotic bounds for transmission eigenvalue problem for anisotropic media.

- LAKSHTANOV-VAINBERG (2012) - *SIAM J. Math. Analysis* - *Inverse Problems*

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Spectral Theory of the Transmission Eigenvalue Problem for Spherically Stratified Media

RUTGERS

Spherically Stratified Medium

The transmission eigenvalue problem for spherically stratified media is to find nontrivial $v, w \in L^2(D)$, $v - w \in H_0^2(D)$ such that

$$\begin{aligned} \Delta v + k^2 v &= 0 && \text{in} \quad B \\ \Delta w + k^2 n w &= 0 && \text{in} \quad B \\ w &= v && \text{on} \quad \partial B \\ \frac{\partial w}{\partial r} &= \frac{\partial v}{\partial r} && \text{on} \quad \partial B \end{aligned}$$

where $B := \{x : |x| < a\}$.

Transmission eigenvalues are non-scattering frequencies.

The far field operator is not injective and does not have dense range.

Spherically Stratified Medium

Restricting to spherically stratified solutions, we make the ansatz

$$v(r) = a_0 \frac{\sin kr}{kr} \qquad w(r) = b_0 \frac{y(r)}{r}$$

where $y(r)$ is the unique solution of the ODE

$$y'' + k^2 n(r) y = 0$$
$$y(0) = 0, \quad y'(0) = 1$$

Since $y(a) = a_0 j_0(a)$, $y'(a) = a_0 j_0'(a)$ we have that transmission eigenvalues are solutions to

$$d(k) := Det \begin{vmatrix} y(a) & \dfrac{\sin ka}{k} \\ y'(a) & \cos ka \end{vmatrix} = 0.$$

Spherically Stratified Medium

$d(k)$ is an entire function of k that is real for real k and is bounded on the real axis. Hence if $d(k)$ is not a constant then there exist a countably infinite set of transmission eigenvalues.

Theorem (Aktosun-Gintides-Papanicolaou)

If $d(k) \equiv 0$ then $n(r) \equiv 1$.

We now assume that $n(r) \not\equiv 1$. Then from the asymptotic expression

$$d(k) = \frac{1}{ka^2}\left[\frac{1}{[n(0)n(a)]^{1/4}}\sin(k\delta)\cos(ka) - \left[\frac{n(a)}{n(0)}\right]^{1/4}\cos(k\delta)\sin(ka)\right] + O\left(\frac{1}{k^2}\right)$$

as $k \to \infty$, if

$$\delta = \int_0^a \sqrt{n(\rho)}d\rho \neq a$$

there exist an infinite number of positive transmission eigenvalues.

Complex Transmission Eigenvalues

Example: Let $n(r) = n_0^2$ where $0 < n_0 \neq 1$ is a constant.

- When $n_0 = \dfrac{2}{3}$ we have that

$$d(k) = -\frac{1}{k}\sin^3\left(\frac{ka}{2}\right)\left[3 + 2\cos\left(\frac{2ka}{3}\right)\right]$$

$d(k)$ has an infinite set of real and complex zeros.

- For $n_0 = \dfrac{1}{2}$ and we have that

$$d(k) = -\frac{2}{k}\sin^3\left(\frac{ka}{2}\right)$$

$d(k)$ has an infinite set of real zeros and no complex zeros.

Note: There are always complex eigenvalues for n constant if non-spherical eigenfunctions are included.
(very recently proved by Colton-Leung)

Entire Functions - Definitions

Definition

Let $M(r)$ denote the maximum modulus of the entire function $f(z)$ on $|z| = r$. Then $f(z)$ is of order ρ if

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \rho.$$

Roughly $|f(z)| \leq Ae^{\tau|z|^\rho}$

Definition

The entire function $f(z)$ of order $\rho = 1$ is called a function of exponential type τ if

$$\limsup_{r\to\infty} \frac{\log M(r)}{r} = \tau.$$

Spherically Stratified Medium

Now assume that $n(a) = 1$ and $n'(a) = 0$.

1. $d(k)$ is an even entire function of k of order (at most) one.
2. If $\int_0^a \rho^2[1 - n(\rho)]d\rho \neq 0$, $d(k)$ has a zero of order two at $k = 0$.

Thus, by the Hadamard factorization theorem, we have that

$$d(k) = c\, k^2 \prod_{j=1}^{\infty} \left(1 - k^2/k_j^2\right)$$

where $\{k_j\}$ are the zeros of $d(k)$ (including multiplicities) and c is a constant. From

$$d(k) = \frac{1}{k[n(0)]^{1/4}} \left\{ \sin k \left(a - \int_0^a \sqrt{n(\rho)}\, d\rho \right) + O\left(\frac{1}{k}\right) \right\}$$

as $k \to \infty$ along the positive real axis we have that $c[n(0)]^{1/4}$ is known. Hence, under the above assumptions, the transmission eigenvalues (real and complex!) determine $[n(0)]^{1/4} d(k)$.

The Inverse Spectral Problem

As we have just seen, under appropriate assumptions the transmission eigenvalues determine $[n(0)]^{1/4}d(k)$. In order to determine $n(r)$ from $[n(0)]^{1/4}d(k)$ we need an integral representation of the solution to

$$y'' + k^2 n(r) y = 0$$
$$y(0) = 0, \quad y'(0) = 1.$$

Using the Liouville transformation

$$\xi := \int_0^r \sqrt{n(\rho)}\, d\rho$$
$$z(\xi) := [n(r)]^{1/4} y(r)$$

We arrive at

$$z'' + [k^2 - p(\xi)]z = 0$$
$$z(0) = 0, \quad z'(0) = [n(0)]^{-1/4}$$

where

$$p(\xi) := \frac{n''(r)}{4[n(r)]^2} - \frac{5}{16}\frac{[n'(r)]^2}{[n(r)]^3}.$$

The Inverse Spectral Problem

The solution of

$$z'' + [k^2 - p(\xi)]z = 0$$
$$z(0) = 0, \quad z'(0) = [n(0)]^{-1/4}$$

can be represented in the form

$$z(\xi) = \frac{1}{[n(0)]^{1/4}} \left[\frac{\sin k\xi}{k} + \int_0^{\xi} K(\xi, t) \frac{\sin kt}{k} \, dt \right]$$

for $0 \leq \xi \leq \delta$ where $\delta = \int_0^a \sqrt{n(\rho)} \, d\rho$, and $K(\xi, t)$ is the unique solution of the Goursat problem

$$K_{\xi\xi} - K_{tt} - p(\xi)K = 0, \qquad 0 < t < \xi < \delta$$
$$K(\xi, 0) = 0, \quad 0 \leq \xi \leq \delta$$
$$K(\xi, \xi) = \frac{1}{2} \int_0^{\xi} p(s) \, ds, \quad 0 \leq \xi \leq \delta$$

A. Kirsch (2011), *An Introduction to the Mathematical Theory of Inverse Problems, Springer*.

The Inverse Spectral Problem

Theorem (Rundell-Sacks)

Let $K(\xi, t)$ satisfy the above Goursat problem. Then $p \in C^1[0, \delta]$ is uniquely determined by the Cauchy data $K(\delta, t)$, $K_\xi(\delta, t)$.

Now recall the determinant

$$d(k) := Det \begin{vmatrix} y(a) & -\dfrac{\sin ka}{k} \\ y'(a) & -\cos ka \end{vmatrix} = 0.$$

From the Liouville transformation and the representation for $z(\xi)$ we have that

$$y(a) = \frac{1}{[n(0)]^{1/4}} \left[\frac{\sin\, k\delta}{k} + \int_0^\delta K(\delta, t) \frac{\sin\, kt}{k}\, dt \right]$$

$$y'(a) = \frac{1}{[n(0)]^{1/4}} \left[\cos\, k\delta + \frac{\sin\, k\delta}{2k} \int_0^\delta p(s) ds + \int_0^\delta K_\xi(\delta, t) \frac{\sin\, kt}{k}\, dt \right]$$

The Inverse Spectral Problem

Note that the asymptotic formulas for $d(k)$ gives us δ. The above formula now gives us

$$\frac{\ell\pi}{a} d\left(\frac{\ell\pi}{a}\right) = \frac{(-1)^{\ell+1}}{[n(0)]^{1/4}}\left[\sin\frac{\ell\pi\delta}{a} + \int_0^\delta K(\delta,t)\sin\frac{\ell\pi t}{a}\,dt\right] \qquad (1)$$

and

$$\begin{aligned}\frac{\ell\pi}{a} d\left(\frac{\ell\pi}{\delta}\right) &= -y(a)\frac{\ell\pi}{\delta}\cos\frac{\ell\pi a}{\delta} \\ &+ \frac{\sin\frac{\ell\pi a}{\delta}}{[n(0)]^{1/4}}\left[(-1)^\ell + \frac{\delta}{\ell\pi}\int_0^\delta K_\xi(\delta,t)\sin\frac{\ell\pi t}{\delta}\,dt\right] \qquad (2)\end{aligned}$$

The Inverse Spectral Problem

- Since $\left\{\sin \frac{\ell \pi t}{a}\right\}$ is complete in $L^2[0, \delta]$ if $\delta \leq a$ we have from (1) that $K(\delta, t)$ (and hence $y(a)$) is known.
- From (2) and the completeness of $\sin \frac{\ell \pi t}{\delta}$ in $L^2[0, \delta]$ we have that $K_\xi(\delta, t)$ is known.

The Rundell-Sacks Theorem now implies that $p(\xi)$ is uniquely determined for $0 \leq \xi \leq \delta$ from a knowledge of $[n(0)]^{1/4} d(k)$. From this we can now easily determine $n(r)$.

The Inverse Spectral Problem

Theorem (Colton-Leung)

Assume that $n \in C^3[0, a]$, $n(a) = 1$ and $n'(a) = 0$. If $0 < n(r) < 1$ for $0 < r < a$ the transmission eigenvalues (including multiplicity) with spherically symmetric eigenfunctions, uniquely determine $n(r)$.

Theorem (Cakoni-Colton-Gintides)

Assume that $n \in C^1[0, \infty)$, $0 < n(r) < 1$ or $n(r) > 1$, and that $n(0)$ is known. All the transmission eigenvalues uniquely determine $n(r)$.

The only extension of the above theorem to the case of more general domains D is for n constant. More specifically, n is uniquely determined from a knowledge of the smallest positive transmission eigenvalue provided it is known a priori that either $n > 1$ or $0 < n < 1$.

Complex Transmission Eigenvalues Again

The previous result on the inverse spectral problem requires that $n(a) = 1$ and $n'(a) = 0$. However, our previous example on the existence of complex transmission eigenvalues was for n constant, i.e. having a jump across the boundary.

Recall that if $\delta = \int_0^a \sqrt{n(\rho)} d\rho \neq a$ real transmission eigenvalues always exist.

We now examine the existence of complex transmission eigenvalues when $n(a) = 1$ and $n'(a) = 0$.

Complex Transmission Eigenvalues Again

Let $n_+(r)$ denote the number of zeros of an entire function $f(z)$ in the right half plane with $|z| < r$.

Theorem (Cartwright-Levinson)

Let the entire function $f(z)$ of exponential type be such that

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1 + x^2}\, dx < \infty$$

and suppose that

$$\limsup_{y \to \pm\infty} \frac{|f(iy)|}{|y|} = \tau.$$

Then

$$\lim_{r \to \infty} \frac{n_+(r)}{r} = \frac{\tau}{\pi}.$$

Complex Transmission Eigenvalues Again

Definition

The number τ/π is called the density of zeros in the right half plane.

We now again consider

$$y'' + k^2 n(r) y = 0, \qquad y(0) = 0, \quad y'(0) = 1$$

and use the Liouville transformation

$$\xi := \int_0^r \sqrt{n(\rho)}\, d\rho, \qquad z(\xi) := [n(r)]^{1/4} y(r).$$

As previously, we have the representation

$$z(\xi) = \frac{1}{[n(0)]^{1/4}} \left[\frac{\sin k\xi}{k} + \int_0^\xi K(\xi, t) \frac{\sin kt}{k}\, dt \right]$$

and again define

$$d(k) := Det \begin{vmatrix} y(a) & -\dfrac{\sin ka}{k} \\ y'(a) & -\cos ka \end{vmatrix}$$

Complex Transmission Eigenvalues Again

Integrating by parts in the expression for $z(\xi)$ now yields

$$d(k) = \frac{-1}{k[n(0)]^{1/4} n(a)^{1/4}} \left[\sin\left((\delta - a)k\right) - \frac{K(\delta,\delta)}{k} \cos\left((\delta - a)k\right) + \frac{K_\tau(\delta,\delta) - K_\xi(\delta,\delta)}{2k^2} \sin\left((\delta - a)k\right) + \frac{n''(a)}{8k^2} \sin\left((\delta + a)k\right) + O\left(\frac{1}{k^3}\right) \right]$$

$$\text{where again} \qquad \delta := \int_0^a \sqrt{n(\rho)}\, d\rho.$$

Thus $d(k)$ is of type $(\delta + a)$ and the leading term $\sin\left((\delta - a)k\right)$ generates an infinite set of positive real zeros with density $|\delta - a|/\pi$. However, if $n''(a) \neq 0$, from the Cartwright-Levinson theorem the density of all zeros in the right half plane is $(\delta + a)/\pi$.

Theorem (Colton-Leung-Meng)

Suppose that $n \in C^2[0, a]$ with $n(a) = 1$ and $n'(a) = 0$ and $\delta \neq 1$. Then, under the extra assumption that $n''(a) \neq 0$, there exist infinitely many real and infinitely many complex transmission eigenvalues.