Passive sensor imaging
using cross correlations of ambient noise signals

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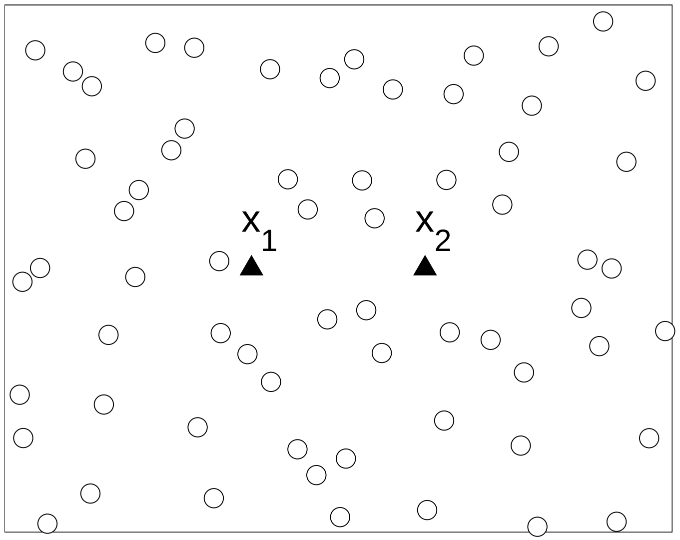
In this talk:

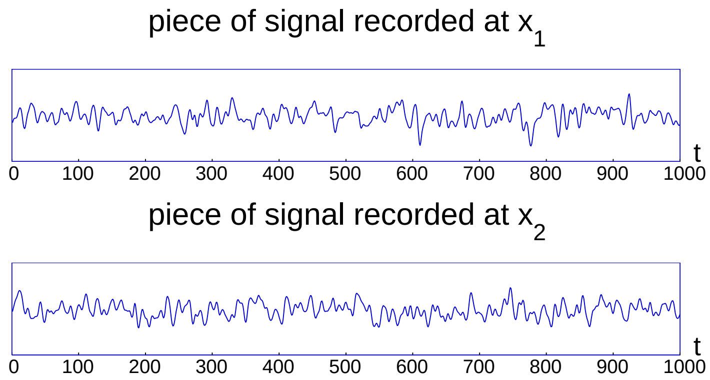
Part I: Travel time estimation for background velocity estimation.

Part II: Passive sensor imaging of reflectors.

Travel time estimation by cross correlation

- Ambient noise sources ($\circ$) emit stationary random signals.
- The waves propagate in the (inhomogeneous) medium.
- The signals $u(t, \boldsymbol{x}_1)$ and $u(t, \boldsymbol{x}_2)$ are recorded at two sensors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.





- What information (about the medium) can possibly be in these signals ?

Travel time estimation by cross correlation

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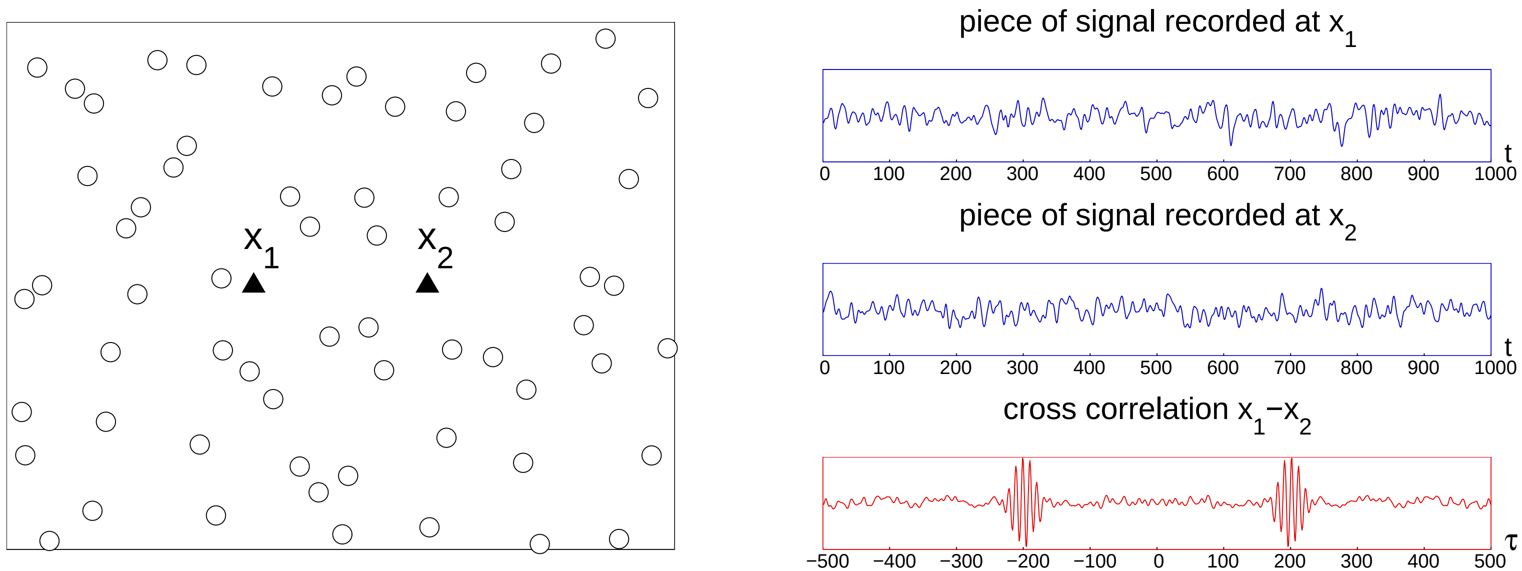
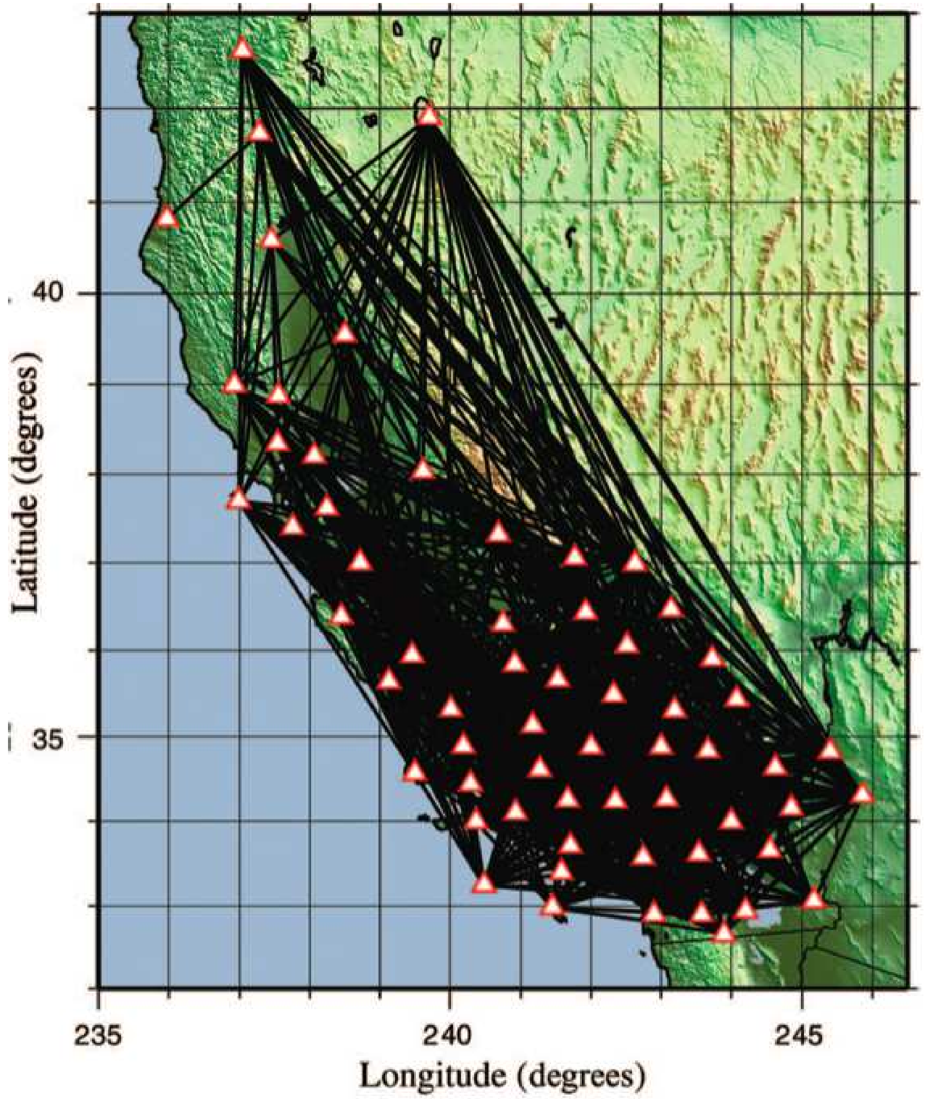


- Compute the empirical cross correlation:

$$C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{T} \int_0^T u(t, \boldsymbol{x}_1) u(t + \tau, \boldsymbol{x}_2) dt$$

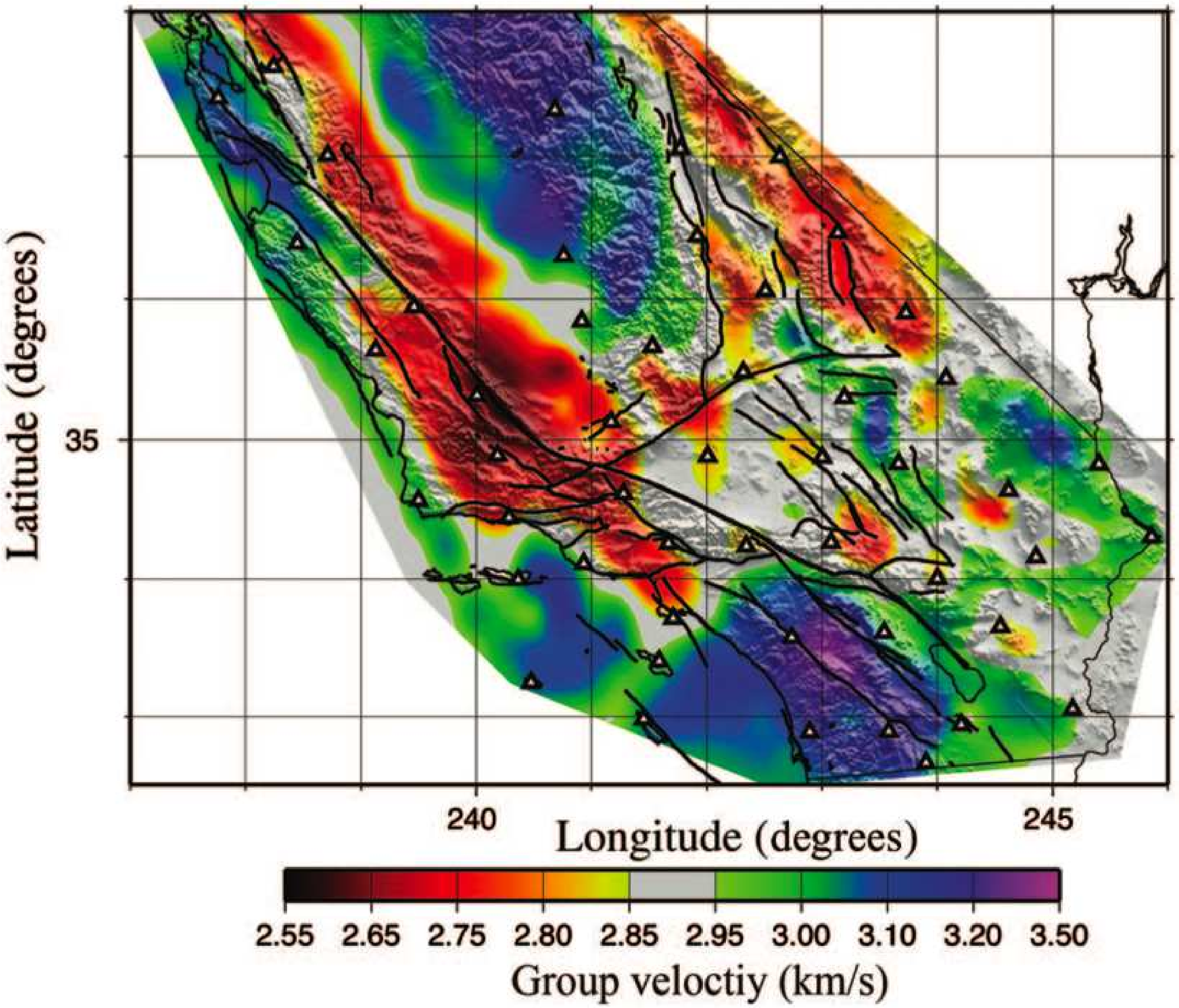
- $C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)$ contains two pseudo-peaks separated by twice the travel time from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$.

Estimations of travel times between pairs of sensors



Surface (Rayleigh) waves [from Shapiro, Campillo, et al, Science 307 (2005), 1615]

Background velocity estimation from travel time estimates



[from Shapiro, Campillo, et al, Science 307 (2005), 1615]

The wave equation with noise sources

- Consider the scalar wave model with noise sources:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u}{\partial t^2}(t,\vec{\boldsymbol{x}}) - \Delta_{\vec{\boldsymbol{x}}} u(t,\vec{\boldsymbol{x}}) = n(t,\vec{\boldsymbol{x}})$$

$n(t,\vec{\boldsymbol{x}})$: source.
$c(\vec{\boldsymbol{x}})$: propagation speed (parameter of the medium), assumed to be constant outside a domain with compact support.

- In the Fourier domain, we have

$$\hat{u}(\omega,\vec{\boldsymbol{x}}) = \int \hat{G}(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\hat{n}(\omega,\vec{\boldsymbol{y}})d\vec{\boldsymbol{y}}$$

where the time-harmonic Green's function $\hat{G}(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})$ is the solution of the Helmholtz equation

$$\Delta_{\vec{\boldsymbol{x}}}\hat{G} + \frac{\omega^2}{c^2(\vec{\boldsymbol{x}})}\hat{G} = -\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}),$$

with the Sommerfeld radiation condition ($c(\vec{\boldsymbol{x}}) = c_0$ at infinity):

$$\lim_{|\vec{\boldsymbol{x}}|\to\infty} |\vec{\boldsymbol{x}}|\Big(\frac{\vec{\boldsymbol{x}}}{|\vec{\boldsymbol{x}}|}\cdot\nabla_{\vec{\boldsymbol{x}}} - i\frac{\omega}{c_0}\Big)\hat{G}(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) = 0$$

Green's function estimation with ambient noise sources (1/3)

$$\frac{1}{c^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u}{\partial t^2}(t,\vec{\boldsymbol{x}}) - \Delta_{\vec{\boldsymbol{x}}} u(t,\vec{\boldsymbol{x}}) = n(t,\vec{\boldsymbol{x}})$$

• Sources $n(t,\vec{\boldsymbol{x}})$: Gaussian random process, stationary in time, with mean zero and covariance

$$\big\langle n(t_1,\vec{\boldsymbol{y}}_1) n(t_2,\vec{\boldsymbol{y}}_2)\big\rangle = F(t_2-t_1) K(\vec{\boldsymbol{y}}_1)\delta(\vec{\boldsymbol{y}}_1-\vec{\boldsymbol{y}}_2)$$

$\langle\cdot\rangle$: statistical average.
The function $\hat{F}$ is the power spectral density of the sources.
The function K characterizes the spatial support of the sources.

• The field $u(t,\vec{\boldsymbol{x}})$ is stationary in time. The mean field $\langle u(t,\vec{\boldsymbol{x}})\rangle$ is zero. The information is carried by the correlations $\langle u(t_1,\vec{\boldsymbol{x}}_1)u(t_2,\vec{\boldsymbol{x}}_2)\rangle$.

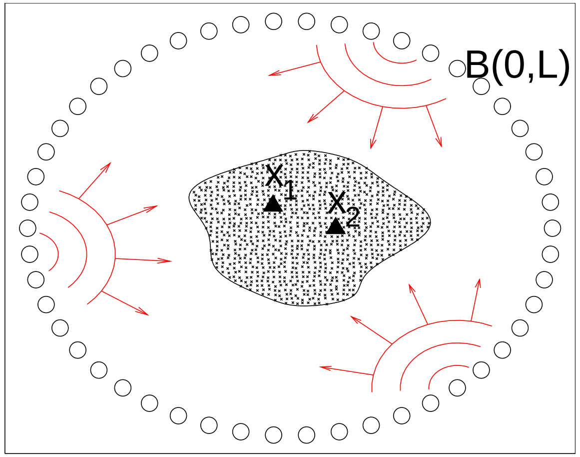
• The empirical cross correlation:

$$C_T(\tau,\vec{\boldsymbol{x}}_1,\vec{\boldsymbol{x}}_2) = \frac{1}{T}\int_0^T u(t,\vec{\boldsymbol{x}}_1)u(t+\tau,\vec{\boldsymbol{x}}_2)dt$$

converges in probability as $T\to\infty$ to the statistical cross correlation $C^{(1)}$ given by

$$\begin{aligned} C^{(1)}(\tau,\vec{\boldsymbol{x}}_1,\vec{\boldsymbol{x}}_2) &= \big\langle u(0,\vec{\boldsymbol{x}}_1)u(\tau,\vec{\boldsymbol{x}}_2)\big\rangle \\ &= \frac{1}{2\pi}\int d\vec{\boldsymbol{y}}\int d\omega \overline{\hat{G}}(\omega,\vec{\boldsymbol{x}}_1,\vec{\boldsymbol{y}})\hat{G}(\omega,\vec{\boldsymbol{x}}_2,\vec{\boldsymbol{y}})K(\vec{\boldsymbol{y}})\hat{F}(\omega)e^{-i\omega\tau} \end{aligned}$$

Green's function estimation with ambient noise sources (2/3)



Cross correlation with noise sources distributed on a closed surface $\partial B(\mathbf{0}, L)$:

$$C^{(1)}(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{1}{2\pi} \int d\omega \int_{\partial B(\mathbf{0},L)} d\sigma(\vec{\boldsymbol{y}}) \overline{\hat{G}}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{y}}) \hat{G}(\omega, \vec{\boldsymbol{x}}_2, \vec{\boldsymbol{y}}) \hat{F}(\omega) e^{-i\omega\tau}$$

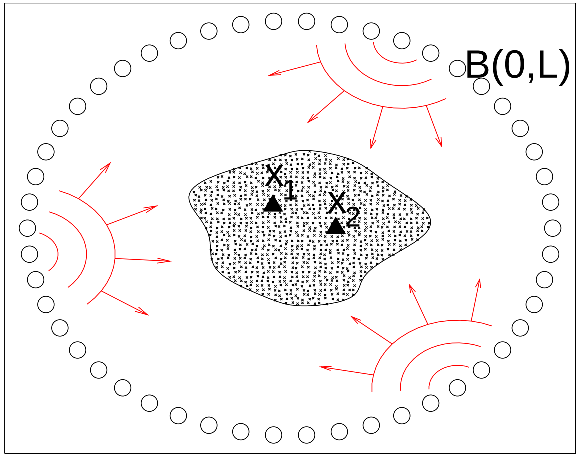
By Helmholtz-Kirchhoff identity,

$$\hat{G}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) - \overline{\hat{G}}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{2i\omega}{c_0} \int_{\partial B(\mathbf{0},L)} d\sigma(\vec{\boldsymbol{y}}) \overline{\hat{G}}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{y}}) \hat{G}(\omega, \vec{\boldsymbol{x}}_2, \vec{\boldsymbol{y}})$$

we have

$$C^{(1)}(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{c_0}{4\pi} \int \frac{\hat{F}(\omega)}{\omega} \mathrm{Im}\big(\hat{G}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2)\big) e^{-i\omega\tau} d\omega$$

Green's function estimation with ambient noise sources (3/3)

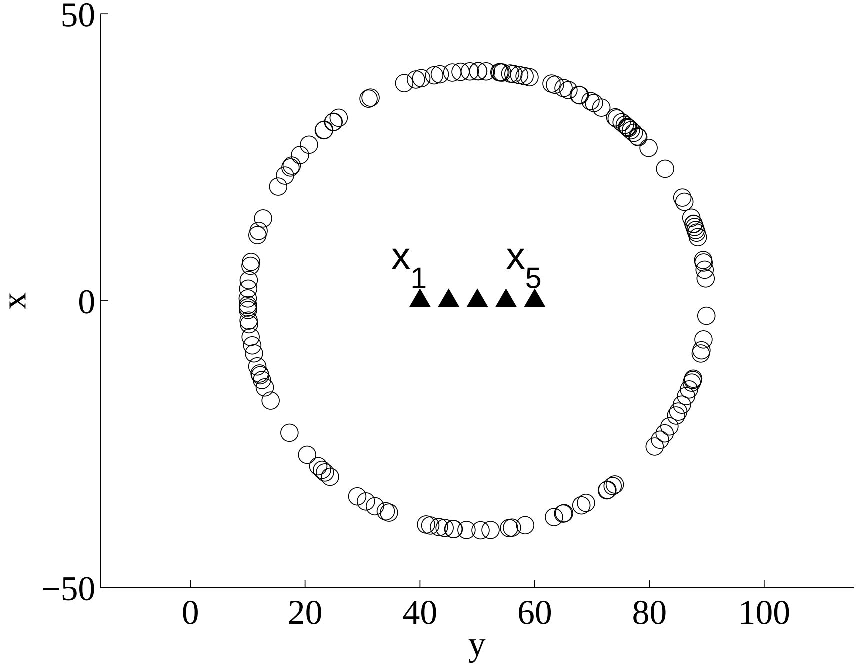


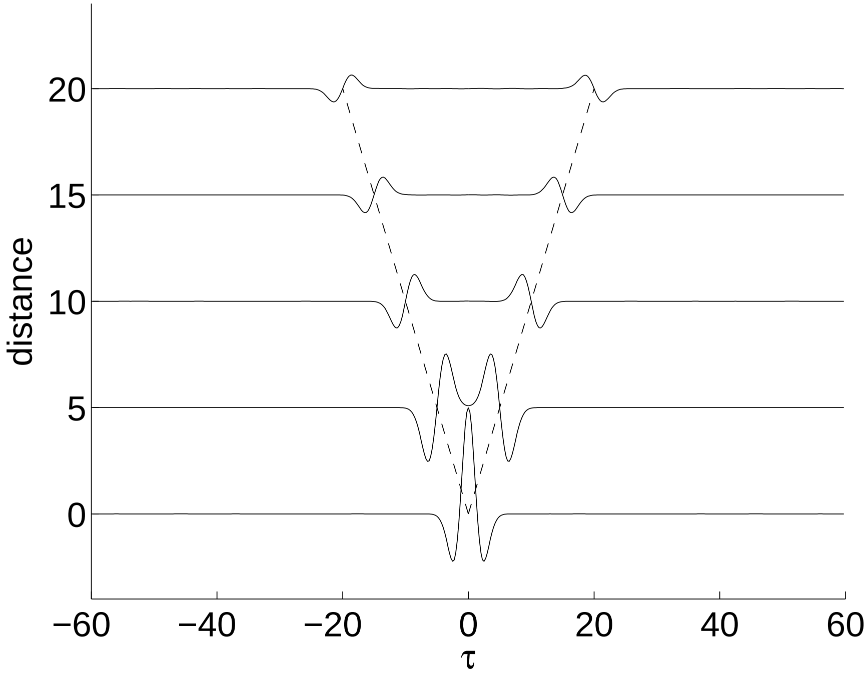
$$\partial_\tau C^{(1)}(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = -\frac{ic_0}{4\pi}\int \hat{F}(\omega)\mathrm{Im}\big(\hat{G}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2)\big)e^{-i\omega\tau}d\omega$$

$$= -\frac{c_0}{2}\Big(F *_\tau G(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) - F *_\tau G(-\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2)\Big)$$

• The cross correlation of noise signals recorded by two passive sensors is related to the Green's function between the sensors.
↪ the passive sensors can be transformed into virtual sources (known in seismology).

• This proof requires the sources to surround the region of interest.
Other proofs can justify that travel time estimation is possible with cross correlations of ambient noise signals (in a bounded cavity, ...).





Left: The circles are the noise sources and the triangles are the sensors (the distance between two successive sensors is 5). Here $\hat{F}(\omega) = \omega^2 \hat{G}(\omega)$, $\hat{G}(\omega) = \exp(-\omega^2)$, $c_0 = 1$.
Right: Cross correlation $\tau \to C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$ between the pairs of sensors $(\boldsymbol{x}_1, \boldsymbol{x}_j)$, $j = 1, \dots, 5$, versus the distance $|\boldsymbol{x}_j - \boldsymbol{x}_1|$. In theory

$$
\begin{aligned}
C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j) &= \frac{c_0}{8\pi|\boldsymbol{x}_1 - \boldsymbol{x}_j|} \Big[G'\Big(\tau - \frac{|\boldsymbol{x}_j - \boldsymbol{x}_1|}{c_0}\Big) - G'\Big(\tau + \frac{|\boldsymbol{x}_j - \boldsymbol{x}_1|}{c_0}\Big)\Big], \quad j \geq 2 \\
C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_1) &= -\frac{1}{4\pi} G''(\tau)
\end{aligned}
$$

Peaks in the form of the first derivative of a Gaussian centered at $\pm|\boldsymbol{x}_j - \boldsymbol{x}_1|/c_0$ can be clearly distinguished for $j \geq 2$.

A quick introduction to geometric optics (1/2)

We look for an approximate expression as $\varepsilon \to 0$ for $\hat{G}\big(\frac{\omega}{\varepsilon}, \boldsymbol{x}, \boldsymbol{y}\big)$ solution of

$$\Delta_{\boldsymbol{x}} \hat{G}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}, \boldsymbol{y}\Big) + \frac{\omega^2}{c^2(\boldsymbol{x})\varepsilon^2} \hat{G}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}, \boldsymbol{y}\Big) = -\delta(\boldsymbol{x} - \boldsymbol{y})$$

Note that, if $c(\boldsymbol{x}) = c_0$, then

$$\hat{G}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}, \boldsymbol{y}\Big) = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} e^{i \frac{\omega}{\varepsilon} \frac{|\boldsymbol{x} - \boldsymbol{y}|}{c_0}}$$

Consider a smoothly varying $c(\boldsymbol{x})$ and look for an expansion of the form:

$$\hat{G}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}, \boldsymbol{y}\Big) = e^{i \frac{\omega}{\varepsilon} \mathcal{T}(\boldsymbol{x}, \boldsymbol{y})} \sum_{j=0}^{\infty} \frac{\varepsilon^j \mathcal{A}_j(\boldsymbol{x}, \boldsymbol{y})}{\omega^j}$$

Substitute the ansatz into Helmholtz equation and collect the terms with the same powers in ε:

$$O\Big(\frac{1}{\varepsilon^2}\Big): \qquad |\nabla_{\boldsymbol{x}} \mathcal{T}|^2 - \frac{1}{c^2(\boldsymbol{x})} = 0$$

$$O\Big(\frac{1}{\varepsilon}\Big): \qquad 2\nabla_{\boldsymbol{x}} \mathcal{T} \cdot \nabla_{\boldsymbol{x}} \mathcal{A}_0 + \mathcal{A}_0 \Delta_{\boldsymbol{x}} \mathcal{T} = 0$$

$\hookrightarrow$ Eikonal equation for the quantity $\mathcal{T}$ (that turns out to be the travel time) + transport equation for the amplitude $\mathcal{A}_0$.

Solve by method of characteristics (ray equations).

A quick introduction to geometric optics (2/2)

Geometric optics approximation of the Green's function:

$$\hat{G}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}, \boldsymbol{y}\Big) \sim \mathcal{A}(\boldsymbol{x}, \boldsymbol{y}) e^{i\frac{\omega}{\varepsilon}\mathcal{T}(\boldsymbol{x}, \boldsymbol{y})}$$

valid when $\varepsilon \ll 1$, where the travel time is

$$\mathcal{T}(\boldsymbol{x}, \boldsymbol{y}) = \inf \Big\{ T \text{ s.t. } \exists (\boldsymbol{X}_t)_{t\in[0,T]} \in \mathcal{C}^1 \,,\, \boldsymbol{X}_0 = \boldsymbol{x} \,,\, \boldsymbol{X}_T = \boldsymbol{y} \,,\, \Big|\frac{d\boldsymbol{X}_t}{dt}\Big| = c(\boldsymbol{X}_t) \Big\}$$

The curve(s) that minimizes this functional are called ray(s).

Simple geometry hypothesis: $c(\boldsymbol{x})$ is smooth and there is a unique ray between any pair of points (in the region of interest).

In the homogeneous case $c(\boldsymbol{x}) \equiv c_0$:

$$\hat{G}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}, \boldsymbol{y}\Big) = \mathcal{A}(\boldsymbol{x}, \boldsymbol{y}) e^{i\frac{\omega}{\varepsilon}\mathcal{T}(\boldsymbol{x}, \boldsymbol{y})}, \text{ with } \mathcal{A}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}, \quad \mathcal{T}(\boldsymbol{x}, \boldsymbol{y}) = \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}$$

High-frequency analysis

- We assume that the ratio ε of the decoherence time of the sources over the typical travel time between sensors is small.

$\hookrightarrow$ The time correlation function of the sources is of the form

$$F^\varepsilon(t_2-t_1) = F\Big(\frac{t_2-t_1}{\varepsilon}\Big) \Longrightarrow \hat{F}^\varepsilon(\omega) = \varepsilon \hat{F}(\varepsilon\omega)$$

$$\begin{aligned} C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) &= \frac{1}{2\pi}\int d\boldsymbol{y} \int d\omega\, \overline{\hat{G}}(\omega, \boldsymbol{x}_1, \boldsymbol{y})\hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y})K(\boldsymbol{y})\varepsilon\hat{F}(\varepsilon\omega)e^{-i\omega\tau} \\ &\overset{\omega \to \frac{\omega}{\varepsilon}}{=} \frac{1}{2\pi}\int d\boldsymbol{y} \int d\omega\, \overline{\hat{G}}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}_1, \boldsymbol{y}\Big)\hat{G}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}_2, \boldsymbol{y}\Big)K(\boldsymbol{y})\hat{F}(\omega)e^{-i\frac{\omega}{\varepsilon}\tau} \end{aligned}$$

Geometric optics approximation for $\hat{G}$:

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi}\int d\boldsymbol{y} \int d\omega\, \mathcal{A}(\boldsymbol{x}_1, \boldsymbol{y})\mathcal{A}(\boldsymbol{x}_2, \boldsymbol{y})K(\boldsymbol{y})\hat{F}(\omega)e^{i\frac{\omega}{\varepsilon}T(\boldsymbol{y})}$$

with the rapid phase

$$\omega T(\boldsymbol{y}) = \omega[\mathcal{T}(\boldsymbol{x}_2, \boldsymbol{y}) - \mathcal{T}(\boldsymbol{x}_1, \boldsymbol{y}) - \tau]$$

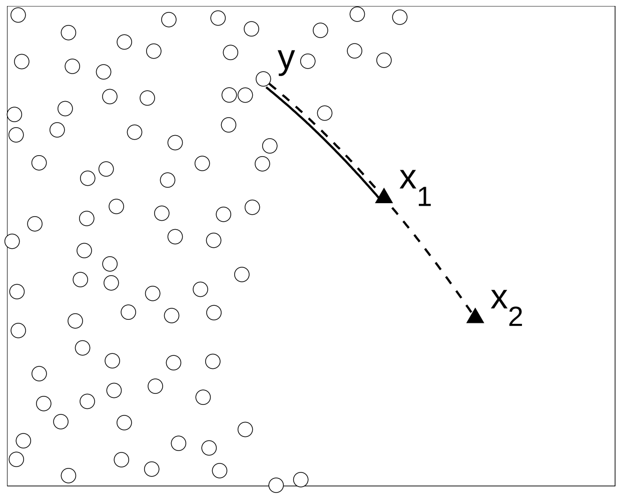
Use of the stationary phase theorem. The dominant contribution comes from the stationary points $(\omega, \boldsymbol{y})$ satisfying:

$$\nabla_{\boldsymbol{y}}\Big(\omega T(\boldsymbol{y})\Big) = \boldsymbol{0}\,, \quad \partial_\omega\Big(\omega T(\boldsymbol{y})\Big) = 0$$

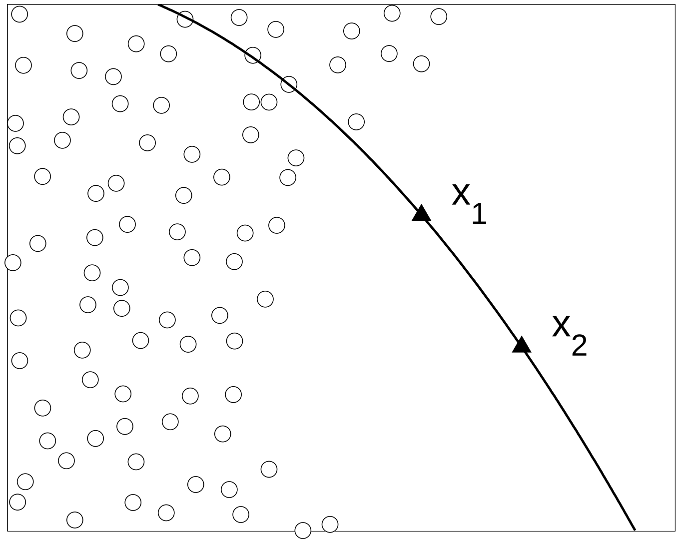
$\hookrightarrow$ two conditions:

$$\nabla_{\boldsymbol{y}}\mathcal{T}(\boldsymbol{y},\boldsymbol{x}_2) = \nabla_{\boldsymbol{y}}\mathcal{T}(\boldsymbol{y},\boldsymbol{x}_1)\,, \quad \mathcal{T}(\boldsymbol{x}_2,\boldsymbol{y}) - \mathcal{T}(\boldsymbol{x}_1,\boldsymbol{y}) = \tau$$

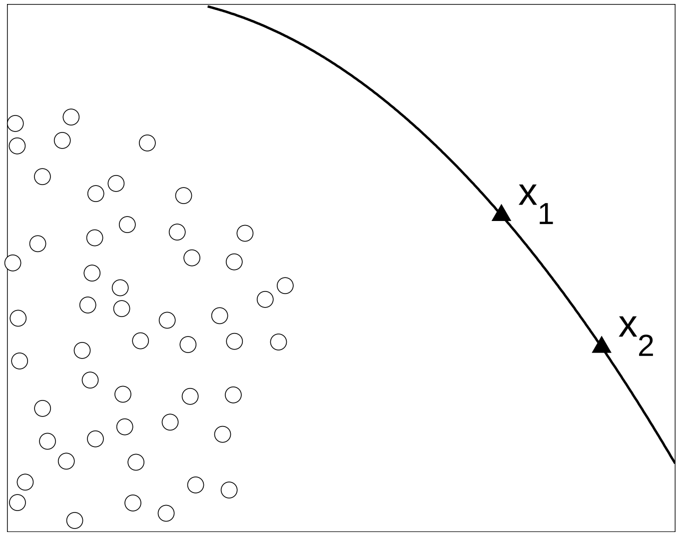
$\Longrightarrow$ $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are on the same ray issuing from $\boldsymbol{y}$
$\Longrightarrow$ $\tau = \pm\mathcal{T}(\boldsymbol{x}_1,\boldsymbol{x}_2)$.



Also: $\boldsymbol{y}$ should be in the support of K.



Singular component at $\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$



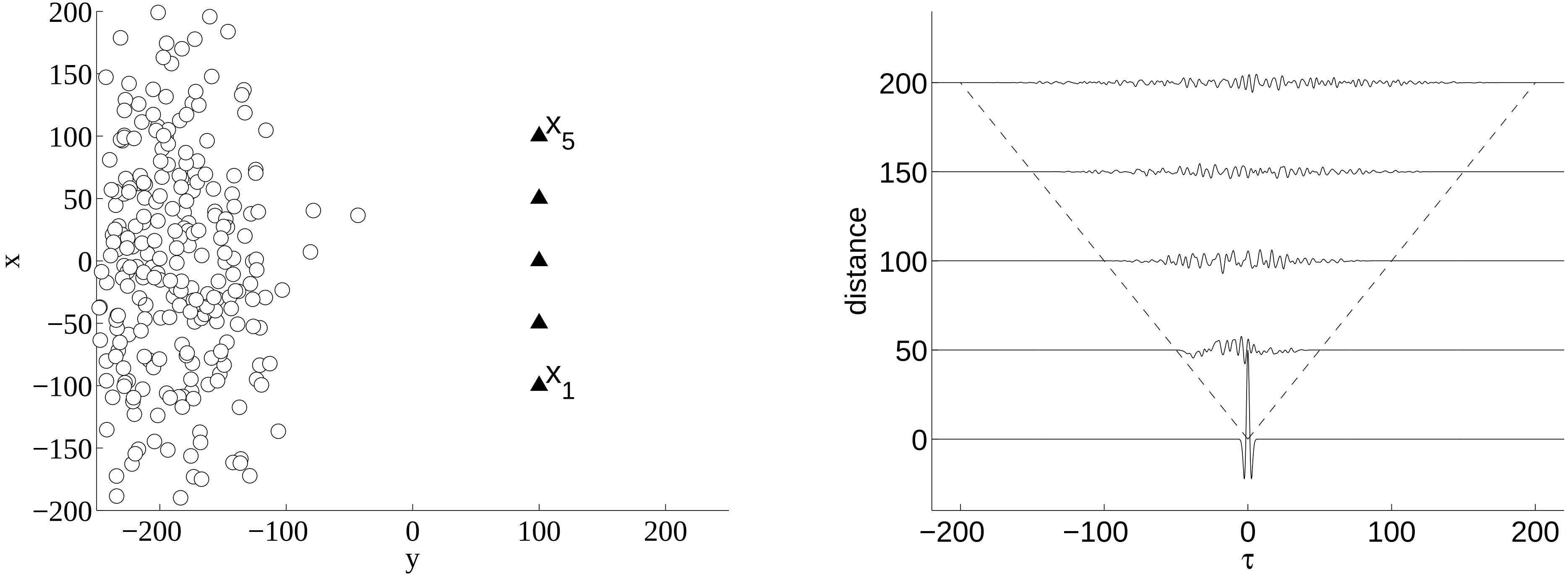
No singular component

Conclusion: The cross correlation $C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)$ has singular components iff the ray joining $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ reaches into the source region (i.e. the support of K). Then there are one or two singular components at $\tau = \pm\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$.

[More exactly:

the rays $\boldsymbol{y} \to \boldsymbol{x}_1 \to \boldsymbol{x}_2$ contribute to the singular component at $\tau = \mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$,

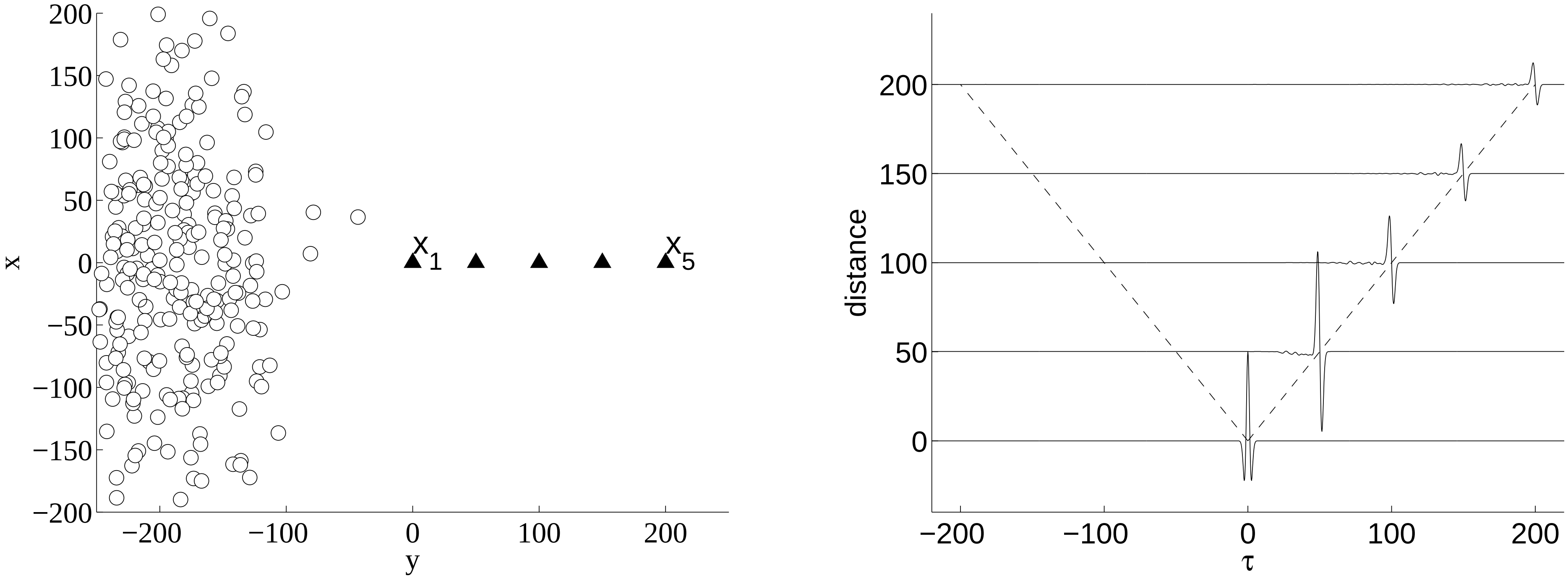
the rays $\boldsymbol{y} \to \boldsymbol{x}_2 \to \boldsymbol{x}_1$ contribute to the singular component at $\tau = -\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$.]



Left: The circles are the noise sources and the triangles are the sensors (the distance between two successive sensors is 50).

Right: Cross correlation $\tau \to C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$ between the pairs of sensors $(\boldsymbol{x}_1, \boldsymbol{x}_j)$, $j = 1, \ldots, 5$, versus the distance $|\boldsymbol{x}_j - \boldsymbol{x}_1|$.

No peak can be distinguished for $j \geq 2$.



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Peaks in the form of the first derivative of a Gaussian centered at $+|\boldsymbol{x}_j - \boldsymbol{x}_1|/c_0$ can be clearly distinguished for $j \geq 2$.

High-frequency analysis

- As ε tends to zero, in a homogeneous medium with background velocity c_0:

$$\begin{aligned}\partial_\tau C^{(1)}(\tau,\boldsymbol{x}_1,\boldsymbol{x}_2) &= \frac{c_0}{2}\mathcal{A}(\boldsymbol{x}_1,\boldsymbol{x}_2)\Big[\mathcal{K}(\boldsymbol{x}_2,\boldsymbol{x}_1)F_\varepsilon\big(\tau+\mathcal{T}(\boldsymbol{x}_1,\boldsymbol{x}_2)\big)\\ &\quad -\mathcal{K}(\boldsymbol{x}_1,\boldsymbol{x}_2)F_\varepsilon\big(\tau-\mathcal{T}(\boldsymbol{x}_1,\boldsymbol{x}_2)\big)\Big],\end{aligned}$$

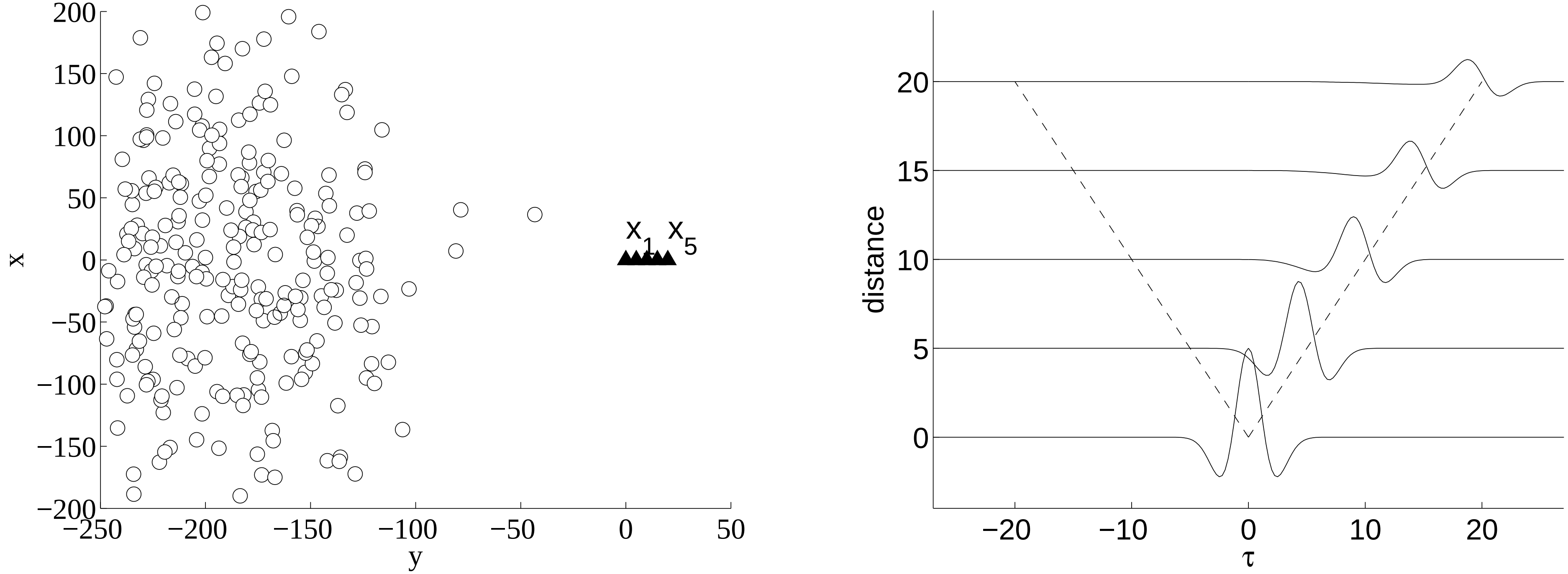
where $\mathcal{A}(\boldsymbol{x}_1,\boldsymbol{x}_2)=1/(4\pi|\boldsymbol{x}_1-\boldsymbol{x}_2|)$, $\mathcal{T}(\boldsymbol{x}_1,\boldsymbol{x}_2)=|\boldsymbol{x}_1-\boldsymbol{x}_2|/c_0$,

$$\mathcal{K}(\boldsymbol{x}_1,\boldsymbol{x}_2)=\int_0^\infty K\Big(\boldsymbol{x}_1+\frac{\boldsymbol{x}_1-\boldsymbol{x}_2}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}l\Big)dl,$$

- $\mathcal{K}(\boldsymbol{x}_1,\boldsymbol{x}_2)$ is the power released by the noise sources located along the ray starting from $\boldsymbol{x}_1$ with the direction of $\boldsymbol{x}_1-\boldsymbol{x}_2$.

- It is possible to extract the travel times between pairs of sensors (with a resolution equal to the inverse of the noise bandwidth).

- It is difficult to extract the amplitude $\mathcal{A}$ of the high-frequency Green's function as it comes with a multiplicative term that depends on the source distribution.



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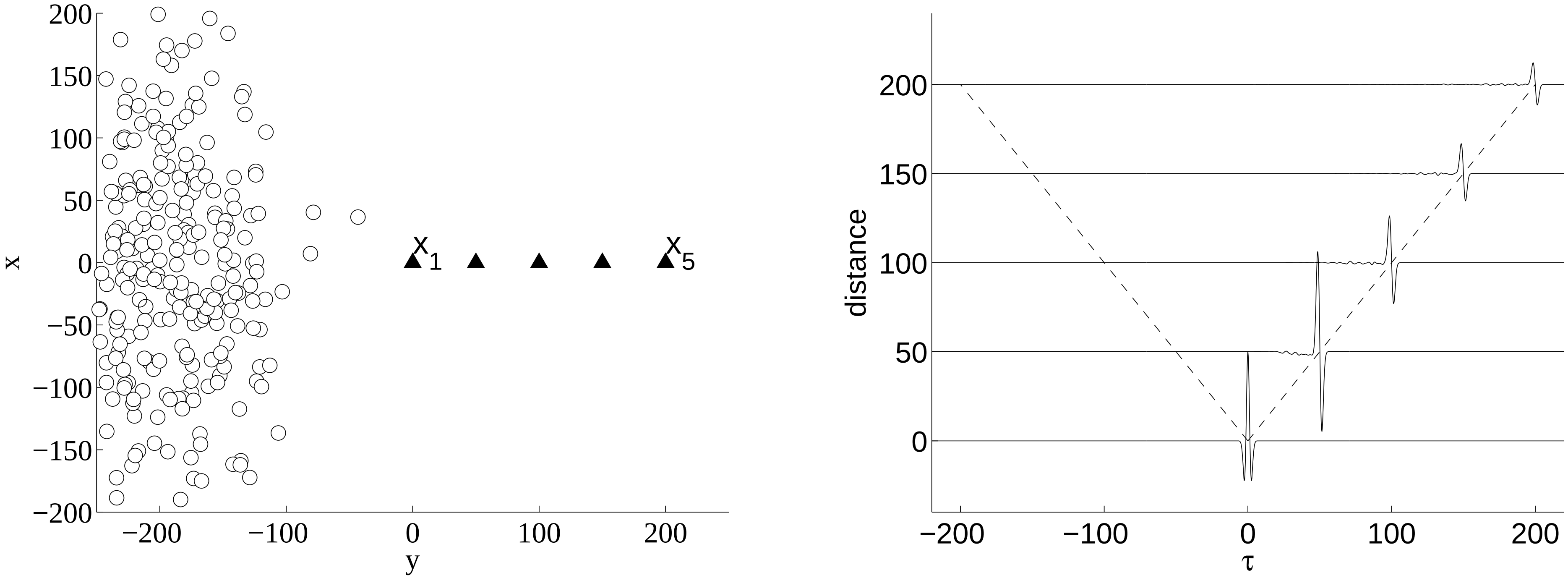
Right: Cross correlation $\tau \to C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$ between the pairs of sensors $(\boldsymbol{x}_1, \boldsymbol{x}_j)$, $j = 1, \ldots, 5$, versus the distance $|\boldsymbol{x}_j - \boldsymbol{x}_1|$.

Peaks centered at $+|\boldsymbol{x}_j - \boldsymbol{x}_1|/c_0$ can be clearly distinguished for $j \geq 2$, but their forms are not exactly the first derivative of a Gaussian for $j = 2, 3$ (distance $5, 10$), and become of this form for $j = 4, 5$ (distance $15, 20$).



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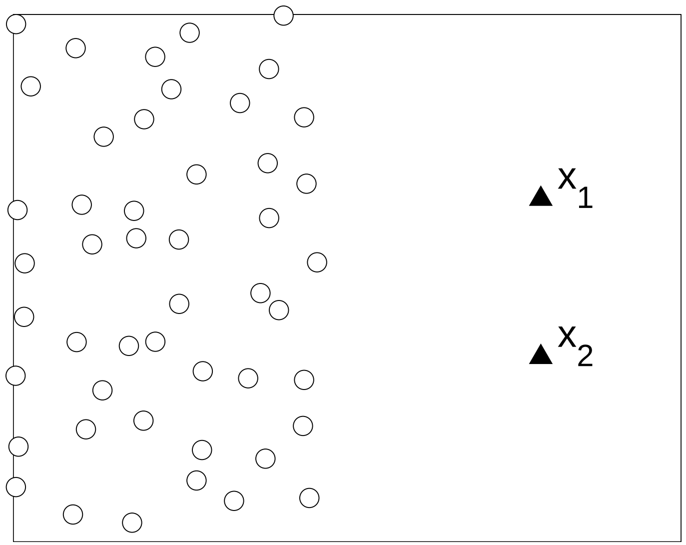
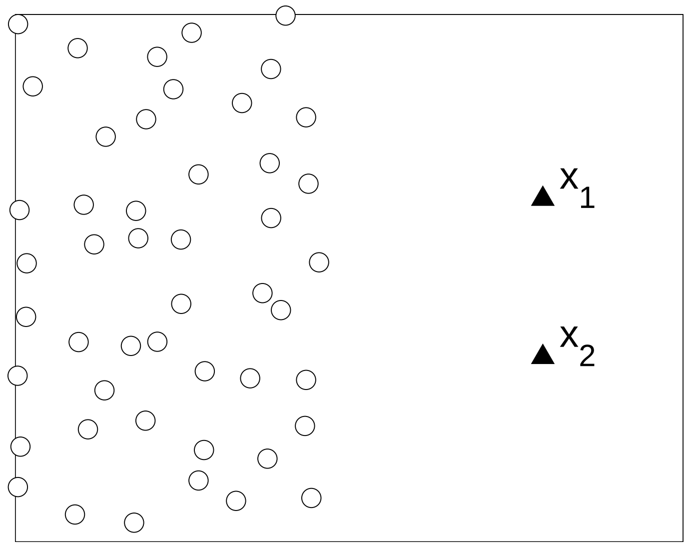
Peaks in the form of the first derivative of a Gaussian centered at $+|\boldsymbol{x}_j - \boldsymbol{x}_1|/c_0$ can be clearly distinguished for $j \geq 2$.



• Here, the cross correlation method does not allow for travel time estimation, because there is not enough "directional diversity".

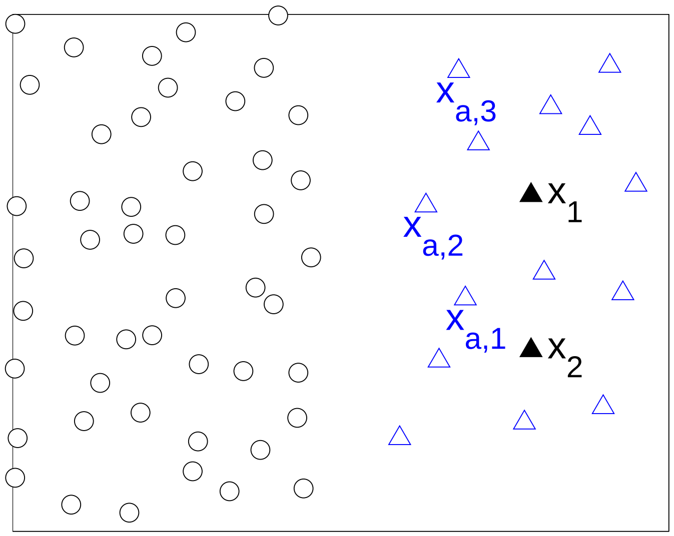


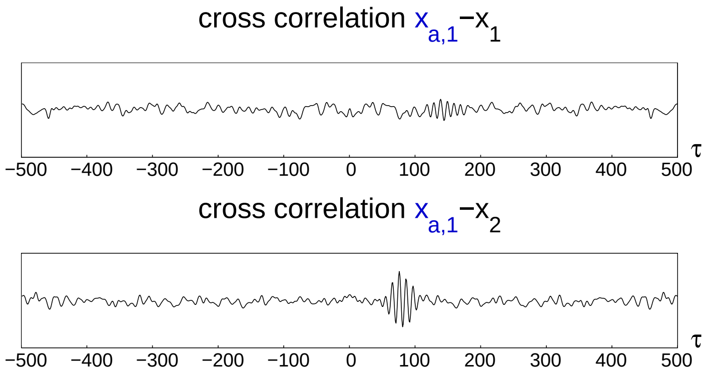
• Here, the cross correlation method does not allow for travel time estimation, because there is not enough "directional diversity".

• Idea (first suggested by M. Campillo [1]): exploit the scattering properties of the medium and use the scatterers as "secondary noise sources".

[1] M. Campillo and L. Stehly, *Eos Trans. AGU* **88**(52) (2007), Fall Meet. Suppl., Abstract S51D-07.

Fourth-order cross correlations for travel time estimation



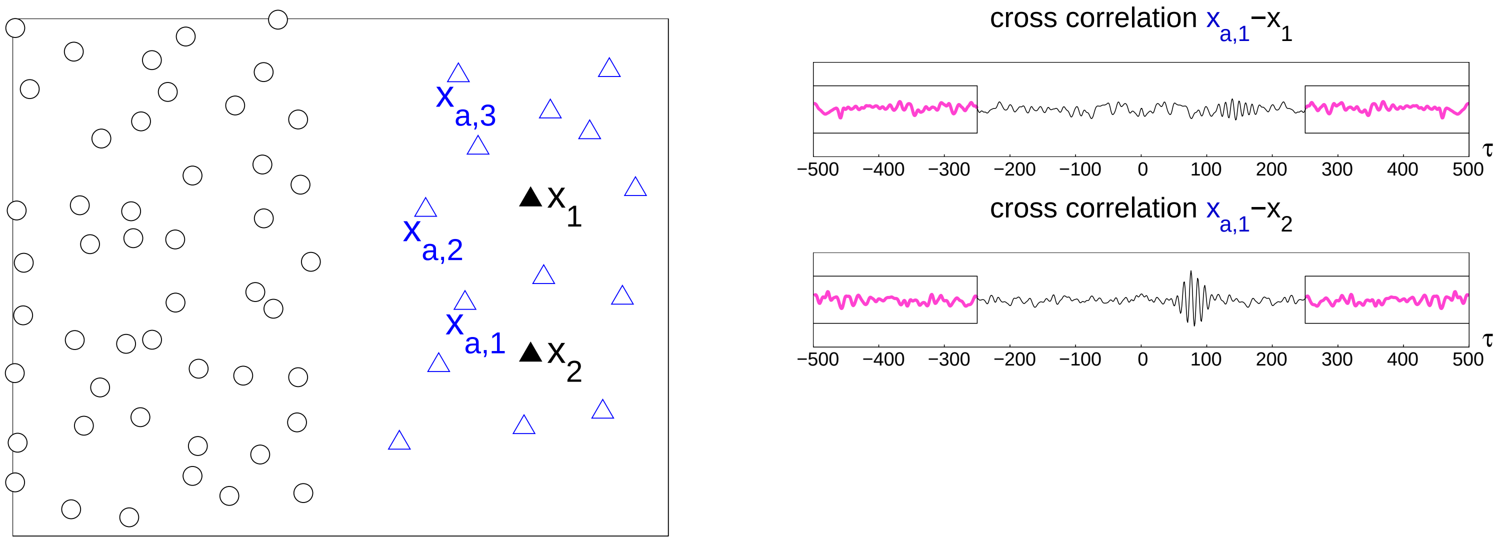


Use of auxiliary sensors $\boldsymbol{x}_{a,j}$, $j = 1, \ldots, N$. Algorithm:

1) for each j, compute the cross correlations $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_1)$ and $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_2)$:

$$C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_l) = \frac{1}{T} \int_0^T u(t, \boldsymbol{x}_{a,j}) u(t + \tau, \boldsymbol{x}_l) dt \,, \quad l = 1, 2$$

Fourth-order cross correlations for travel time estimation



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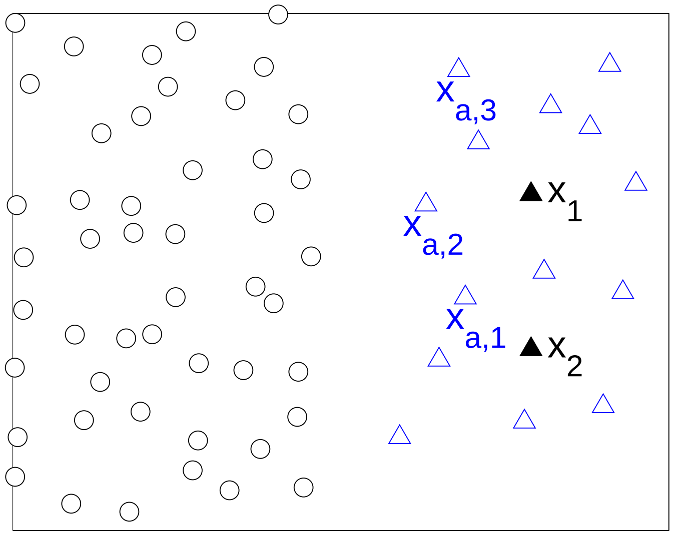
1) for each j, compute the cross correlations $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_1)$ and $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_2)$:

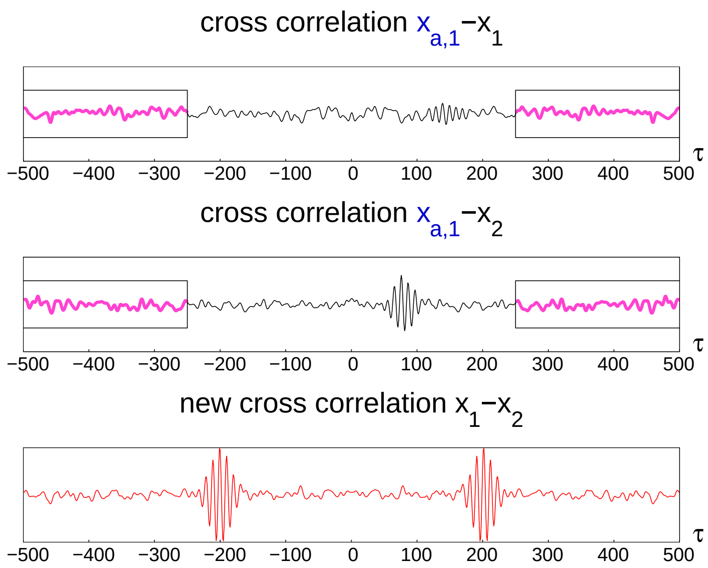
$$C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_l) = \frac{1}{T} \int_0^T u(t, \boldsymbol{x}_{a,j}) u(t+\tau, \boldsymbol{x}_l) dt \,, \quad l = 1, 2$$

2) consider the tails of $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_1)$ and $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_2)$:

$$C_{T,\mathrm{coda}}(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_l) = C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_l) \big[\mathbf{1}_{(-T_{c2}, -T_{c1})}(\tau) + \mathbf{1}_{(T_{c1}, T_{c2})}(\tau) \big], \quad l = 1, 2$$

Fourth-order cross correlations for travel time estimation





Use of auxiliary sensors $\boldsymbol{x}_{a,j}$, $j = 1, \ldots, N$. Algorithm:

1) for each j, compute the cross correlations $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_1)$ and $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_2)$:

$$C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_l) = \frac{1}{T}\int_0^T u(t, \boldsymbol{x}_{a,j}) u(t+\tau, \boldsymbol{x}_l) dt\,, \quad l = 1, 2$$

2) consider the tails of $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_1)$ and $C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_2)$:

$$C_{T,\mathrm{coda}}(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_l) = C_T(\tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_l)\big[\mathbf{1}_{(-T_{c2},-T_{c1})}(\tau) + \mathbf{1}_{(T_{c1},T_{c2})}(\tau)\big], \quad l = 1, 2$$

3) compute the cross correlations between the tails and sum over j:

$$C_T^{(3)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_{j=1}^N \int_{-\infty}^{\infty} C_{T,\mathrm{coda}}(\tau', \boldsymbol{x}_{a,j}, \boldsymbol{x}_1) C_{T,\mathrm{coda}}(\tau' + \tau, \boldsymbol{x}_{a,j}, \boldsymbol{x}_2) d\tau'$$

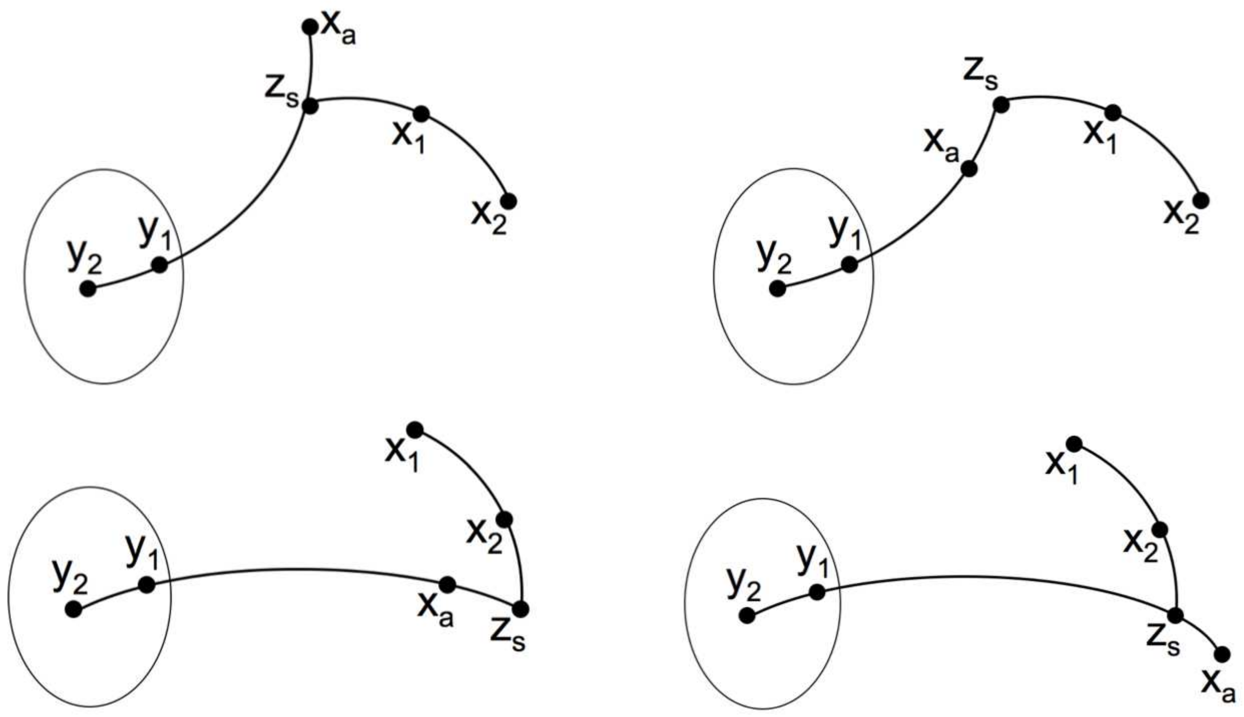
Analysis of the fourth-order cross correlation $C^{(3)}$

- Self-averaging property for $C^{(3)}$ when $T \to \infty$.
- Born (single scattering) approximation for the scattering medium.
- Geometric optics approximation for the background Green's function.

$\hookrightarrow$ expression of $C^{(3)}$ with a fast phase parameterized by a frequency ω, an auxiliary sensor $\boldsymbol{x}_a$, two sources $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, a scatterer $\boldsymbol{z}_s$ (and the main sensors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$).

- Stationary phase analysis: five conditions for the stationary points.

↪ There are stationary points:



Conclusion: $C^{(3)}$ has singular components if:

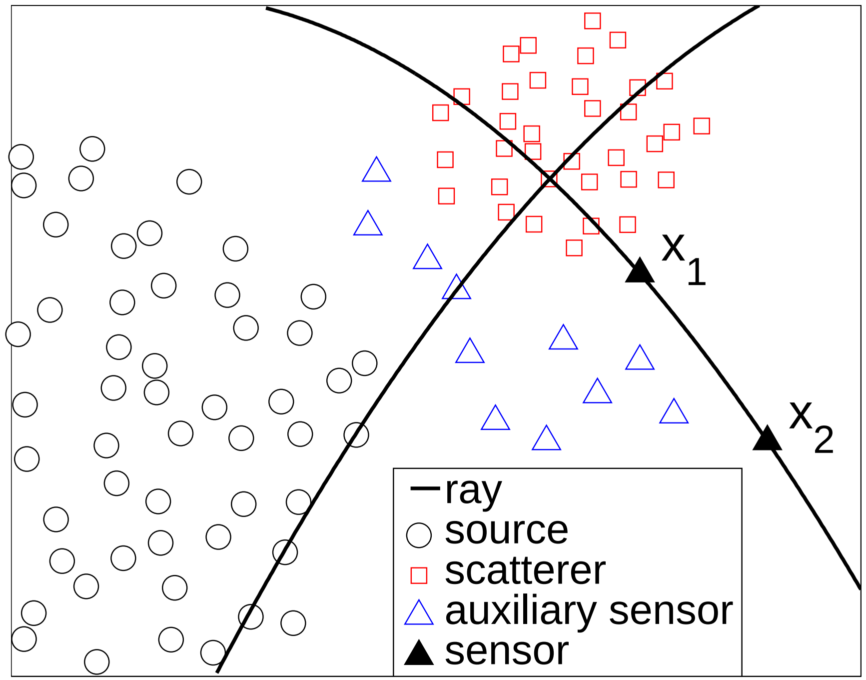
1) there are scatterers along the ray joining $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

2) there are auxiliary sensors along rays joining sources and scatterers.

These singular components are at $\tau = \pm\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$.

It is not required that the ray joining $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ reaches into the source region !

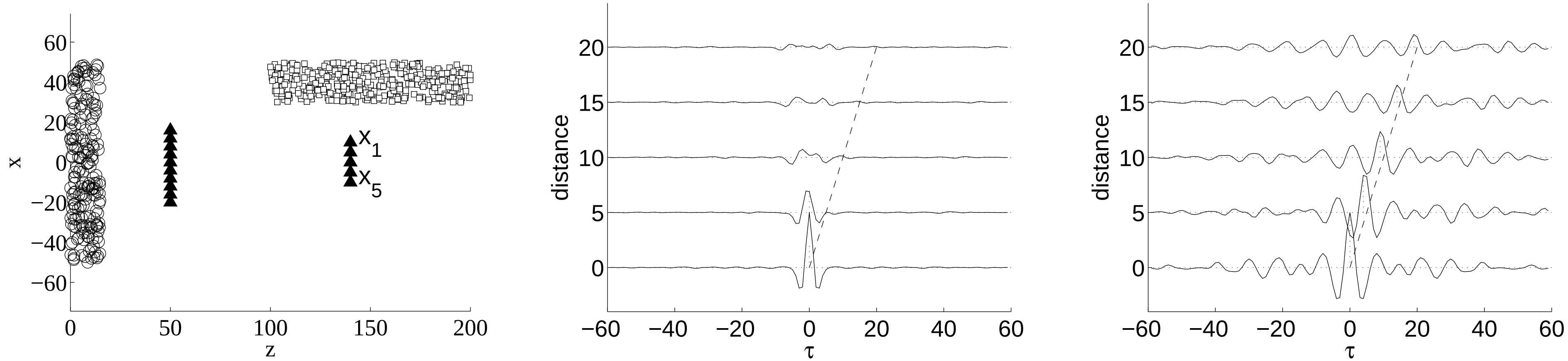
If the scattering region covers the region of interest or surrounds it, then $C^{(3)}$ has singular components at $\tau = \pm\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$!



Here:

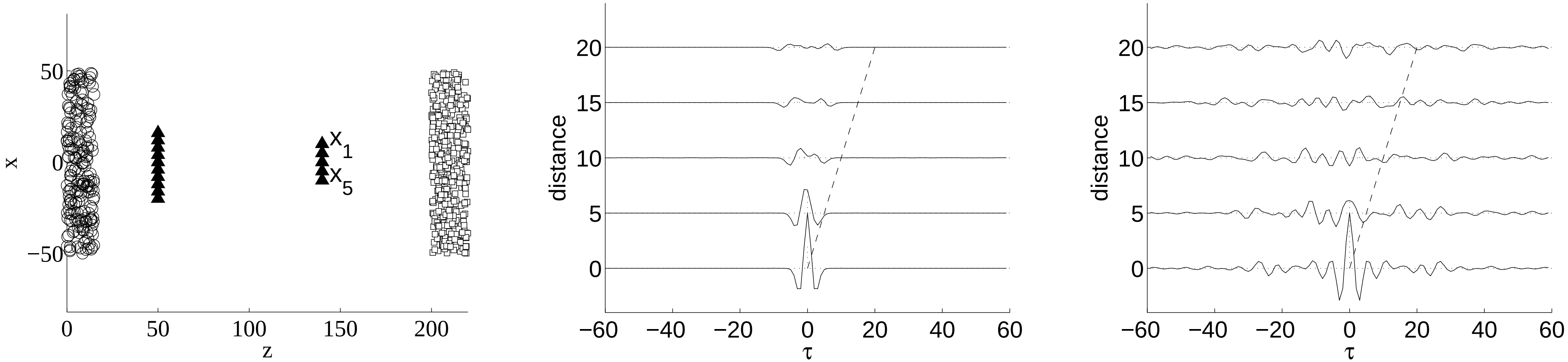
It is not possible to extract the travel time $\tau(\boldsymbol{x}_1, \boldsymbol{x}_2)$ from $C^{(1)}$

It is possible to extract the travel time $\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ from $C^{(3)}$



Configuration $\qquad$ $C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$ $\qquad$ $C^{(3)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$

The circles are the noise sources, the squares are the scatterers, and the triangles are the sensors.



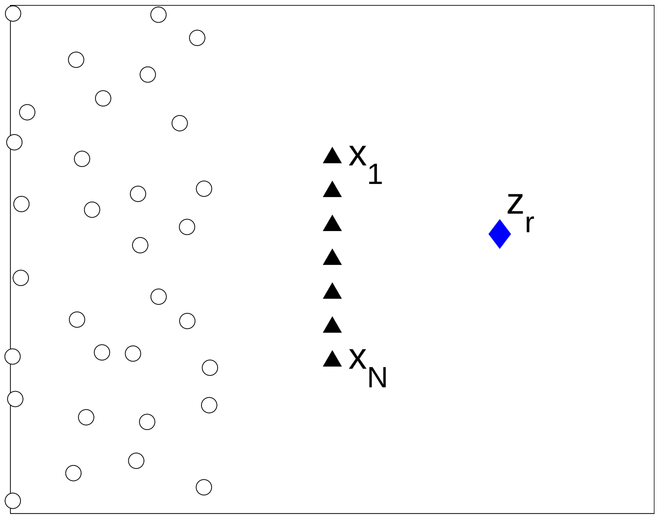
Configuration $\quad C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j) \quad C^{(3)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$

The circles are the noise sources, the squares are the scatterers, and the triangles are the sensors.

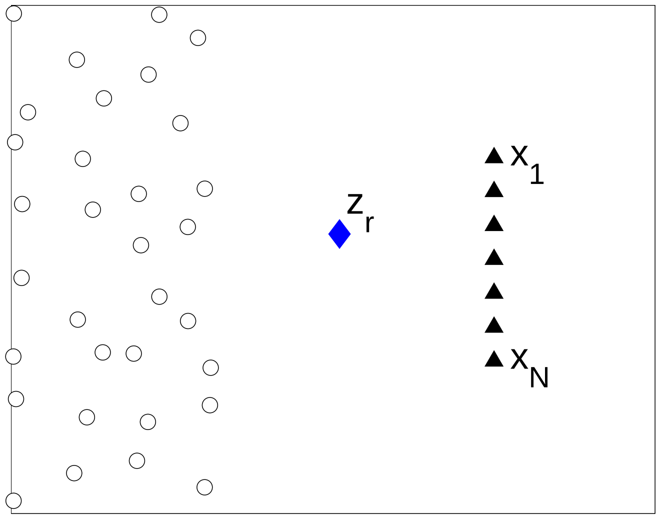
Imaging of reflectors by cross correlation of ambient noise signals

- Array of passive sensors $\boldsymbol{x}_j$, $j = 1, \ldots, N$
- Ambient noise sources emitting stationary random signals
- Target at $\boldsymbol{z}_{\rm r}$ (small reflector to be imaged)

- Two different illumination configurations



Daylight configuration



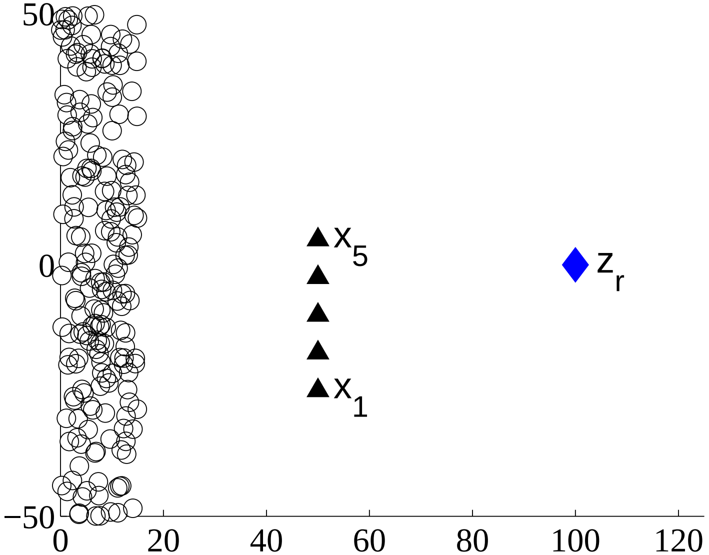
Backlight configuration

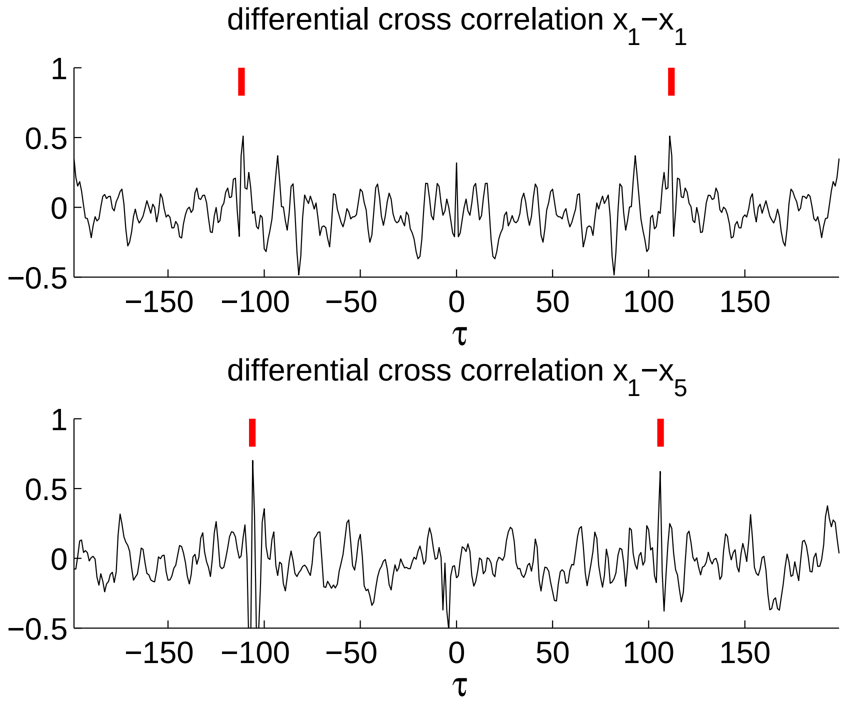
- Two types of situations:
- Data in the absence (C_0) and in the presence (C) of the reflector.
- Data only in the presence of the reflector (C).

- Note: The travel times between the sensors and points in the search region (around $\boldsymbol{z}_{\rm r}$) are supposed to be known.

Daylight configuration

- Data in the absence ($C_0(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$, $j, l = 1, \ldots, 5$) and in the presence ($C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$, $j, l = 1, \ldots, 5$) of the reflector





Differential cross correlation:

$$\Delta C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l) = (C - C_0)(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)\,, \quad j, l = 1, \ldots, 5$$

Theory (high-frequency analysis): $\Delta C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$ *has singular components at* $\tau = \pm[\mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z}_{\rm r}) + \mathcal{T}(\boldsymbol{x}_l, \boldsymbol{z}_{\rm r})]$.

High-frequency analysis

We still have

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi} \int d\boldsymbol{y} \int d\omega \, \overline{\hat{G}}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}_1, \boldsymbol{y}\Big) \hat{G}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}_2, \boldsymbol{y}\Big) K(\boldsymbol{y}) \hat{F}(\omega) e^{-i\frac{\omega}{\varepsilon}\tau}$$

Geometric optics approximatinon:

$$\hat{G}\Big(\frac{\omega}{\varepsilon}, \boldsymbol{x}_1, \boldsymbol{x}_2\Big) \sim \mathcal{A}(\boldsymbol{x}_1, \boldsymbol{x}_2) e^{i\frac{\omega}{\varepsilon}\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)} + \frac{\omega^2}{\varepsilon^2} \mathcal{A}_{\mathrm{r}}(\boldsymbol{x}_1, \boldsymbol{x}_2) e^{i\frac{\omega}{\varepsilon}\mathcal{T}_{\mathrm{r}}(\boldsymbol{x}_1, \boldsymbol{x}_2)}$$

Here

$$\mathcal{T}_{\mathrm{r}}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \mathcal{T}(\boldsymbol{x}_1, \boldsymbol{z}_{\mathrm{r}}) + \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_2) \, ,$$

Point-like reflector:

$$\mathcal{A}_{\mathrm{r}}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{\sigma_{\mathrm{r}} l_{\mathrm{r}}^3}{c_0^2} \mathcal{A}(\boldsymbol{x}_1, \boldsymbol{z}_{\mathrm{r}}) \mathcal{A}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_2).$$

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \simeq C_0^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) + C_{\rm I}^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) + C_{\rm II}^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)\,,$$

with

$$\begin{aligned}
C_0^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) &= \frac{1}{2\pi} \int d\boldsymbol{y} K(\boldsymbol{y}) \int d\omega\ \hat{F}(\omega) \overline{\mathcal{A}}(\boldsymbol{x}_1, \boldsymbol{y}) \mathcal{A}(\boldsymbol{x}_2, \boldsymbol{y}) e^{i\frac{\omega}{\varepsilon}\mathcal{T}_0(\boldsymbol{y})}\,,\\
C_{\rm I}^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) &= \frac{1}{2\pi\varepsilon^2} \int d\boldsymbol{y} K(\boldsymbol{y}) \int d\omega\ \omega^2 \hat{F}(\omega) \overline{\mathcal{A}_{\rm r}}(\boldsymbol{x}_1, \boldsymbol{y}) \mathcal{A}(\boldsymbol{x}_2, \boldsymbol{y}) e^{i\frac{\omega}{\varepsilon}\mathcal{T}_{\rm I}(\boldsymbol{y})}\,,\\
C_{\rm II}^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) &= \frac{1}{2\pi\varepsilon^2} \int d\boldsymbol{y} K(\boldsymbol{y}) \int d\omega\ \omega^2 \hat{F}(\omega) \overline{\mathcal{A}}(\boldsymbol{x}_1, \boldsymbol{y}) \mathcal{A}_{\rm r}(\boldsymbol{x}_2, \boldsymbol{y}) e^{i\frac{\omega}{\varepsilon}\mathcal{T}_{\rm II}(\boldsymbol{y})}\,,
\end{aligned}$$

and

$$\begin{aligned}
\omega\mathcal{T}_0(\boldsymbol{y}) &= \omega[\mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_1) - \tau]\\
\omega\mathcal{T}_{\rm I}(\boldsymbol{y}) &= \omega[\mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_2) - \mathcal{T}_{\rm r}(\boldsymbol{y}, \boldsymbol{x}_1) - \tau]\\
&= \omega[\mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{y}, \boldsymbol{z}_{\rm r}) - \mathcal{T}(\boldsymbol{z}_{\rm r}, \boldsymbol{x}_1) - \tau]\,,\\
\omega\mathcal{T}_{\rm II}(\boldsymbol{y}) &= \omega[\mathcal{T}_{\rm r}(\boldsymbol{y}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_1) - \tau]\\
&= \omega[\mathcal{T}(\boldsymbol{y}, \boldsymbol{z}_{\rm r}) + \mathcal{T}(\boldsymbol{z}_{\rm r}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_1) - \tau]\,.
\end{aligned}$$

- The term $C_0^{(1)}$ is of the same form as the function $C^{(1)}$ without reflector. It has singular components only if $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{y}$ are on the same ray. These singular components are supported on $\pm\mathcal{T}(\boldsymbol{x}_1, \boldsymbol{x}_2)$.

$$
\begin{aligned}
C_{\mathrm{II}}^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) &= \frac{1}{2\pi\varepsilon^2} \int d\boldsymbol{y} K(\boldsymbol{y}) \int d\omega\, \omega^2 \hat{F}(\omega) \overline{\mathcal{A}}(\boldsymbol{x}_1, \boldsymbol{y}) \mathcal{A}_{\mathrm{r}}(\boldsymbol{x}_2, \boldsymbol{y}) e^{i\frac{\omega}{\varepsilon}\mathcal{T}_{\mathrm{II}}(\boldsymbol{y})}, \\
\omega \mathcal{T}_{\mathrm{II}}(\boldsymbol{y}) &= \omega[\mathcal{T}(\boldsymbol{y}, \boldsymbol{z}_{\mathrm{r}}) + \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_1) - \tau].
\end{aligned}
$$

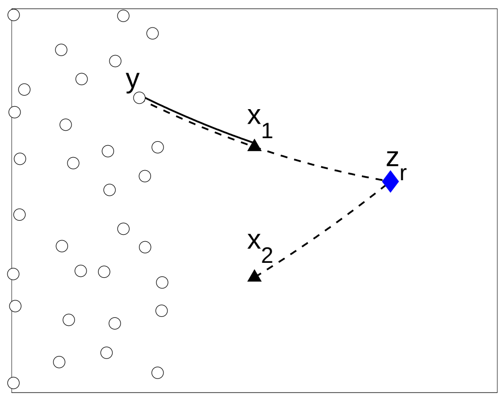
The dominant contribution to the term $C_{\mathrm{II}}^{(1)}$ comes from the stationary points $(\omega, \boldsymbol{y})$ satisfying

$$
\partial_\omega \Big(\omega \mathcal{T}_{\mathrm{II}}(\boldsymbol{y}) \Big) = 0, \qquad \nabla_{\boldsymbol{y}} \Big(\omega \mathcal{T}_{\mathrm{II}}(\boldsymbol{y}) \Big) = \boldsymbol{0},
$$

which gives the conditions

$$
\mathcal{T}(\boldsymbol{y}, \boldsymbol{z}_{\mathrm{r}}) + \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_1) = \tau, \qquad \nabla_{\boldsymbol{y}} \mathcal{T}(\boldsymbol{y}, \boldsymbol{z}_{\mathrm{r}}) = \nabla_{\boldsymbol{y}} \mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_1).
$$

The second condition implies that $\boldsymbol{x}_1$ and $\boldsymbol{z}_{\mathrm{r}}$ are on the same side of a ray issuing from $\boldsymbol{y}$. If the points are aligned along the ray as $\boldsymbol{y} \to \boldsymbol{x}_1 \to \boldsymbol{z}_{\mathrm{r}}$ (daylight configuration), then the first condition is equivalent to $\tau = \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_2) + \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_1)$.



$$C_{\mathrm{I}}^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi\varepsilon^2} \int d\boldsymbol{y} K(\boldsymbol{y}) \int d\omega\, \omega^2 \hat{F}(\omega) \overline{\mathcal{A}_{\mathrm{r}}}(\boldsymbol{x}_1, \boldsymbol{y}) \mathcal{A}(\boldsymbol{x}_2, \boldsymbol{y}) e^{i\frac{\omega}{\varepsilon}\mathcal{T}_{\mathrm{I}}(\boldsymbol{y})},$$
$$\omega \mathcal{T}_{\mathrm{I}}(\boldsymbol{y}) = \omega[\mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{y}, \boldsymbol{z}_{\mathrm{r}}) - \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_1) - \tau]$$

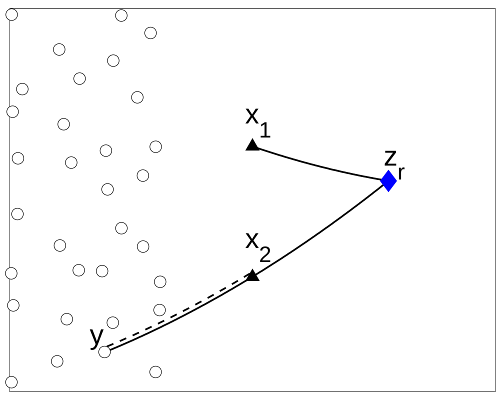
- The dominant contribution to the term $C_{\mathrm{I}}^{(1)}$ comes from the stationary points $(\omega, \boldsymbol{y})$ satisfying

$$\partial_\omega \Big(\omega \mathcal{T}_{\mathrm{I}}(\boldsymbol{y}) \Big) = 0\,, \qquad \nabla_{\boldsymbol{y}} \Big(\omega \mathcal{T}_{\mathrm{I}}(\boldsymbol{y}) \Big) = \boldsymbol{0}\,,$$

which gives the conditions

$$\mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{y}, \boldsymbol{z}_{\mathrm{r}}) - \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_1) = \tau\,, \qquad \nabla_{\boldsymbol{y}} \mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_2) = \nabla_{\boldsymbol{y}} \mathcal{T}(\boldsymbol{y}, \boldsymbol{z}_{\mathrm{r}})\,.$$

The second condition implies that $\boldsymbol{x}_2$ and $\boldsymbol{z}_{\mathrm{r}}$ are on the same side of a ray issuing from $\boldsymbol{y}$. If the points are aligned along the ray as $\boldsymbol{y} \to \boldsymbol{x}_2 \to \boldsymbol{z}_{\mathrm{r}}$ (daylight configuration), then the first condition is equivalent to $\tau = -\mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_1)$.



Daylight configuration - migration

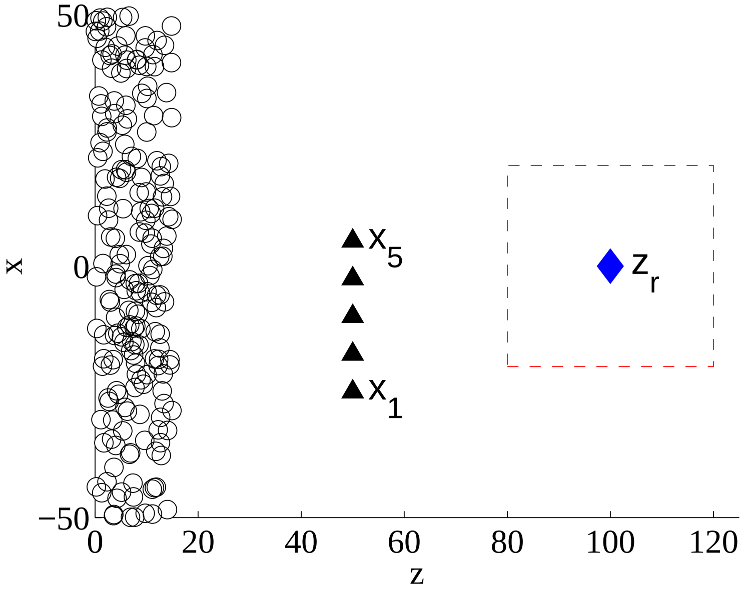
$$\Delta C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l) = (C - C_0)(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$$

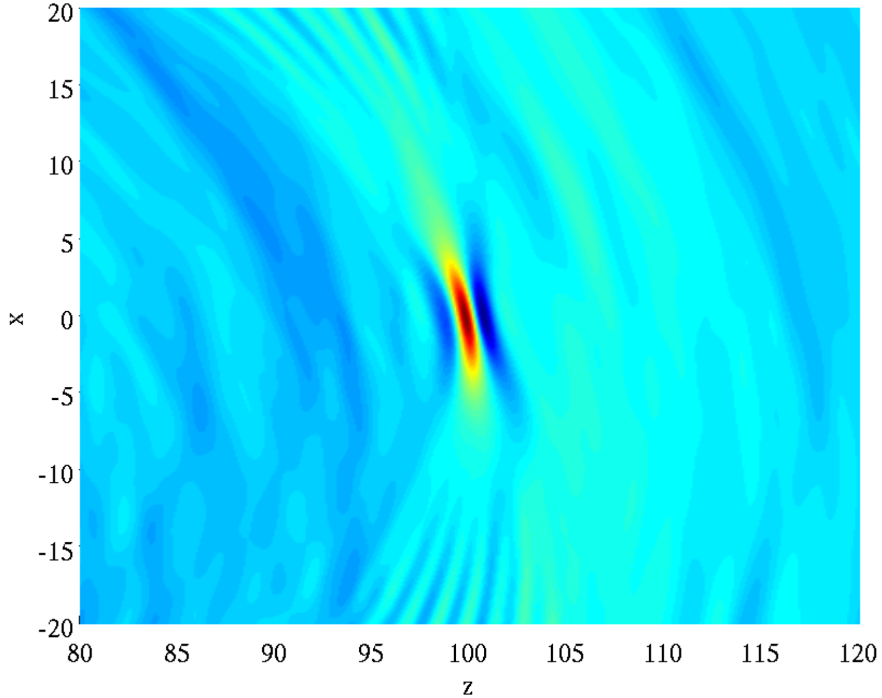
Theory: $\Delta C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$ *has singular components at* $\tau = \pm[\mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z}_{\rm r}) + \mathcal{T}(\boldsymbol{x}_l, \boldsymbol{z}_{\rm r})]$.

- Migration of the differential cross correlations ΔC.

Migration functional for the search point $\boldsymbol{z}^S$:

$$\mathcal{I}^{\rm D}(\boldsymbol{z}^S) = \sum_{j,l=1}^{N} \Delta C\big(\mathcal{T}(\boldsymbol{z}^S, \boldsymbol{x}_j) + \mathcal{T}(\boldsymbol{z}^S, \boldsymbol{x}_l), \boldsymbol{x}_j, \boldsymbol{x}_l\big)$$





Daylight configuration - resolution analysis

Migration functional:

$$\mathcal{I}^{\rm D}(\boldsymbol{z}^S) = \sum_{j,l=1}^{N} \Delta C\big(\mathcal{T}(\boldsymbol{z}^S, \boldsymbol{x}_j) + \mathcal{T}(\boldsymbol{z}^S, \boldsymbol{x}_l), \boldsymbol{x}_j, \boldsymbol{x}_l\big)$$

Analogy with Kirchhoff Migration [1] for array imaging using an array of active sensors $(\boldsymbol{x}_j)_{j=1,\ldots,N}$ emitting broadband pulses. The data is then the impulse response matrix $(u(t, \boldsymbol{x}_j; \boldsymbol{x}_l))_{j,l=1,\ldots,N}$ and the Kirchhoff Migration functional is

$$\mathcal{I}^{\rm KM}(\boldsymbol{z}^S) = \sum_{j,l=1}^{N} u\big(\mathcal{T}(\boldsymbol{z}^S, \boldsymbol{x}_j) + \mathcal{T}(\boldsymbol{z}^S, \boldsymbol{x}_l), \boldsymbol{x}_j; \boldsymbol{x}_l\big)$$

$\hookrightarrow$ Passive imaging using ambient noise has the same resolution as array imaging using active sources !

$\rightarrow$ Range resolution $\simeq c_0/B$, where B is the bandwidth.

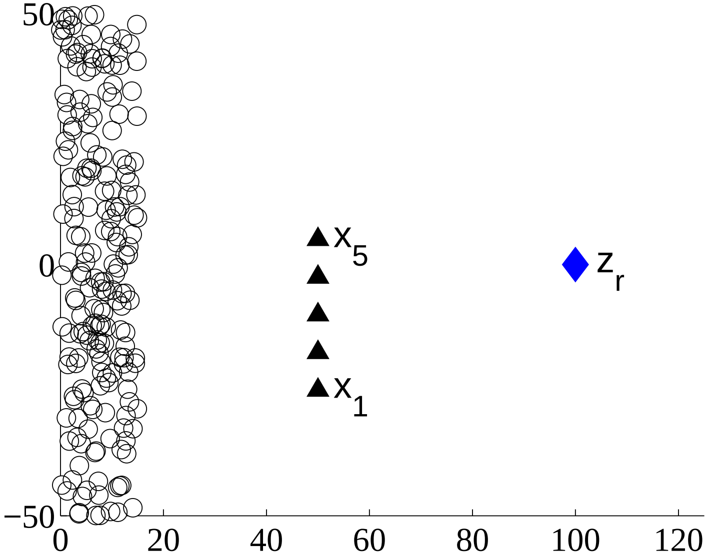
$\rightarrow$ Cross range resolution (for a linear array) $\simeq \lambda_0 L/a$, where λ_0 is the carrier wavelength, L is the distance from the array to the reflector, a the diameter of the array.

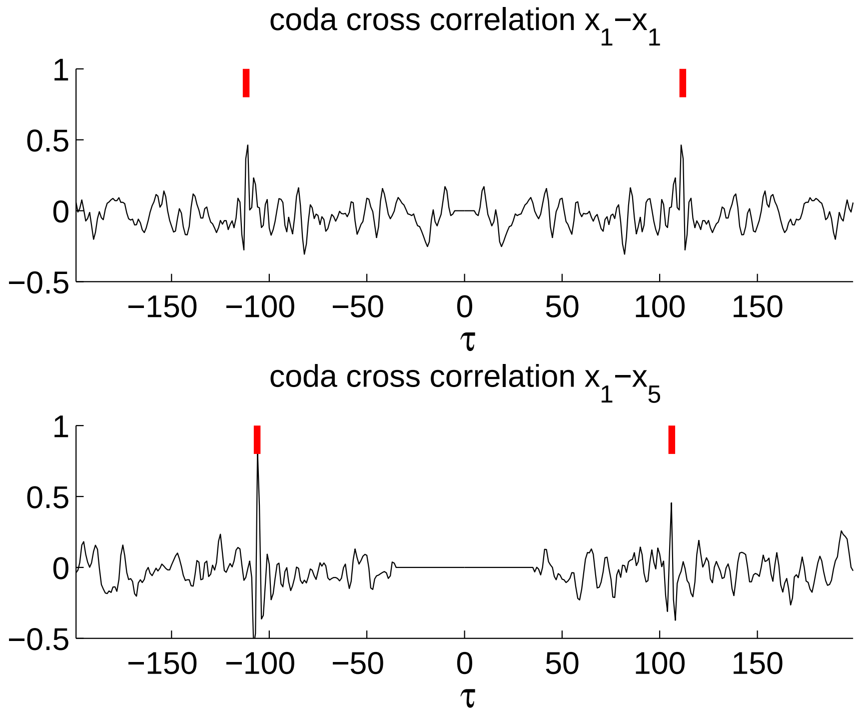
$\rightarrow$ Cross range resolution (for a distributed network) $\simeq c_0/B$ (triangulation).

[1] N. Bleistein, J. K. Cohen, and J. W. Stockwell Jr, Mathematics of seismic imaging, Springer, 2001.

Daylight configuration

- Data *only* in the presence of the reflector: $C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$, $j, l = 1, \ldots, 5$.





Coda cross correlation:

$$C_{\rm coda}(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l) = C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)\mathbf{1}_{(\mathcal{T}(\boldsymbol{x}_j, \boldsymbol{x}_l), \infty)}(|\tau|)$$

Daylight configuration - migration

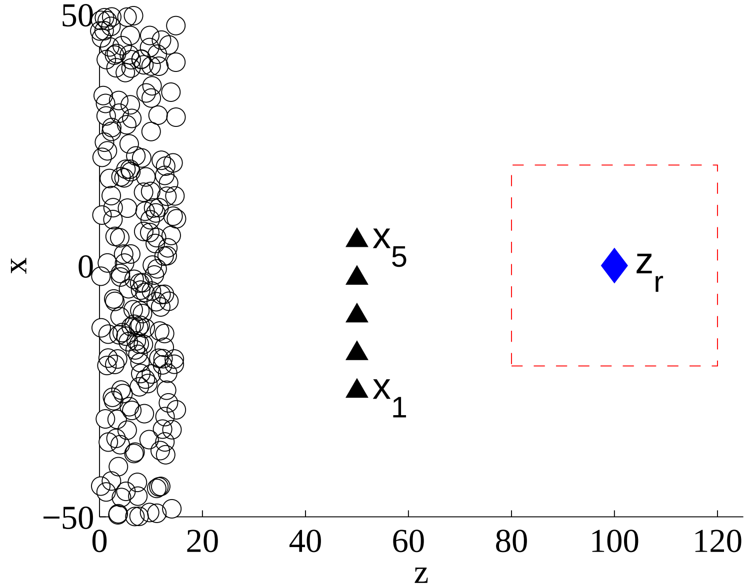
$$C_{\rm coda}(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l) = C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)\mathbf{1}_{(\mathcal{T}(\boldsymbol{x}_j,\boldsymbol{x}_l),\infty)}(|\tau|)$$

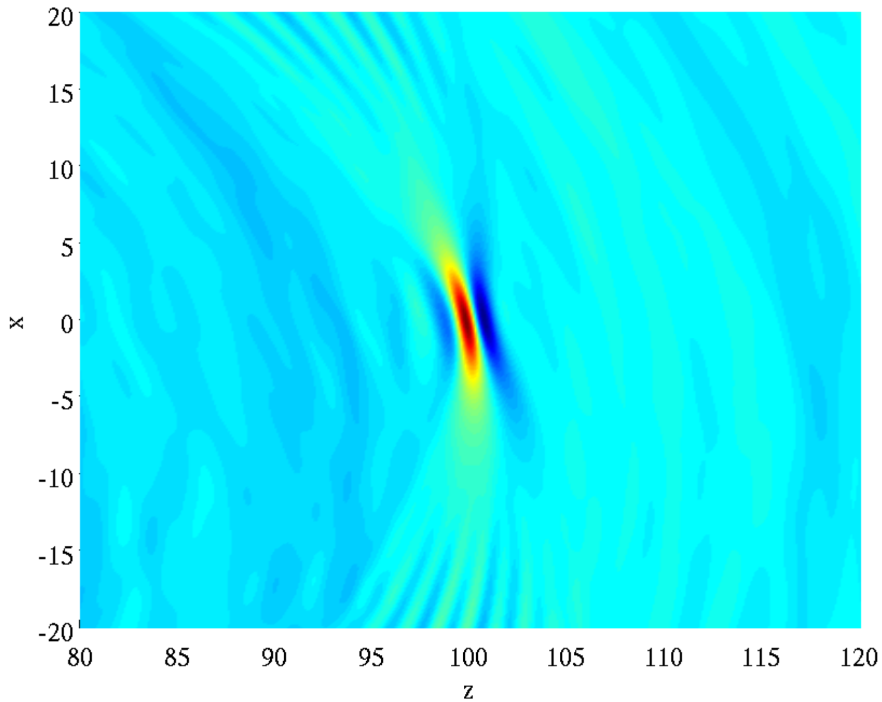
Theory: $C_{\rm coda}(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$ *has singular components at* $\tau = \pm[\mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z}_{\rm r}) + \mathcal{T}(\boldsymbol{x}_l, \boldsymbol{z}_{\rm r})]$.
Triangular inequality: $|\mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z}_{\rm r}) + \mathcal{T}(\boldsymbol{x}_l, \boldsymbol{z}_{\rm r})| \geq \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{x}_l) \Longrightarrow$ *singular components in* $C_{\rm coda}$.

- Migration of the coda cross correlations.

Migration functional for the search point $\boldsymbol{z}^S$:

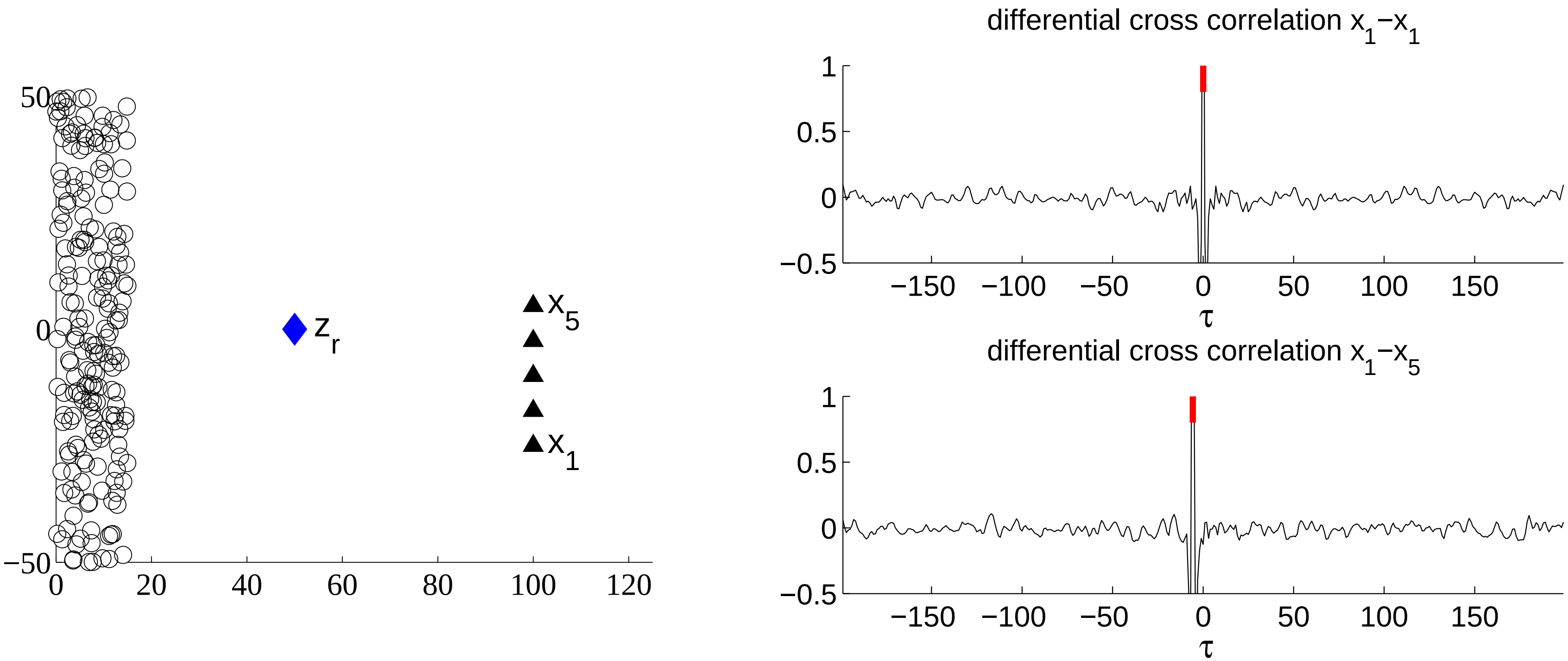
$$\mathcal{I}^{\rm D}(\boldsymbol{z}^S) = \sum_{j,l=1}^{N} C_{\rm coda}\big(\mathcal{T}(\boldsymbol{z}^S, \boldsymbol{x}_j) + \mathcal{T}(\boldsymbol{z}^S, \boldsymbol{x}_l), \boldsymbol{x}_j, \boldsymbol{x}_l\big)$$





Backlight configuration

- Data in the absence (C_0) and in the presence (C) of the reflector



Differential cross correlation:

$$\Delta C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l) = (C - C_0)(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$$

Theory: $\Delta C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$ *has singular components at* $\tau = \mathcal{T}(\boldsymbol{x}_l, \boldsymbol{z}_\mathrm{r}) - \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z}_\mathrm{r})$.

$$
\begin{aligned}
C_{\mathrm{I}}^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) &= \frac{1}{2\pi\varepsilon^2} \int d\boldsymbol{y} K(\boldsymbol{y}) \int d\omega\, \omega^2 \hat{F}(\omega) \overline{\mathcal{A}_{\mathrm{r}}}(\boldsymbol{x}_1, \boldsymbol{y}) \mathcal{A}(\boldsymbol{x}_2, \boldsymbol{y}) e^{i \frac{\omega}{\varepsilon} \mathcal{T}_{\mathrm{I}}(\boldsymbol{y})}, \\
\omega \mathcal{T}_{\mathrm{I}}(\boldsymbol{y}) &= \omega[\mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{y}, \boldsymbol{z}_{\mathrm{r}}) - \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_1) - \tau]
\end{aligned}
$$

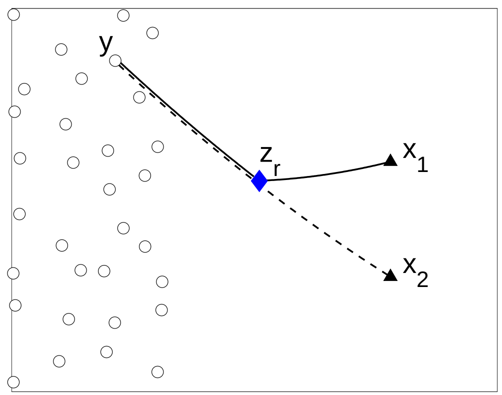
- The dominant contribution to the term $C_{\mathrm{I}}^{(1)}$ comes from the stationary points $(\omega, \boldsymbol{y})$ satisfying

$$
\partial_\omega \Big(\omega \mathcal{T}_{\mathrm{I}}(\boldsymbol{y}) \Big) = 0, \qquad \nabla_{\boldsymbol{y}} \Big(\omega \mathcal{T}_{\mathrm{I}}(\boldsymbol{y}) \Big) = \mathbf{0},
$$

which gives the conditions

$$
\mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{y}, \boldsymbol{z}_{\mathrm{r}}) - \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_1) = \tau, \qquad \nabla_{\boldsymbol{y}} \mathcal{T}(\boldsymbol{y}, \boldsymbol{x}_2) = \nabla_{\boldsymbol{y}} \mathcal{T}(\boldsymbol{y}, \boldsymbol{z}_{\mathrm{r}}).
$$

The second condition implies that $\boldsymbol{x}_2$ and $\boldsymbol{z}_{\mathrm{r}}$ are on the same side of a ray issuing from $\boldsymbol{y}$. If the points are aligned along the ray as $\boldsymbol{y} \to \boldsymbol{z}_{\mathrm{r}} \to \boldsymbol{x}_2$ (backlight configuration), then the first condition is equivalent to $\tau = \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_2) - \mathcal{T}(\boldsymbol{z}_{\mathrm{r}}, \boldsymbol{x}_1)$.



Backlight configuration - migration

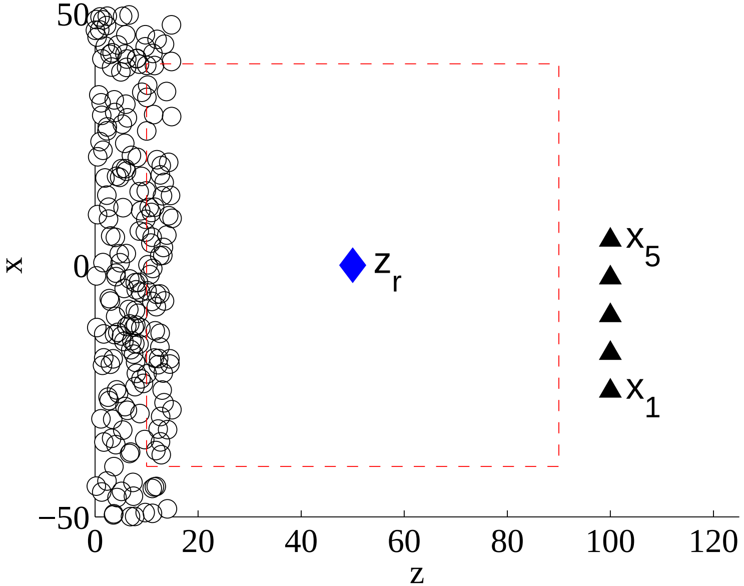
$$\Delta C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l) = (C - C_0)(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$$

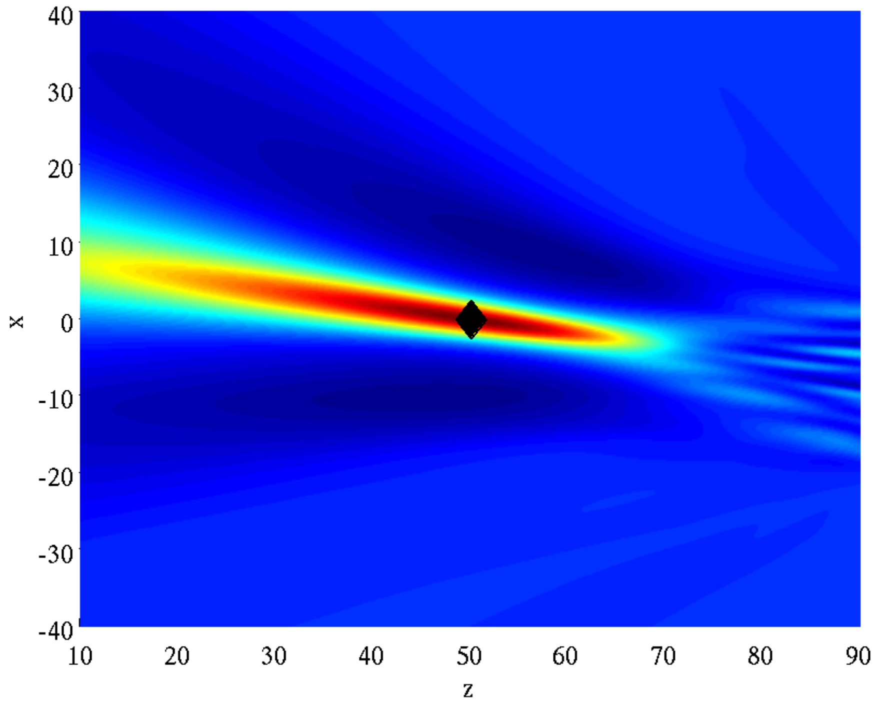
Theory: $\Delta C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$ *has singular components at* $\tau = \mathcal{T}(\boldsymbol{x}_l, \boldsymbol{z}_{\rm r}) - \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z}_{\rm r})$.

- Migration of the differential cross correlations ΔC.

Migration functional for the search point $\boldsymbol{z}^S$:

$$\mathcal{I}^{\rm B}(\boldsymbol{z}^S) = \sum_{j,l=1}^{N} \Delta C\big(\mathcal{T}(\boldsymbol{x}_l, \boldsymbol{z}^S) - \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z}^S), \boldsymbol{x}_j, \boldsymbol{x}_l\big)$$





Backlight configuration - resolution analysis

Migration functional

$$\mathcal{I}^{\mathrm{B}}(\boldsymbol{z}^S) = \sum_{j,l=1}^{N} \Delta C\big(\mathcal{T}(\boldsymbol{x}_l, \boldsymbol{z}^S) - \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z}^S), \boldsymbol{x}_j, \boldsymbol{x}_l\big)$$

$$= \frac{1}{2\pi} \int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\mathcal{T}(\boldsymbol{z}^S,\boldsymbol{x}_l)-\mathcal{T}(\boldsymbol{z}^S,\boldsymbol{x}_j)]} \widehat{\Delta C}\big(\mathcal{T}(\boldsymbol{x}_l, \boldsymbol{z}^S) - \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z}^S), \boldsymbol{x}_j, \boldsymbol{x}_l\big)$$

Analogy with Incoherent Interferometry imaging [1], used when $\boldsymbol{z}_{\mathrm{r}}$ is a source emitting an impulse that is recorded by passive sensors at $(\boldsymbol{x}_j)_{j=1,\ldots,N}$. The data is then the vector $(u(t, \boldsymbol{x}_j))_{j=1,\ldots,N}$. The MF functional is

$$\mathcal{I}^{\mathrm{MF}}(\boldsymbol{z}^S) = \frac{1}{2\pi} \int d\omega \Big| \sum_{l=1}^{N} e^{-i\omega\mathcal{T}(\boldsymbol{z}^S,\boldsymbol{x}_l)} \hat{u}(\omega, \boldsymbol{x}_l) \Big|^2$$

$$= \frac{1}{2\pi} \int d\omega \sum_{j,l=1}^{N} e^{-i\omega[\mathcal{T}(\boldsymbol{z}^S,\boldsymbol{x}_l)-\mathcal{T}(\boldsymbol{z}^S,\boldsymbol{x}_j)]} \hat{u}(\omega, \boldsymbol{x}_l)\overline{\hat{u}(\omega, \boldsymbol{x}_j)}$$

$\rightarrow$ Backlight cross correlation imaging with passive sensor arrays provides poor range resolution, as in Incoherent Interferometry imaging.

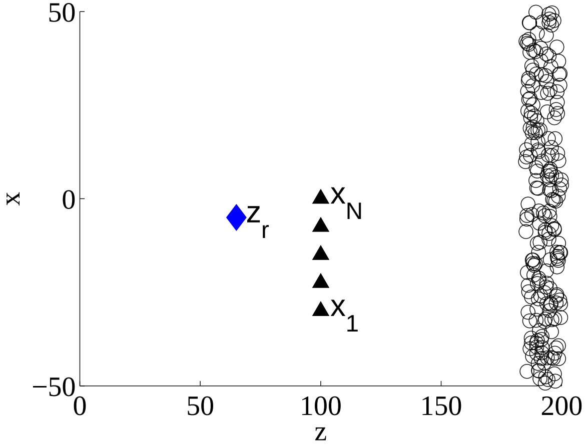
[1] L. Borcea, G. Papanicolaou, and C. Tsogka, *Inverse Problems* **19**, S134 (2003).

Backlight configuration

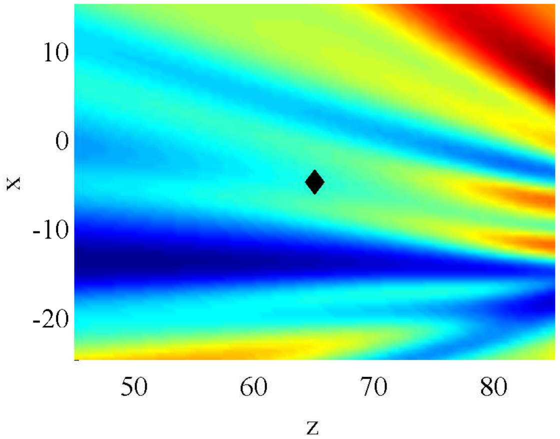
- Data *only* in the presence (C) of the reflector: $C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$, $j, l = 1, \ldots, 5$.

Theory: $C(\tau, \boldsymbol{x}_j, \boldsymbol{x}_l)$ *has singular components at* $\tau = \mathcal{T}(\boldsymbol{x}_l, \boldsymbol{z}_{\rm r}) - \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z}_{\rm r})$.
Triangular inequality $|\mathcal{T}(\boldsymbol{x}_l, \boldsymbol{z}_{\rm r}) - \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z}_{\rm r})| \leq \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{x}_l) \Longrightarrow$ *the singular components of the scattered waves are buried in the components of the direct waves*
$\hookrightarrow$ the coda cross correlation technique cannot be applied.

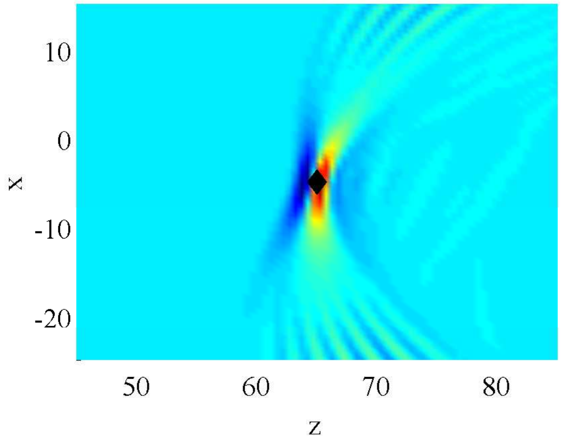
Imaging with daylight illumination



Configuration



$\mathcal{I}^{\mathrm{B}}$

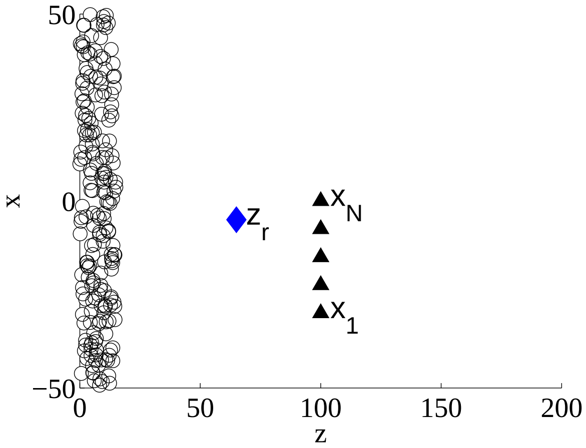


$\mathcal{I}^{\mathrm{D}}$

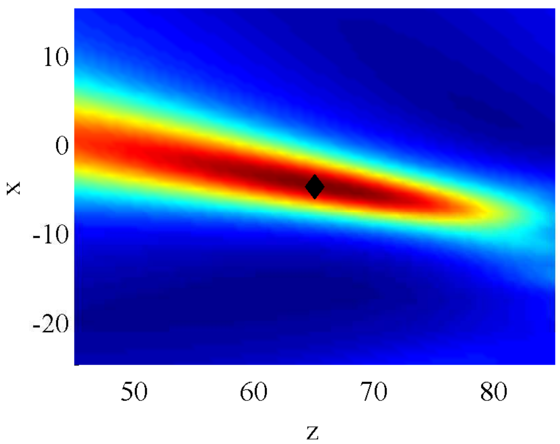
Passive sensor imaging using the differential cross correlation technique in a homogeneous medium. The daylight illumination configuration is plotted in the left figure: the circles are the noise sources and the triangles are the sensors.

Good range resolution ($\sim c_0/B$), good cross range resolution ($\sim \lambda_0 L/a$) for $\mathcal{I}^{\mathrm{D}}$ (sum of travel times is used in $\mathcal{I}^{\mathrm{D}}$, very sensitive to range).

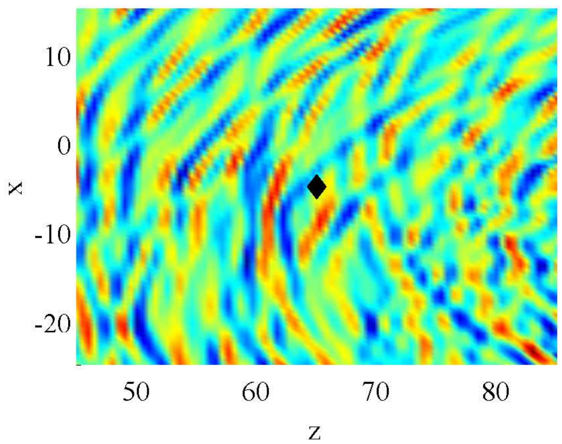
Imaging with backlight illumination



Configuration



$\mathcal{I}^{\mathrm{B}}$



$\mathcal{I}^{\mathrm{D}}$

Passive sensor imaging using the differential cross correlation technique in a homogeneous medium. The backlight illumination configuration is plotted in left figure: the circles are the noise sources and the triangles are the sensors.

Poor range resolution ($\sim \lambda_0 L^2/a^2$), good cross range resolution ($\sim \lambda_0 L/a$) for $\mathcal{I}^{\mathrm{B}}$ (difference of travel times is used in $\mathcal{I}^{\mathrm{B}}$, not sensitive to range).

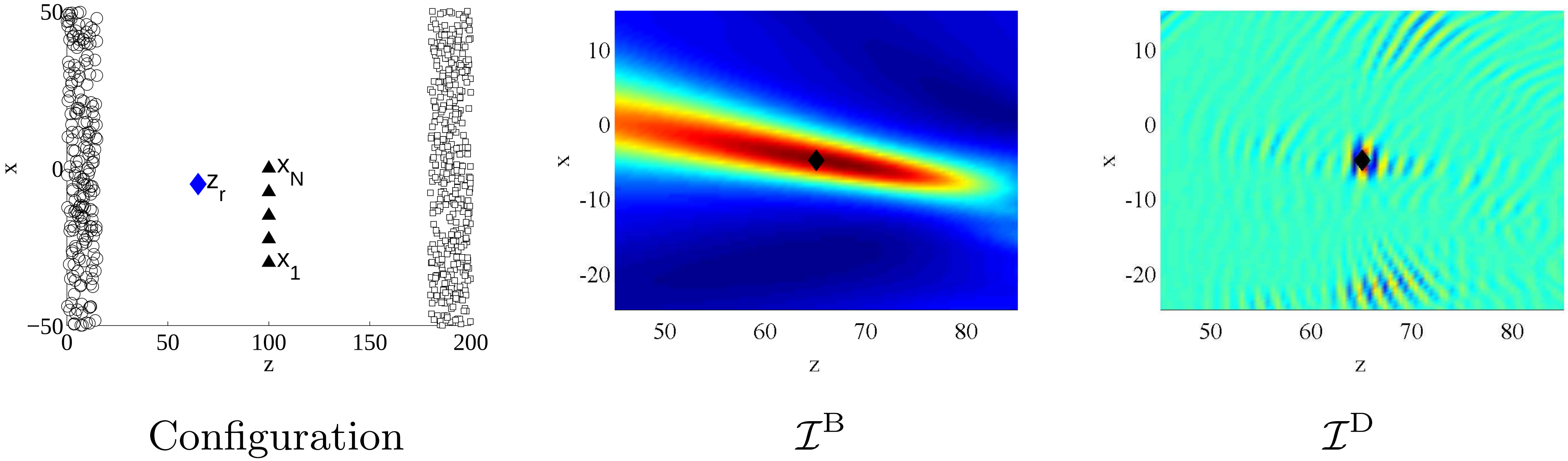
Imaging in a randomly scattering medium

What if the background is not homogeneous (or smoothly varying) but scattering ?

A scattering medium plays a dual role: it enhances the directional diversity of the illumination but it blurs the image.

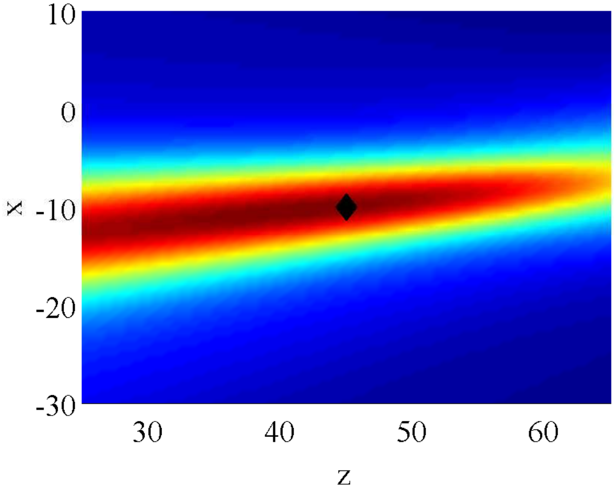
Migration of $C^{(1)}$ and/or $C^{(3)}$ with backlight and/or daylight imaging functionals.

Imaging with backlight illumination and with strong scattering

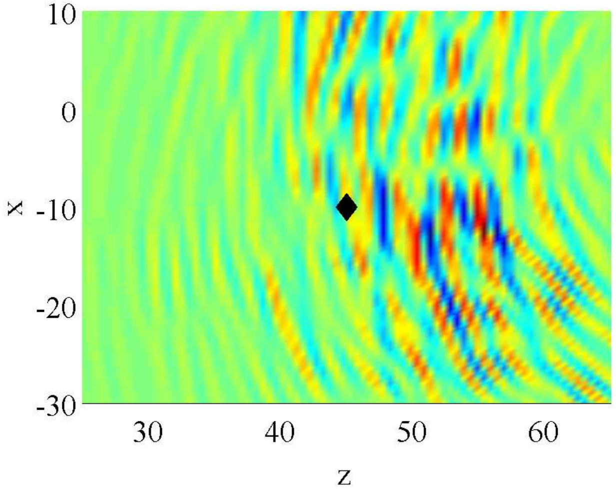


Passive sensor imaging using the differential cross correlation technique in a strongly scattering medium. The configuration is plotted in the left figure: the circles are the noise sources, the squares are the strong scatterers.

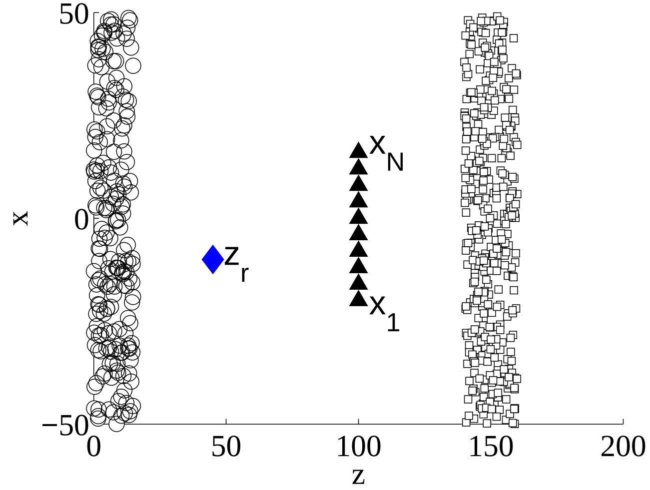
Imaging with backlight illumination and with <u>weak</u> scattering



$\mathcal{I}^{\mathrm{B}}$ with $\Delta C^{(1)}$

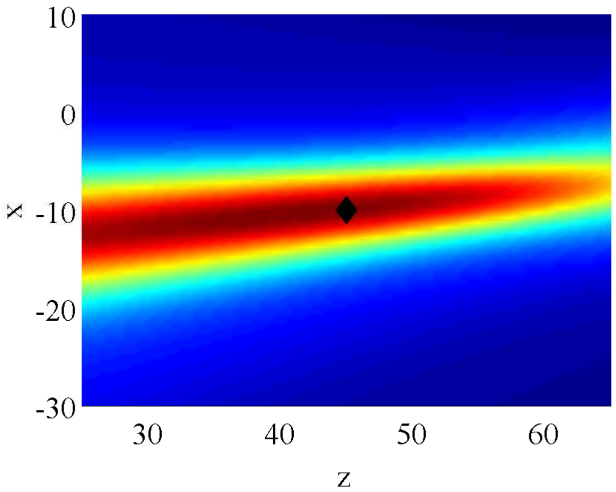


$\mathcal{I}^{\mathrm{D}}$ with $\Delta C^{(1)}$

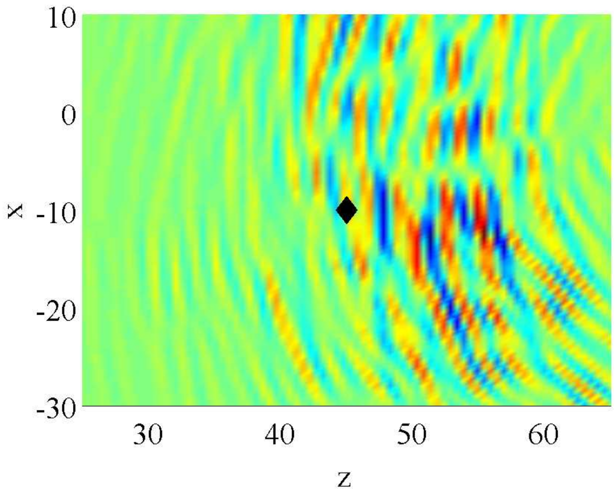


Configuration

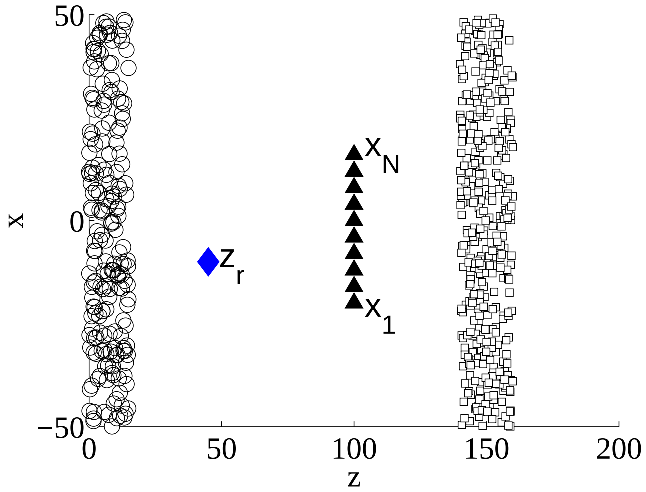
Imaging with backlight illumination and with weak scattering



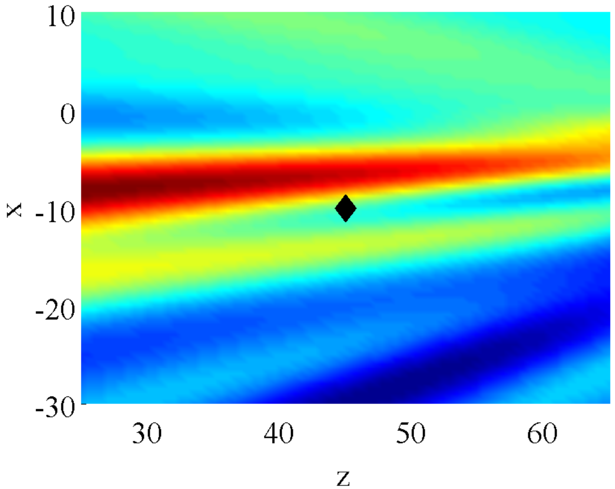
$\mathcal{I}^{\mathrm{B}}$ with $\Delta C^{(1)}$



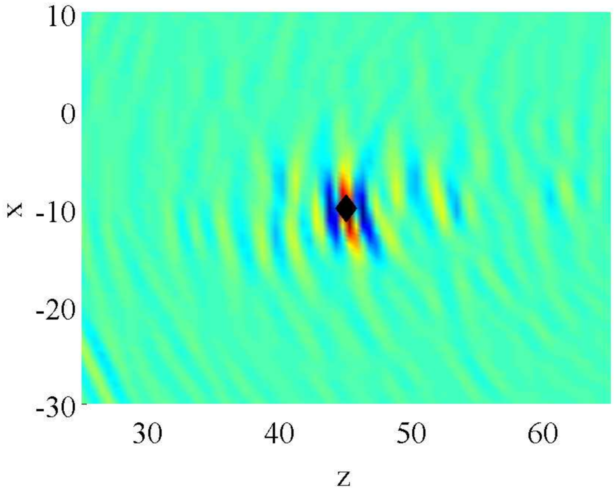
$\mathcal{I}^{\mathrm{D}}$ with $\Delta C^{(1)}$



Configuration



$\mathcal{I}^{\mathrm{B}}$ with $\Delta C^{(3)}$



$\mathcal{I}^{\mathrm{D}}$ with $\Delta C^{(3)}$

Conclusion

• Travel time estimation and imaging of reflectors are possible using cross correlation of ambient noise signals.

• It is possible to exploit the scattering properties of the medium for travel time estimation using special fourth-order cross correlation.

• It is possible to exploit the scattering properties of the medium for imaging by cross correlating and migrating the coda cross correlations ($\mathcal{I}^{\mathrm{D}}$ and/or $\mathcal{I}^{\mathrm{B}}$ migration with $C^{(1)}$ and/or $C^{(3)}$).

• Other propagation regimes can be analyzed: parabolic approximation, radiative transfer, randomly layered media.

• Main applications in geophysics (global, regional, and local scales: volcano monitoring[1], oil reservoir monitoring). Also in microwave imaging.

Cf:

J. Garnier and G. Papanicolaou, *SIAM J. Imaging Sciences* **2**, 396 (2009).

J. Garnier and G. Papanicolaou, *Passive Imaging with Ambient Noise*, CUP, in press.

[1] F. Brenguier, N. M. Shapiro, M. Campillo, et al, *Nature Geoscience* **1**, 126 (2008).